

A comparative study of numerical integration based on mixed quadrature rule and Haar wavelets *

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Abstract This note deals with the problem of determining the approximate solution of real definite integrals. Approximating the solution of given real definite integrals here is done by means of mixed quadrature rule which is then compared with the Haar wavelets for single, double and improper integrals. The main advantage of the proposed numerical integration method over Haar wavelets is its efficiency, lesser functional evaluation and simple applicability. Some numerical examples are provided to illustrate the accuracy and relative error of the proposed rule.

Key words Numerical integration, Mixed quadrature, Haar wavelets, Maclaurin's theorem, Degree of precision.

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1 Introduction

In the recent years different types of wavelet approaches have been used in the field of numerical integration for the benefits of science and technology. Numerical integration is the approximate computation of an integer involving integral equations and finite element methods. Siraj-ul- Islam et al. [1] have established wavelet and hybrid functions for numerical evaluation of the real definite integrals. Majid Rostami et al. [2] have proposed numerical integration rules based on block-pulse functions and Chebyshev wavelet to approximate the value of real definite integrals. A. Chakrabarti and S.C. Martha [3] have developed a straight method for numerical integration of Fredholm integral equation of the second kind using the least squares method. The Haar wavelet and the mixed quadrature methods have gained popularity among researchers due to their useful properties. Haar wavelets have been used in different numerical approximation problems due to their orthonormal properties with compact support. Lepik [4] has proposed Haar wavelets in solving differential equation, Lepik and Tamme [5] have applied the Haar wavelets for solving linear and non linear integral equations. Maleknejad and Mirzaee [6] have used linear integral equation via Haar wavelets. Lepik [7] used the Haar wavelets in solving non linear integro-differential equations and partial differential equations,

Mixed quadrature has faster convergence than the Haar wavelets due to its higher degree of precision for the evaluation of real definite integral. The mixed quadrature of higher precision has been established with linear convex combination of Gaussian and Newtonian type rule of lower precision each for

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single and double variables. Recently mixed quadrature rules have been successfully employed for the numerical solution of integral equations as well as finite element methods. Jena and Nayak [8] have applied the mixed quadrature rule to find the approximate solution of non linear Fredholm integral equation of the second kind, Jena and Dash [9] have established mixed quadrature sphere with finite element approach. Dash and Das [10] have proposed identification of some Clenshaw- Curtis rules with Fejer rules. Tripathy et al. [11] have used a mixed quadrature of Lobatto four point rule with Gaussian quadrature for the approximate evaluation of real definite integrals.

Motivated by the excellent performance of these methods, here we will apply the same technique with less functional evaluation for numerical integration of real definite integrals for single as well as double integrals for $(h \geq 4)$ and $(h < 4)$ for improper integrals respectively and compare the result with [1]. The organization of this paper is as follows. In section 2 numerical integration using Haar wavelets is described and in section 3 mixed quadrature rule is used for numerical integration. Error analysis for mixed quadrature is given in section 4 and lastly, numerical results are reported in section 5.

2 Numerical integration using Haar wavelets

In this section we consider numerical integration for single and double integrals using Haar wavelets. The scaling function for the family of Haar wavelet functions defined on the interval $[\alpha, \beta)$ is

$$\varphi_1(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta) \\ 0 & \text{elsewhere} \end{cases} \quad (2.1)$$

The mother wavelet for the Haar wavelets family defined on the interval $[\alpha, \beta)$ is

$$\varphi_2(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \frac{\alpha+\beta}{2}) \\ -1 & \text{for } x \in [\frac{\alpha+\beta}{2}, \beta) \\ 0 & \text{elsewhere} \end{cases} \quad (2.2)$$

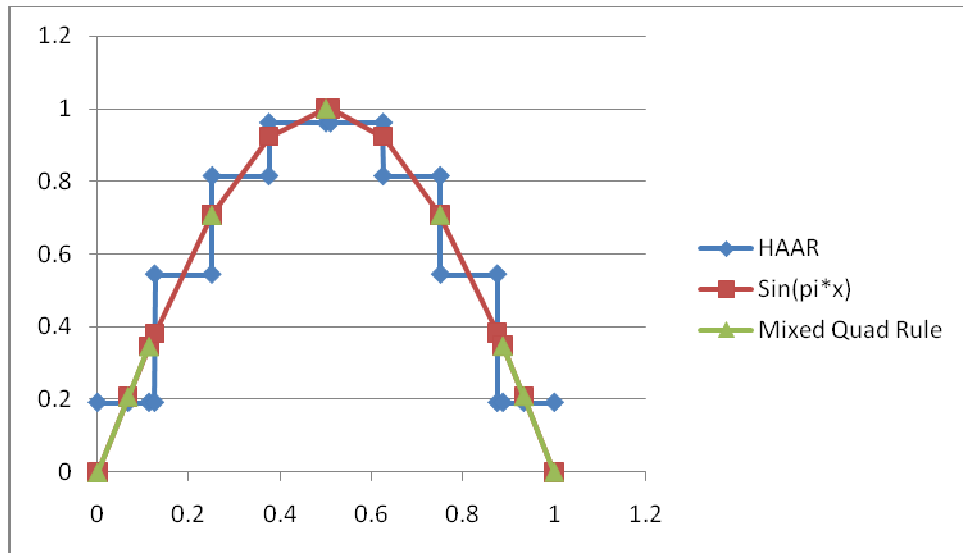


Fig. 1: Comparison of Exact result, Haar wavelets and Mixed quadrature of function $g(x) = \sin(\pi x)$ for $M = 4$.

2.1 Numerical integration for single integrals

We consider the integral

$$\int_a^b g(x) dx \quad (2.3)$$

over the interval $[\alpha, \beta]$, the function $g(x)$ can be approximated using Haar wavelets as

$$\int_{\alpha}^{\beta} g(x) dx \approx \frac{(\beta - \alpha)}{2M} \sum_{k=1}^{2M} g\left(\alpha + \frac{(\beta - \alpha)(k - 0.5)}{2M}\right) \quad (2.4)$$

where the maximal level of resolution is the integer h , $M = 2^h$, where $h = 0, 1, \dots, J$, $J = 2^M$.

2.2 Numerical technique for double integrals

We now consider the integral

$$\int_{\sigma}^{\delta} \int_{\alpha}^{\beta} g(x, y) dx dy \quad (2.5)$$

The function $g(x, y)$ can be approximated using Haar wavelets as

$$\int_{\sigma}^{\delta} \int_{\alpha}^{\beta} g(x, y) dx dy \approx \frac{(\beta - \alpha)(\delta - \sigma)}{(2M)^2} \sum_{l=1}^{2M} \sum_{k=1}^{2M} g(x_k, y_l) \quad (2.6)$$

$$\int_{\sigma}^{\delta} \int_{\alpha}^{\beta} g(x, y) dx dy \approx \frac{(\beta - \alpha)(\delta - \sigma)}{(2M)^2} \sum_{l=1}^{2M} \sum_{k=1}^{2M} g\left(\alpha + \frac{(\beta - \alpha)(k - 0.5)}{2M}, \sigma + \frac{(\delta - \sigma)(l - 0.5)}{2M}\right) \quad (2.7)$$

where

$$\begin{aligned} x_k &= \alpha + \frac{(\beta - \alpha)(k - 0.5)}{2M} \\ y_l &= \sigma + \frac{(\delta - \sigma)(l - 0.5)}{2M} \quad k, l = 1, 2, \dots, 2M \end{aligned} \quad (2.8)$$

In both cases the Haar wavelets approximation converges rapidly to the exact function by increasing the value of M .

3 Mixed Quadrature Rule

In this section we propose the numerical integration of single and double variables using mixed quadrature rules.

3.1 Numerical technique for single integrals

Lemma 3.1. *If the smooth function $f(x)$ is defined on $-1 \leq x \leq 1$, then the mixed quadrature rule $R_{GL32F5}(f)$ and error due to mixed quadrature for the numerical integration is*

$$I(f) = R_{GL32F5}(f) + E_{GL32F5}(f).$$

Proof. Gauss Legendre 3-point rule is

$$R_{GL3}(f) = \left[\frac{5}{9} \left\{ f\left(-\sqrt{\frac{3}{5}}\right) \right\} + 8f(0) + \frac{5}{9} \left\{ f\left(\sqrt{\frac{3}{5}}\right) \right\} \right]$$

Fejer's five point second rule for single integral is

$$R_{2F5}(f) = \frac{2}{45} \left[7 \left(f\left(\frac{\sqrt{3}}{2}\right) + f\left(-\frac{\sqrt{3}}{2}\right) \right) + 13f(0) + 9 \left(f\left(\frac{1}{2}\right) + f\left(-\frac{1}{2}\right) \right) \right]$$

each of the constituent rule $R_{GL3}(f)$ and $R_{2F5}(f)$ is of precision five. Let $E_{GL3}(f)$ and $E_{2F5}(f)$ denote the errors in approximating the integral $I(f)$ by the rules $R_{GL3}(f)$ and $R_{2F5}(f)$ respectively.

$$I(f) = R_{GL3}(f) + E_{GL3}(f) \quad (3.1)$$

$$I(f) = R_{2F5}(f) + E_{2F5}(f). \quad (3.2)$$

Using Maclaurin's expansion in (3.1) and (3.2), we get

$$E_{GL3}(f) \cong \frac{8}{175 \times 6!} f^{vi}(0) + \frac{88}{1125 \times 8!} f^{viii}(0) + \dots$$

$$E_{2F5}(f) \cong \frac{1}{67200} f^{vi}(0) + \frac{1}{1814400} f^{viii}(0) + \dots$$

Multiplying (3.1) by $(\frac{1}{64})$ and (3.2) by $(\frac{1}{15})$ respectively and subtracting them we have

$$I(f) = \int_{-1}^1 f(x) dx \cong \frac{1}{49} [64 R_{2F5}(f) - 15 R_{GL3}(f)] + \frac{1}{49} [64 E_{2F5}(f) - 15 E_{GL3}(f)]$$

$$I(f) = R_{GL32F5}(f) + E_{GL32F5}(f)$$

where

$$R_{GL32F5}(f) = \frac{1}{49} [64 R_{2F5}(f) - 15 R_{GL3}(f)] \quad (3.3)$$

and

$$E_{GL32F5}(f) = \frac{1}{49} [64 E_{2F5}(f) - 15 E_{GL3}(f)] \quad (3.4)$$

(3.3) and (3.4) are the required mixed quadrature rule and error due to the mixed quadrature rule. $\square$

Lemma 3.2. (For double integrals): The quadrature rule and error for the smooth function $f(x, y)$ which is defined on $0 \leq x \leq 1$ and $0 \leq y \leq 1$ is obtained by convex combination of $R_{GL3}(f)$ rules and Fejer's five point second rule respectively $E_{GL3}(f)$ and $E_{2F5}(f)$

$$I(f) = R_{GL32F5}(f) + E_{GL32F5}(f).$$

Proof. Gauss Legendre 3-point rule is

$$R_{GL3}(f) = \frac{1}{81} \left[\begin{aligned} &5 \left\{ 5f\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) + 8f\left(-\sqrt{\frac{3}{5}}, 0\right) + 5f\left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) \right\} + \\ &8 \left\{ 5f\left(0, -\sqrt{\frac{3}{5}}\right) + 8f(0, 0) + 5f\left(0, \sqrt{\frac{3}{5}}\right) \right\} + \\ &5 \left\{ 5f\left(\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) + 8f\left(\sqrt{\frac{3}{5}}, 0\right) + 5f\left(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) \right\} \end{aligned} \right] \quad (3.5)$$

Fejer's five point second rule is

$$R_{2F5}(f) = \frac{4}{2025} \left[\begin{aligned} &\left\{ 49f\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) + 49f\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right) + 91f\left(\frac{\sqrt{3}}{2}, 0\right) + 63f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + 63f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \right\} + \\ &\left\{ 49f\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) + 49f\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right) + 91f\left(-\frac{\sqrt{3}}{2}, 0\right) + 63f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + 63f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \right\} + \\ &\left\{ 91f\left(0, \frac{\sqrt{3}}{2}\right) + 91f\left(0, -\frac{\sqrt{3}}{2}\right) + 169f(0, 0) + 117f\left(0, \frac{1}{2}\right) + 117f\left(0, -\frac{1}{2}\right) \right\} + \\ &\left\{ 63f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + 63f\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) + 117f\left(\frac{1}{2}, 0\right) + 81f\left(\frac{1}{2}, \frac{1}{2}\right) + 81f\left(\frac{1}{2}, -\frac{1}{2}\right) \right\} + \\ &\left\{ 63f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + 63f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) + 117f\left(-\frac{1}{2}, 0\right) + 81f\left(-\frac{1}{2}, \frac{1}{2}\right) + 81f\left(-\frac{1}{2}, -\frac{1}{2}\right) \right\} + \end{aligned} \right] \quad (3.6)$$

each of the constituent rule $R_{GL3}(f)$ and $R_{2F5}(f)$ is of precision five. Let $E_{GL3}(f)$ and $E_{2F5}(f)$ denotes the errors in approximating the integral $I(f)$ by the rules $R_{GL3}(f)$ and $R_{2F5}(f)$ respectively.

$$I(f) = R_{GL3}(f) + E_{GL3}(f) \quad (3.7)$$

$$I(f) = R_{2F5}(f) + E_{2F5}(f) \quad (3.8)$$

Using Maclaurin's expansion in (3.7) and (3.8)

$$E_{GL3}(f) = \frac{16}{175 \times 6!} [f_{6,0}(0, 0) + f_{0,6}(0, 0)] + \frac{64}{75 \times 8!} [f_{6,2}(0, 0) + f_{2,6}(0, 0)] + \frac{286256}{703125 \times 8!} [f_{8,0}(0, 0) + f_{0,8}(0, 0)] + \dots$$

$$E_{2F5}(f) = \frac{3}{140 \times 6!} [f_{6,0}(0, 0) + f_{0,6}(0, 0)] + \frac{1}{201600} [f_{6,2}(0, 0) + f_{2,6}(0, 0)] + \frac{2}{45 \times 8!} [f_{8,0}(0, 0) + f_{0,8}(0, 0)] + \dots$$

Multiplying (3.7) by $(\frac{3}{4})$ and (3.8) by $(\frac{16}{5})$ respectively and subtracting them, we have

$$I(f) = \int_{-1}^1 \int_{-1}^1 f(x, y) dx dy \cong \frac{1}{49} [64 R_{2F5}(f) - 15 R_{GL3}(f)] + \frac{1}{49} [64 E_{2F5}(f) - 15 E_{GL3}(f)]$$

$$I(f) = R_{GL32F5}(f) + E_{GL32F5}(f)$$

where

$$R_{GL32F5}(f) = \frac{1}{49} [64 R_{2F5}(f) - 15 R_{GL3}(f)] \quad (3.9)$$

and

$$E_{GL32F5}(f) = \frac{1}{49} [64 E_{2F5}(f) - 15 E_{GL3}(f)] \quad (3.10)$$

(3.9) and (3.10) give the required mixed quadrature rule and the estimated error.

□

4 Error Analysis

Lemma 4.1. Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1, 1]$. The bounds of truncational error for single integrals $E_{GL32F5}(f)$ associated with the rule $R_{GL32F5}(f)$ is given by

$$|E_{GL32F5}(f)| \cong \frac{1}{7937967} |f^{viii}(0)|.$$

Lemma 4.2. Let $f(x, y)$ be sufficiently differentiable function in the closed interval $[-1, 1]$. The bounds of truncational error for double integrals $E_{GL32F5}(f)$ associated with the rule $R_{GL32F5}(f)$ is given by

$$|E_{GL32F5}(f)| \cong \frac{458768}{6890625 \times 8!} |f_{8,0}(0, 0) + f_{0,8}(0, 0)|.$$

Lemma 4.3. The error bound for the truncation error is given by $|E_{GL32F5}(f)| \leq \frac{8K}{1715 \times 5!}$ where $K = \max_{-1 \leq x \leq 1} |f^{vii}(x)|$.

Proof.

$$E_{GL3}(f) \cong \frac{8}{175 \times 6!} f^{vi}(\eta_1),$$

$$E_{2F5}(f) = \frac{1}{67200} f^{vi}(\eta_2) \quad \text{where } \eta_1, \eta_2 \in [-1, 1].$$

As $f^{vi}(x)$ is continuous and bounded in $[-1, 1]$, so there exist points c and d in the interval $[-1, 1]$ such that $k_1 = f^{vi}(c)$ and $k_2 = f^{vi}(d)$ where $k_1 = \max_{-1 \leq x \leq 1} |f^{vi}(x)|$ and $k_2 = \min_{-1 \leq x \leq 1} |f^{vi}(x)|$.

$$\begin{aligned} |E_{GL32F5}(f)| &\cong \frac{4}{1715 \times 5!} \{f^{vi}(c) - f^{vi}(d)\} \\ &= \frac{4}{1715 \times 5!} \int_c^d f^{vii}(x) dx \\ &= \frac{4}{1715 \times 5!} (d - c) f^{vii}(\zeta) \\ &\quad \text{for some } \zeta \in [-1, 1]. \end{aligned}$$

By Mean Value Theorem $|c - d| \leq 2$ (Conte and Boor [12]), then

$$|E_{GL32F5}(f)| \leq \frac{8}{1715 \times 5!} f^{vii}(\zeta)$$

$$|E_{GL32F5}(f)| \leq \frac{8K}{1715 \times 5!} \quad \text{where } K = \max_{-1 \leq x \leq 1} |f^{vii}(x)|.$$

□

Lemma 4.4. The bounds for the truncational error $|E_{GL32F5}(f)| \leq \frac{96K}{1715 \times 6!} (\eta_2 - \eta_1)$, where $\eta_1, \eta_2 \in [-1, 1]$ and $K = \max_{\substack{-1 \leq x \leq 1 \\ -1 \leq y \leq 1}} |f_{7,0}(x, *) + f_{0,7}(*, y)|$.

Proof. $E_{2F5}(f) = \frac{3}{140 \times 6!} [f_{6,0}(\eta_1) + f_{0,6}(\eta_1)]$ $E_{GL3}(f) = \frac{16}{175 \times 6!} [f_{6,0}(\eta_2) + f_{0,6}(\eta_2)]$ where $\eta_1, \eta_2 \in [-1, 1]$.

$$\begin{aligned}
E_{GL32F5}(f) &= \frac{1}{49} [64E_{2F5}(f) - 15E_{GL3}(f)] \\
&= \frac{48}{1715 \times 6!} [f_{6,0}(\eta_2, 0) + f_{0,6}(0, \eta_2) - f_{6,0}(\eta_1, 0) - f_{0,6}(0, \eta_1)] \\
&= \frac{48}{1715 \times 6!} \left[\int_{\eta_1}^{\eta_2} f_{7,0}(x, 0) dx + \int_{\eta_1}^{\eta_2} f_{0,7}(0, y) dy \right] \\
&= \frac{48}{1715 \times 6!} \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} [f_{7,0}(x, *) + f_{0,7}(*, y)] dx dy \\
|E_{GL32F5}(f)| &\leq \frac{48K}{1715 \times 6! (\eta_2 - \eta_1)} |\eta_2 - \eta_1|^2 \\
&\leq \frac{48K}{1715 \times 6!} (\eta_2 - \eta_1)
\end{aligned}$$

where $K = \max_{\substack{-1 \leq x \leq 1 \\ -1 \leq y \leq 1}} |f_{7,0}(x, *) + f_{0,7}(*, y)|$,

which gives only the truncational error bound on η_1, η_2 which are known points in $[-1, 1]$. From the above it is clear that the error in approximation will be less if the points η_1, η_2 are close to each other. $\square$

Corollary 4.5. *The error bound for the truncational error $E_{GL32F5}(f)$ is given by $|E_{GL32F5}(f)| \leq \frac{96K}{1715 \times 6!}$ when $|\eta_1 - \eta_2| \leq 2$ (Conte and Boor [12]).*

5 Numerical Examples

In this section some numerical examples are taken to validate our proposed work.

Example 5.1.

$$I_1 = \int_0^1 \sin(x^2) dx = \frac{1}{2} \int_{-1}^1 \sin\left(\left(\frac{t+1}{2}\right)^2\right) dt$$

Example 5.2.

$$I_2 = \int_0^5 \sqrt{x^2 - 5x + 31} dx = \frac{5}{2} \sqrt{\left(\frac{5(t+1)}{2}\right)^2 - 5\left(\frac{5(t+1)}{2}\right) + 31} dt$$

Example 5.3. (Improper Integral)

$$I_3 = \int_0^1 \frac{e^{-\frac{1}{x}}}{x^2} dx = \int_1^\infty e^{-u} du = \int_0^{e^{-1}} dt = \frac{1}{2} \int_{-1}^1 e^{-1} dv$$

Example 5.4.

$$I_4 = \int_0^{\frac{\pi}{2}} \int_0^\pi \sin(x+y) dx dy = \frac{\pi^2}{8} \int_{-1}^1 \int_{-1}^1 \sin\left(\frac{\pi}{2}(t+1) + \frac{\pi}{4}(z+1)\right) dt dz$$

Example 5.5. (Improper Integral)

$$I_5 = \int_0^1 \int_0^1 \frac{1}{(x^2 + y^2)} dy dx = \left(\frac{\delta - \sigma}{2}\right) \left(\frac{\beta - \alpha}{2}\right) \sum_{j=1}^m \sum_{i=1}^n c_j c_i g(x_i, y_j)$$

(For Improper integral in double integration only we use (3.5), (3.6) and substitute them in (3.9).)

Table 1: Relative error of Example 5.1

	Siraj-ul- Islam et al. [1]	Due to $R_{GL32F5}(f)$
$J = 4$	1.4177e-04	
$J = 5$	3.5432e-05	2.5865e-06
$J = 6$	8.8574e-06	
$J = 7$	2.2143e-06	

Table 2: Relative error of Example 5.2

	Siraj-ul- Islam et al. [1]	Due to $R_{GL32F5}(f)$
$J = 4$	3.5293e-05	2.4307e-07
$J = 5$	8.8229e-06	
$J = 6$	2.2057e-06	
$J = 7$	5.5143e-07	

Table 3: Relative error of Example 5.3

	Siraj-ul- Islam et al. [1]	Due to $R_{GL32F5}(f)$
$J = 4$	4.0642e-05	0.0000
$J = 5$	1.0173e-05	
$J = 6$	2.5431e-06	
$J = 7$	6.3578e-07	

Table 4: Relative error of Example 5.4

	Siraj-ul- Islam et al. [1]	Due to $R_{GL32F5}(f)$
$J = 4$	5.0215e-04	3.4026e-006
$J = 5$	1.2551e-04	
$J = 6$	3.1375e-05	
$J = 7$	7.8437e-06	

Table 5: Relative error of Example 5.5

	Siraj-ul- Islam et al. [1]	Due to $R_{GL32F5}(f)$
$J = 4$	7.1275e-03	1.4921e-002
$J = 5$	3.5719e-03	
$J = 6$	1.7880e-03	
$J = 7$	8.9450e-04	

References

- [1] Islam, S.U., Aziz, I. and Haq, F. (2010). A comparative study of numerical integration based on Haar wavelets and hybrid functions, *Computers and Mathematics with Applications*, 59, 2026–2036.
- [2] Rostami, M., Hashemizadeh, E. and Heidari, M. (2012). A comparative study of numerical integration based on block-pulse and sinc functions and Chebyshev wavelet, *Mathematical Sciences*, 6(1), 8.
- [3] Chakrabarti, A. and Martha, S.C. (2009). Approximate solution of Fredholm integral equations of second kind, *Applied Mathematics and Computation*, 211, 459–466.
- [4] Lepik, Ü. (2001). Numerical solution of differential equations using Haar wavelets, *Math. Comput. Simulation*, 57, 347–353.
- [5] Lepik, Ü, and Tamme, E. (2004). Application of the Haar wavelets for solution of linear integral equations, *Turkey Dynamical Systems and Applications, Proceedings*, 494–507.
- [6] Maleknejad, K. and Mirzaee, F. (2005). Using rationalized Haar wavelet for solving linear integral equations, *Appl. Math. Comp.*, 160, 579–587.
- [7] Lepik, U. (2005). Haar wavelet method for nonlinear integro-differential equations, *Appl. Math. Comp.*, 185, 695–704.
- [8] Jena, S. and Nayak, D. (2015). Hybrid quadrature for numerical treatment of nonlinear Fredholm Integral equation with separable kernel, *Int. J. Appl. Math. Stat.*, 53, 83–89.
- [9] Jena, S. and Dash, P. (2014). Approximation of real definite integrals via hybrid quadrature domain, *Int. J. Sci. Engg. Tech. Res.*, 3, 3188–3191.
- [10] Dash, R.B. and Das, D. (2011). Identification of some Clenshaw- Curtis quadrature rule as mixed quadrature of Fejer and Newton-Cote type of rules, *Int. J. of Mathematical Sciences and Applications*, 1, 1493–1496.
- [11] Tripathy, A.K., Dash, R.B. and Baral, A. (2015). A mixed quadrature rule blending Lobatto and Gauss Legendre three point rule for approximate evaluation of real definite integrals, *Journal of Computing Science and Mathematics*, 6, 366–377.
- [12] Conte, S. and Boor, C. De. (1980). Elementary Numerical Analysis, Tata Mac-Graw Hill, New Delhi.