



Spline collocation approach to study two-dimensional and axisymmetric unsteady flow*

Pinky M. Shah¹, Jyotindra C. Prajapati²

1. Department of Mathematics, Veer Narmad South Gujarat University,
Surat, Gujarat-365005, India.

2. Department of Mathematics, Sardar Patel University,
Vallabh Vidyanagar, Anand, Gujarat-388120, India.

1. E-mail: pinkyshah2302@gmail.com , 2. E-mail: jyotindra18@rediffmail.com

Abstract An analysis is carried out in this paper to study two-dimensional and axisymmetric flow of a viscous incompressible fluid between two parallel plates. The governing non-linear equation of the flow problem is transformed into a linear differential equation using quasilinearization technique (Bellman and Kalaba (1965), Quasilinearization and non-linear boundary value problems, American Elsevier Publishing Company Inc., New York) and quintic spline collocation method is applied to solve the linear problem (Bickley (1968), Piecewise cubic interpolation and two-point boundary value problems, Comp. J., 11, 206–208). The numerical results obtained by this method are compared with homotopy perturbation method (HPM) and the Runge-Kutta fourth order method. The physical interpretation is discussed and the results are demonstrated graphically.

Key words Squeezing flow, Non-linear equation, Spline collocation method, Quasilinearization technique, Homotopy perturbation method.

2010 Mathematics Subject Classification Primary: 34B15, 34G20, 41A15.

1 Introduction

Squeezing flow arises in many engineering and biological systems. Squeezing flow between parallel walls occurs in polymer processing, compression, and injection molding and it also demonstrates the lubrication system. Stefan [17] studied squeezing flow by using lubrication assumption. In 1886 Reynolds [16] deliberated his work for elliptic plates, and Archibald [1] investigated the squeezing flow between the rectangular plates. Squeezing flow model is studied by several researchers to make it more understandable [7, 18, 19, 21]. Analytic solutions for unsteady two-dimensional and axisymmetric squeezing flows between parallel plates are presented by Rashidi et al. [15]. The effects of fluid inertia squeezed films between two circular glasses plates were studied by Kuzma [14]. The homotopy perturbation method (HPM) was introduced by the Chinese researcher J. H. He (see [8–13]). Ariel et al. [2] applied this method to axisymmetric flow over a stretching sheet. The series solution of the two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates was obtained

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).

Received December 28, 2018 / Revised April 12, 2019 / Accepted May 18, 2019. Online First Published on December 24, 2019 at <https://www.bpasjournals.com/>.

Corresponding author Jyotindra C. Prajapati, E-mail: jyotindra18@rediffmail.com

by Ghorbani [6]. In many practical situations the equations describing the flow are non-linear and it is often difficult to find exact solution. To overcome this problem, several numerical schemes are developed to approximate the solution in a better way, one such techniques is the Spline Collocation technique, which is very effective and easy to apply. Bickley [4] considered the use of cubic splines for solving linear two-point boundary value problems, which leads to the solution of a set of linear algebraic equations. The purpose of the present paper is to apply the cubic spline procedure for finding the numerical solution of a linear boundary value problem. The obtained results are compared with those obtained by the Runge-Kutta fourth order method and the HPM.

2 Mathematical Formulation

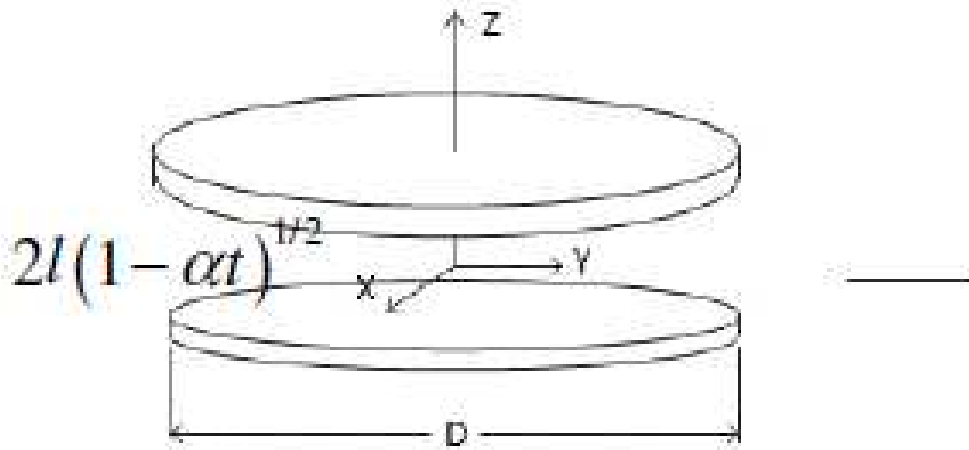


Fig. 1: Schematic Diagram of Unsteady Fluid Problem.

Let the position of the two plates be at $z = \pm l\sqrt{1 - \alpha t}$, where l is the position at time $t = 0$ as shown in the Fig. 1. Let us consider the length l (in the two-dimensional case) or the diameter D (in the axisymmetric case) are significantly larger than the gap width $2z$ at any time. The two plates are squeezed if $\alpha > 0$, until they touch at $t = \frac{1}{\alpha}$, the plates are separated if $\alpha < 0$. The velocity components in the x, y and z directions are u, v and w respectively.

Wang [20] introduced the transformations as mentioned below for the two-dimensional flow,

$$\begin{aligned} u &= \frac{\alpha x}{2(1 - \alpha t)} f'(\eta), \\ w &= -\frac{\alpha l}{2\sqrt{1 - \alpha t}} f(\eta) \end{aligned} \quad (2.1)$$

where, $\eta = \frac{z}{l\sqrt{1 - \alpha t}}$.

Substituting the above transformations into the unsteady two-dimensional Navier-Stokes equation yields

$$f^{iv} + S \{-\eta f''' - 3f'' - f'f'' + ff'''\} = 0 \quad (2.2)$$

where, $S = \frac{\alpha l^2}{2\nu}$ is the squeeze number.

The boundary conditions are given by,

$$\begin{aligned} f(0) &= 0, f''(0) = 0 \\ f(1) &= 1, f'(1) = 0. \end{aligned} \quad (2.3)$$

Similarly, for axisymmetric flow Wang gave the following transforms,

$$\left. \begin{aligned} u &= \frac{\alpha x}{4(1 - \alpha t)} f'(\eta), \\ v &= \frac{\alpha y}{4(1 - \alpha t)} f'(\eta), \\ w &= \frac{-\alpha l}{2\sqrt{1 - \alpha t}} f(\eta). \end{aligned} \right\} \quad (2.4)$$

Using the transformations (2.4), unsteady axisymmetric Navier-Stokes equation reduces to,

$$f^{iv} + S \{-\eta f''' - 3f'' + f f'''\} = 0 \quad (2.5)$$

subject to (2.3).

The following non-linear ordinary differential equation is to be solved

$$f^{iv} + S \{-\eta f''' - 3f'' - \beta f' f'' + f f'''\} = 0 \quad (2.6)$$

where, $\beta = 0$ for the axisymmetric case and $\beta = l$ for the two-dimensional case subject to the boundary condition (2.3).

The governing differential equation of squeezing flow problem was also studied by Dinarvand and Moradi [5].

3 Spline Solution of Unsteady Flows

An application of quasilinearization method to the non-linear boundary value problem (2.6) is made to convert it into a linear boundary value problem. Let i indicate the number of iterations for f_i and its consequent expressions then, using (2.6), the equation for f_{i+1} is described as,

$$f_{i+1}^{iv} + S(-\eta + f_i) f_{i+1}''' - S(3 + \beta f_i') f_{i+1}'' - S\beta f_i'' f_{i+1}' + S f_i''' f_{i+1} = S f_i f_i''' - S\beta f_i'' f_i'$$

and the boundary conditions are

$$f_{i+1}(0) = 0, f_{i+1}'(0) = 0, f_{i+1}(1) = 1, f_{i+1}'(1) = 0.$$

In order to obtain a cubic spline approximation, let us begin with $f_0(\eta) = a\eta^3 + b\eta^2 + c\eta + d$ substituting and satisfying the conditions as an initial guess. This is found to be,

$$f_0(\eta) = -\frac{1}{2}\eta^3 + \frac{3}{2}\eta.$$

Divide $[0, 1]$ into n equal subintervals. The cubic spline for this particular problem is

$$s(\eta) = a_0 + b_0(\eta - \eta_0) + \frac{1}{2}c_0(\eta - \eta_0)^2 + \frac{1}{6}d_0(\eta - \eta_0)^3 + \frac{1}{24}e_0(\eta - \eta_0)^4 + \frac{1}{120} \sum_{k=0}^{n-1} f_k(\eta_i - \eta_k)_+^5$$

The unknowns $a_0, b_0, c_0, d_0, e_0, f_0, f_1, \dots, f_{n-1}$ are to be determined by solving the following system of equations for $i = 0, 1, 2, \dots, n$:

$$\begin{aligned} \sum_{k=0}^{n-1} f_k \left[(\eta_i - \eta_k) + \frac{(-S\eta_i + Sf_i)}{2}(\eta_i - \eta_k)^2 + \frac{(-3S - \beta Sf_i')}{6}(\eta_i - \eta_k)^3 + \frac{(-\beta Sf_i')}{24}(\eta_i - \eta_k)^4 + \right. \\ \left. \frac{(Sf_i'')}{120}(\eta_i - \eta_k)^5 \right] + e_0 \left[1 + (-S\eta_i + Sf_i)(\eta_i - \eta_0) + \frac{(-3S - \beta Sf_i')}{2}(\eta_i - \eta_0)^2 + \frac{(-\beta Sf_i'')}{6}(\eta_i - \eta_0)^3 + \right. \\ \left. \frac{(Sf_i'')}{24}(\eta_i - \eta_0)^4 \right] + d_0 \left[(-S\eta_i + Sf_i) + (-3S - \beta Sf_i')(\eta_i - \eta_0) + \frac{(-\beta Sf_i'')}{2}(\eta_i - \eta_0)^2 + \right. \\ \left. \frac{(Sf_i''')}{6}(\eta_i - \eta_0)^3 \right] + c_0 \left[(-3S - \beta Sf_i') + (-\beta Sf_i'')(\eta_i - \eta_0) + \frac{(Sf_i''')}{2}(\eta_i - \eta_0)^2 \right] + \\ b_0 [(-\beta Sf_i'') + (Sf_i''')(\eta_i - \eta_0)] + a_0 [(Sf_i''')] = S f_i f_i''' - S\beta f_i'' f_i' \end{aligned}$$

The boundary conditions give following equations,

$$\begin{aligned} a_0 = 0, c_0 = 0 \\ a_0 + b_0(1 - \eta_0) + \frac{1}{2}c_0(1 - \eta_0)^2 + \frac{1}{6}d_0(1 - \eta_0)^3 + \frac{1}{24}e_0(1 - \eta_0)^4 + \frac{1}{120} \sum_{k=0}^{n-1} f_k(1 - \eta_k)_+^5 = 1 \\ b_0 + c_0(1 - \eta_0) + \frac{1}{2}d_0(1 - \eta_0)^2 + \frac{1}{6}e_0(1 - \eta_0)^3 + \frac{1}{24} \sum_{k=0}^{n-1} f_k(1 - \eta_k)_+^4 = 0. \end{aligned}$$

We can write the above equations in a matrix form $AX = B$, let consider $h = \frac{1}{6}$ so that $n = 6$. In this case, the matrix A is an (11×11) upper triangular matrix. The quintic spline $s(\eta)$ of the equation is then accomplished for which we present the values at different points in the tables: Table 1 and Table 2.

Table 1: Spline results of $f(\eta)$ at different terms for the Axisymmetric Flow.

S	X	Spline Solution	Runge-Kutta 4	HPM
-1.5	0.2	0.333597	0.333618	0.333618
-1.5	0.4	0.624378	0.624358	0.624358
-1.5	0.6	0.839314	0.839325	0.839325
-1.5	0.8	0.962974	0.962984	0.962984
-0.5	0.2	0.305486	0.305545	0.305545
-0.5	0.4	0.582445	0.582470	0.582470
-0.5	0.6	0.804384	0.804392	0.804392
-0.5	0.8	0.949106	0.949108	0.949108
0.5	0.2	0.288260	0.288260	0.288260
0.5	0.4	0.556145	0.556143	0.556143
0.5	0.6	0.781673	0.781671	0.781671
0.5	0.8	0.939641	0.93964	0.93964
1.5	0.2	0.276353	0.276432	0.276432
1.5	0.4	0.537760	0.537752	0.537752
1.5	0.6	0.765262	0.765249	0.765249
1.5	0.8	0.932471	0.932471	0.932471

4 Summary and Conclusion

The governing non-linear differential equation of fourth order is treated numerically with spline collocation method. For several values of S the tables – Table 1 and Table 2 describe the comparison of the solutions obtained by using the quintic spline approximation, the HPM and the Runge-Kutta fourth order method for the axisymmetric and two-dimensional cases. Figure 2 demonstrates the effect of $f'(\eta)$ with the variation in the positive values of S for the two-dimensional case. Figure 3 displays the influence of negative S on $f'(\eta)$ for the axisymmetric case. Figure 4 and Figure 5 are pictorial representation of skin friction and pressure gradient for axisymmetric and two-dimensional squeezing flow respectively. The solutions obtained by spline function show good agreement and remarkable accuracy with the solutions obtained through the HPM and the Runge-Kutta fourth order method.

Acknowledgements The authors are grateful to Harish D. Doctor for his appreciable remarks for the betterment of this work.

References

- [1] Archibald, F.R. (1956). Load capacity and time relations for squeeze films, *J. Lubr. Technol.*, 78, A231 – A245.
- [2] Ariel, P.D., Hayat, T. and Asghar, S. (2006). Homotopy perturbation method and axisymmetric flow over a stretching sheet, *Int. J. Nonlinear Sci. Numer. Simul.*, 7, 399–406.
- [3] Bellman, R.E. and Kalaba, R.E. (1965). Quasilinearization and non-linear boundary value problems, American Elsevier Publishing Company Inc., New York.
- [4] Bickley, W.G. (1968). Piecewise cubic interpolation and two-point boundary value problems, *Comp. J.*, 11, 206–208.

Table 2: Spline results of $f(\eta)$ at different terms for Two-dimensional Flow.

S	X	Spine Solution	Runge-Kutta 4	HPM
-1.5	0.2	0.319517	0.319526	0.319526
-1.5	0.4	0.603837	0.60383	0.60383
-1.5	0.6	0.822852	0.822876	0.822876
-1.5	0.8	0.956804	0.956801	0.956800
-0.5	0.2	0.302587	0.302582	0.302582
-0.5	0.4	0.578075	0.578082	0.578082
-0.5	0.6	0.800772	0.80078	0.80078
-0.5	0.8	0.947708	0.947702	0.947702
0.5	0.2	0.290291	0.290322	0.290322
0.5	0.4	0.559205	0.559252	0.559252
0.5	0.6	0.784261	0.784303	0.784303
0.5	0.8	0.940684	0.940703	0.940703
1.5	0.2	0.281019	0.28101	0.28101
1.5	0.4	0.544774	0.544779	0.544779
1.5	0.6	0.771361	0.771371	0.771371
1.5	0.8	0.935034	0.935036	0.935036

- [5] Dinarvand, S. and Moradi, A. (2012). Two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates, *Journal of Applied Mathematics*, Article ID 938624, 13 pages.
- [6] Ghorbani, A. (2009). Beyond Adomians polynomials: Hes polynomials, *Chaos Solitons Fract.*, 39, 1486–1492.
- [7] Grimm, R.J. (1976). Squeezing flows of Newtonian liquid films: an analysis includes the fluid inertia, *Appl. Sci. Res.*, 32 (2), 149–166.
- [8] He, J.H. (2000). A new perturbation technique which is also valid for large parameters, *J. Sound Vib.*, 229, 1257–1263.
- [9] He, J.H. (2003). Homotopy perturbation method: a new nonlinear analytical technique, *Applied Mathematics and Computation*, 135, 73–79.
- [10] He, J.H. (2006). Homotopy perturbation method for solving boundary value problems, *Phys. Lett. A.*, 350, 87–88.
- [11] He, J.H. (2000). A coupling method of homotopy technique and perturbation technique for non-linear problems, *International. Journal. Non-Linear Mech.*, 35, 37–43.
- [12] He, J.H. (1999). Homotopy perturbation technique, *Comp Meth. Appl. Mech. Eng.*, 178, 257–262.
- [13] He, J.H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation*, 151, 287–292.
- [14] Kuzma, D.C. (1968). Fluid inertia effects in squeeze films, *Applied Scientific Research*, 18, 15–20.
- [15] Rashidi, M.M., Shahmohamadi, H. and Dinarvand, S. (2008). Analytic approximate solutions for unsteady two-dimensional and axisymmetric squeezing flows between parallel plates, *Mathematical Problems in Engineering*, 1–13. Article ID 935095, <http://dx.doi.org/10.1155/2008/935095>
- [16] Reynolds, O. (1886). On the theory of lubrication and its application to Mr. Beauchamp Towers experiments, including an experimental determination of the viscosity of olive oil, *Philos. Trans. R. Soc. Lond.*, 177, 157–234.

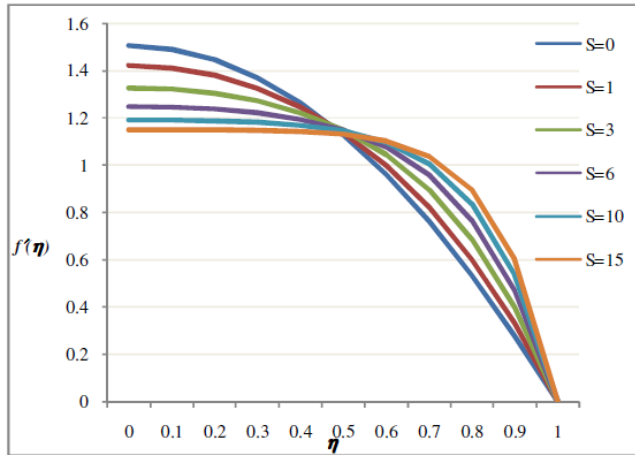


Fig. 2: $f'(\eta)$ for the Two-dimensional flow ($\beta = 1$), when S is positive.

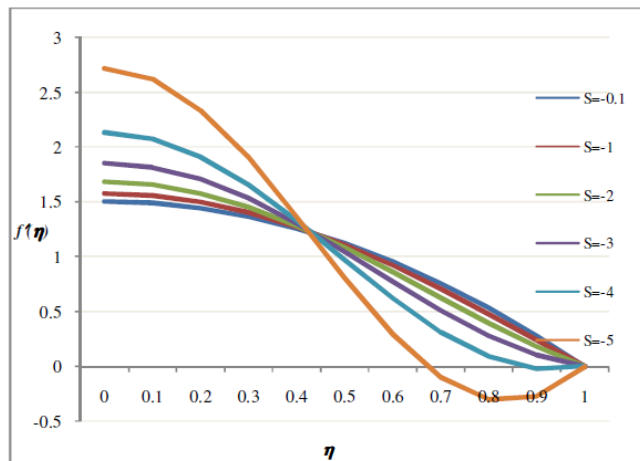


Fig. 3: $f'(\eta)$ for the Axisymmetric flow ($\beta = 0$), when S is negative.

- [17] Stefan, M.J. (1874). Versuch er die scheinbare adhesion, Sitzungsberichte der Akademie der Wissenschaften in Wien, *Mathematik-Naturwissen*, 69, 713–721.
- [18] Tichy, J.A. and Winer, Wo (1970). Inertial considerations in parallel circular squeeze film bearings, *J. Lubr. Techno.*, 92, 588–592.
- [19] Usha, R. and Sridharan, R. (1996). Arbitrary squeezing of a viscous fluid between elliptic plates, *Fluid Dyn. Res.*, 18, 35–51.
- [20] Wang, C.Y. (1976). The squeezing of fluid between two plates, *J. Appl. Mech.*, 43 (4), 579–583.
- [21] Wolfe, W.A. (1965). Squeeze film pressures, *Appl. Sci. Res.*, 14, 77–90.

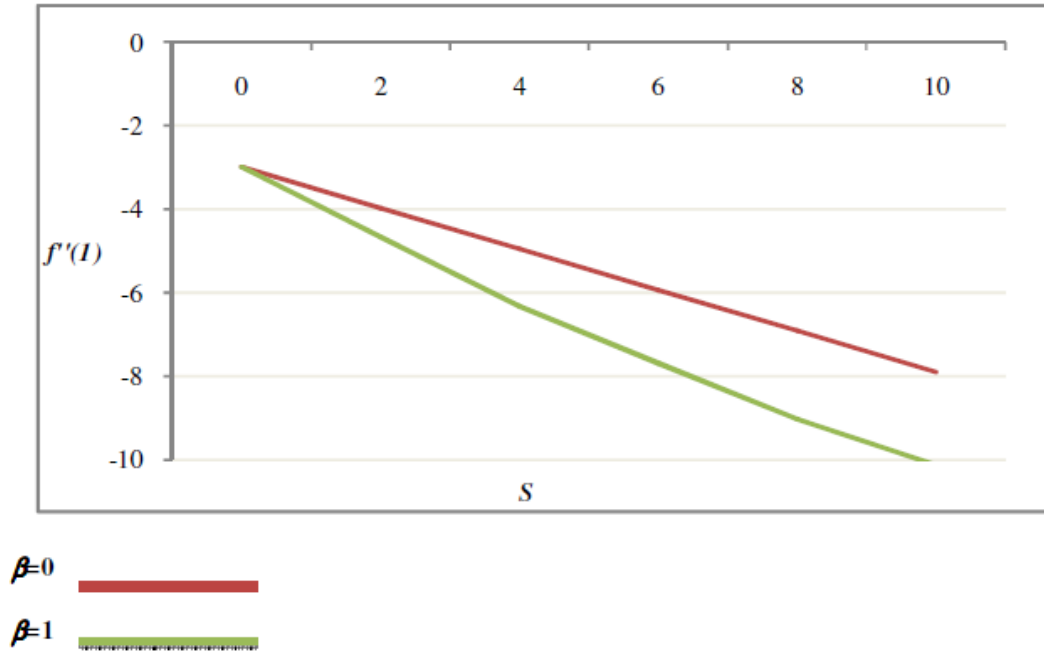


Fig. 4: The Skin Friction $f'(1)$ for the Axisymmetric flow and the Two-dimensional flow.

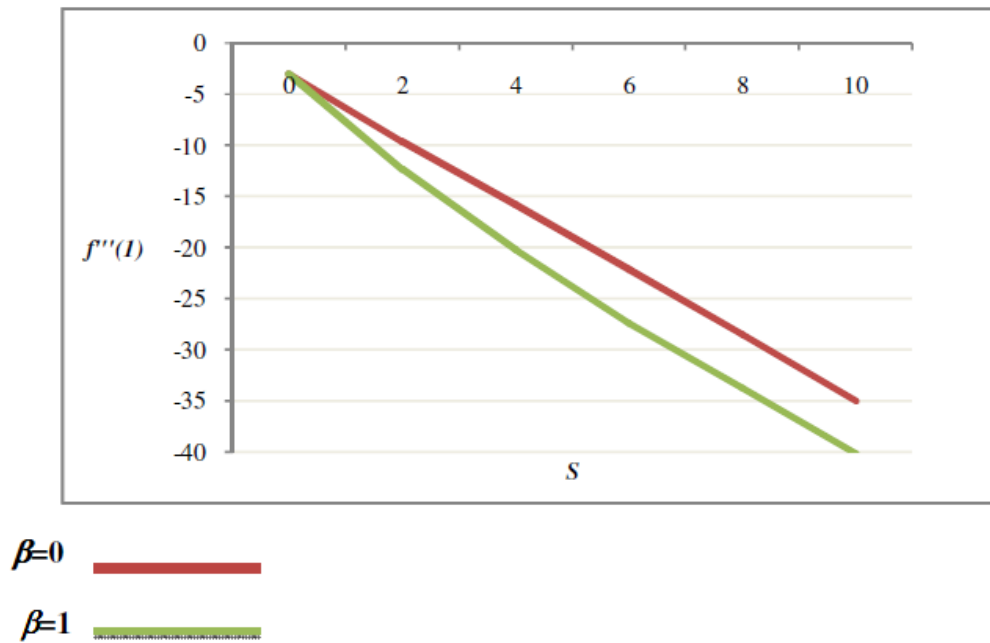


Fig. 5: The Pressure Gradient $f''(1)$ for the Axisymmetric flow and the Two-dimensional flow.