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Fixed width confidence interval for the parameter of $U(a\theta, b\theta)$ distribution *

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Abstract In the literature, an extensive work on sequential fixed width confidence intervals for the parameter θ of the $U(0, \theta)$ distribution is available. In this article, we propose a sequential fixed width $(1-\alpha)$ level confidence interval for θ of $U(a\theta, b\theta)$ distribution, where $0 < a < b$ are increasing and known. Further we obtain the average sample number (ASN) function of the proposed procedure.

Key words Non–coverage probability, sequential procedure, numerical evaluation.

2010 Mathematics Subject Classification Primary: 62L12, Secondary: 62F25.

1 Introduction

Graybill and Connell [6], Cooke [2, 3] and Govindarajulu [4, 5] studied the sequential fixed width $(1-\alpha)$ level confidence intervals for the parameter of $U(0, \theta)$distribution. Akahira and Koike [1] and Koike [7] proposed the $(1-\alpha)$ level fixed width confidence interval for θ in $U(\theta - \zeta/2, \theta + \zeta/2)$ and $U(\theta - a\zeta, \theta + a\zeta)$ distributions respectively, where ζ is unknown and a is a finite known positive number. Patil and Rattihalli [10] gave a sequential fixed width 100% confidence interval for the parameter θ of a model $f(x, \theta) = g(x)/h(\theta)$ for $a(\theta) \leq b(\theta)$, when both limits are strictly increasing. Patil and Rattihalli [11] also considered the problem of obtaining confidence interval having a specified width for the parameter in the $U(\theta, m\theta)$distribution, where $m > 1$ is known and $\theta > 0$. Patil [8, 9] considered the two-stage estimation procedure and a purely sequential procedure for the parameter of $U(\theta, m\theta)$ distribution using large sample approximation of coverage probability.
The $U(a\theta, b\theta)$ distribution is appropriate in the following situation. Consider an agricultural experiment where we want to study the impact of unknown soil fertility gradient θ of a plot on the yield/growth of a certain crop which is an observable random variable, say X, whose range depends on θ, say, $a\theta$ and $b\theta$, where, $0 < a < b$ are increasing and known. It is but natural to assume that both $a\theta$ and $b\theta$ are increasing functions of θ. Assuming that θ is the only unknown entity, the random variable X has $U(a\theta, b\theta)$ distribution. The problem of interest is to find an interval estimation of soil fertility gradient (θ). In this article, we propose a sequential fixed width $(1-\alpha)$ level confidence interval (CI) for the parameter $\theta$of the $U(a\theta, b\theta)$ distribution , where $0 < a < b$ are increasing and known. In section 2, we define a stopping rule to find a fixed width random interval which contains almost surely (a.s.) the parameter θ and also we propose a sequential fixed width $(1-\alpha)$ level confidence interval for the parameter θ of the $U(a\theta, b\theta)$ distribution. In section 3, we obtain the average sample number

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ASN function of the proposed procedure. Section 4 contains the numerical evaluation of the proposed procedure.

2 Sequential Procedure

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d) $U(a\theta, b\theta)$ random variables, where $0 < a < b$ are positive and known. Let $U_k = \min(X_1, X_2, \ldots, X_k)$ and $V_k = \max(X_1, X_2, \ldots, X_k)$, where k is any positive integer. As $a\theta \le U_k \le V_k \le b\theta$ implies $V_k/b \le \theta \le U_k/a$ almost surely (a.s) (that is, the range or 100% confidence interval for θ). As k tends to ∞, V_k/b increases to θ, U_k/a decreases to θ and $U_k/a - V_k/b$ decreases to 0 a.s. Now we have to find CI of fixed width d for θ. For $t > 0$, define the stopping rule $N_t = n$, the least integer such that

$$U_n/a - V_n/b < t. \tag{2.1}$$

As under the rule $N_d : U_{N_d}/a - \quad V_{N_d}/b < d$, that is, $U_{N_d}/a < V_{N_d}/b + d$ and $V_{N_d}/b < \theta < U_{N_d}/a < V_{N_d}/b + d$. Therefore, the random interval $(V_{N_d}/b, \quad V_{N_d}/b + d)$ of length d contains parameter θ a.s. For $t \le d, N_d \le N_t, V_{N_t} \ge V_{N_d}, U_{N_t} \le U_{N_d}$ and $(V_{N_d}/b, \quad V_{N_d}/b + d)$ interval contains parameter θ a.s. Hence in the following we consider the case $t > d$.

Consider the stopping rule $\{N_t : t > 0\}$ defined in (2.1), our interest is to minimize the sample size that is to find largest $c > d$ such that the interval $I_{N_c} = (V_{N_c}/b, V_{N_c}/b + d)$ form a $(1-\alpha)$ level confidence interval for θ. Consider the coverage probability $P(\theta, d, c) = P(\theta \in I_{N_c}) = P(V_{N_c}/b \leqslant \theta \leqslant V_{N_c}/b + d) = P(\theta \leqslant V_{N_c}/b + d) = P(V_{N_c} \geqslant b(\theta - d)) = 1 - P(V_{N_c} \leqslant b(\theta - d)) = 1 - q(\theta, d, c)$, say. Now we have to examine the existence of some $c > d$ for which $P\{\theta, d, c\} \ge 1 - \alpha$ for all θ, that is, the non-coverage probability $q\{\theta, d, c\} \le \alpha$ for all θ.

Consider $q(\theta, d, c) = P(V_{N_c} \leqslant b(\theta - d)) = \sum_{k=1}^{\infty} P(V_k \leqslant b(\theta - d), N_c = k) = \sum_{k=1}^{\infty} q(\theta, d, c, k)$. Note that $N_c > k$ if and only if $U_k/a - V_k/b > c$. Hence, in the following we refer the set $CR = \{(x, y) : x/a - y/b > c\}$ as the continuation region. Since U_k is non- increasing and V_k is nondecreasing, (U_k, V_k) belongs to CR implies (U_{k-1}, V_{k-1}) belongs to CR. Suppose that after $(k-1)^{\text{th}}$observation $(k \ge 3)$, the point (U_{k-1}, V_{k-1}) is in CR. Then under the rule N_c, we stop after k^{th}observation X_k if and only if (i) $X_k/a < V_{k-1}/b + c$ with probability $(V_{k-1}/b + c)\, a - a\theta/\left((b-a)\,\theta\right)$, here $(U_k, V_k) = (X_k, V_{k-1})$, (ii) $X_k > b\,(U_{k-1}/a - c)$ with probability equal to $(b\theta - bd - b\,(U_{k-1}/a - c))\,/\left((b-a)\,\theta\right)$, here $(U_k, V_k) = (U_{k-1}, X_k)$. For the details see the Figure 1.

To obtain $q(\theta, d, c, k)$ we shall consider the following three cases: (i) $a\theta < ac/(b-a)$, (ii) $ac/(b-a) \le a\theta \le abc/(b-a)$ and (iii) $abc/(b-a) \le a\theta$.

(i) $a\theta < ac/(b-a)$

In this case, for $k = 1$ we have $U_1 = V_1 = X_1$ and $U_1/a - V_1/b = (b-a)\,X_1/ab < (b-a)\,\theta/a < c/a$. Hence $N_c = 1$ a.s. and

$$\begin{aligned} q(\theta, d, c, 1) &= P\{X_1 \le b(\theta - d), N_c = 1\} = P\{X_1 \le b(\theta - d), X_1/a - X_1/b < c\} \\ &= P\{X_1 \le b(\theta - d), X_1 < abc/(b-a)\} = P\{X_1 \le b(\theta - d)\} \end{aligned}$$

$$= \begin{cases} (b\theta - bd - a\theta)/((b-a)\theta) & \text{if } \theta > bd/(b-a) \\ 0 & \text{if } \theta \leqslant bd/(b-a). \end{cases}$$

Thus for $\theta > bd/(b-a)$, $q(\theta, d, c, 1) = 1 - bd/\left((b-a)\,\theta\right) < 1 - bd/c$, for all $c > bd$. Hence in this case we have,

$$q(\theta, d, c) < \max\{0, 1 - bd/c\} \tag{2.2}$$

(ii) $ac/(b-a) \le a\theta \le abc/(b-a)$

Here $a\theta \le abc/(b-a)$, that is, $b(\theta - c) < a\theta$. We have,

$$\begin{aligned} \text{non}-\text{coverage region} &= \{(u, v) : u/a - v/b < c, v < b\theta - bd\} \\ &= \begin{cases} \phi & \text{if } b\theta - bd < a\theta, \text{ i.e., } \theta < bd/(b-a) \\ ABD & \text{if } a\theta < b\theta - bd < abc/(b-a), \text{i.e., } bd/(b-a) < \theta < ac/(b-a) + d \\ AGH & \text{if } abc/(b-a) < b\theta - bd, \text{i.e., } ac/(b-a) + d < \theta < abc/(b-a) \end{cases} \end{aligned}$$

Now we consider the following three sub-cases:

(iiA) $ac/(b-a) < a\theta < bd/(b-a)$, (iiB) $bd/(b-a) < a\theta < ac/(b-a)+d$, and
(iiC) $ac/(b-a)+d < a\theta < abc/(b-a)$.
The non-coverage region for the above sub cases and continuity regions are described in the Figure 2.

(iiA) $ac/(b-a) < a\theta < bd/(b-a)$

In this case, as non-coverage region is empty, $q(\theta,d,c)=0$.

(iiB) $bd/(b-a) < a\theta < ac/(b-a)+d$

In this case for $k=1$, we have

$$\begin{aligned}&q(\theta,d,c,1) = P\{X_1 \le b(\theta-d), X_1 < abc/(b-a)\} = P\{X_1 \le b(\theta-d)\}\\ &= (b\theta - bd - a\theta)/((b-a)\theta) = 1 - bd/((b-a)\theta).\end{aligned}$$

For $k=2$, $q(\theta,d,c,2) = P\{V_2 < b\theta - bd, N_c = 2\} = P\{V_2 < b\theta - bd, U_2/a - V_2/b < c, X_1 > abc/(b-a)\}$ $= P\{(U_2,V_2) \,\text{belongs to}\, ABD, X_1 > abc/(b-a)\} = 0$ (because $\theta < X_1 < abc/(b-a)$ a.s.). Similarly $q(\theta,d,c,k) = 0$ for all $k > 2$. Thus we have, $q(\theta,d,c) = \sum_{k=1}^{\infty} q(\theta,d,c,k) = q(\theta,d,c,1) = 1 - bd/(b-a)\theta$. Since $a\theta < ac/(b-a)+d < abc/(b-a)$, we have,

$$q(\theta,d,c) < 1 - d/c. \tag{2.3}$$

(iiC) $ac/(b-a)+d < a\theta < abc/(b-a)$

Here the non-coverage region is $AMHGSA$. In this case for $k=1$, we have,

$$q(\theta,d,c,1) = P\{X_1 \le b(\theta-d), X_1 < abc/(b-a)\} = P(X_1 \le b\theta - bd) = 1 - bd/(b-a)\theta$$

Note that when $abc/(b-a) < X_1 < b\theta - bd$, the point (U_1,V_1) does not belong to $MHGS$. For $k=2$,

$$\begin{aligned}q(\theta,d,c,2) &= P\{V_2 < b\theta - bd, N_c = 2\} = P\{V_2 < b\theta - bd, U_2/a - V_2/b < c, X_1 > abc/(b-a)\}\\ &= P\{(U_2,V_2) \in AMHGSA, X_1 > abc/(b-a)\}\\ &= P\{(U_2,V_2) \in MHGS, abc/(b-a) < X_1 < b\theta - bd\}\end{aligned}$$

$$\begin{aligned}&= \int_{abc/(b-a)}^{b\theta-bd} (b(x/a-c)-a\theta)/(b-a)\theta dx = 1/((b-a)\theta)\int_{abc/(b-a)}^{b\theta-bd} (b(x/a)-(bc+a\theta))\,dx\\ &= 1/((b-a)\theta)\int_{abc/(b-a)}^{b\theta-bd} (b(x/a)-(bc+a\theta))\,dx\\ &= 1/((b-a)\theta)\left[(b(b\theta-bd)^2)/2a - b(abc)^2/(2a(b-a)^2) - (bc+a\theta)((b\theta-bd)-abc/(b-a))\right]\\ &= \frac{1}{2(b-a)\theta}\begin{bmatrix} \frac{b(b\theta-bd)^2}{a} - \frac{b(abc)^2}{a(b-a)^2} + \frac{b(b\theta-bd)(abc)}{a(b-a)} - \frac{b(b\theta-bd)(abc)}{a(b-a)}\\ -2(bc+a\theta)((b\theta-bd)-abc/(b-a))\end{bmatrix}\\ &= \frac{1}{2(b-a)\theta}\left[(b\theta-bd-abc/(b-a))\left((b^2\theta-b^2d)/a + b^2c/(b-a) - 2bc - 2a\theta\right)\right]\\ &= \frac{b((b-a)(\theta-d)-ac)}{2(b-a)^3\theta}\left[b^2((b-a)(\theta-d)+ac)/a - 2(b-a)(bc+a\theta)\right].\end{aligned}$$

Lastly for $k \ge 3$, we have,

$$\begin{aligned}&q(\theta,d,c,k) = P\{(U_k,V_k) \in MHGS, (U_{k-1},V_{k-1}) \in MTH\}\\ &= P\{(U_k,V_k) \in MHGS, (U_{k-1},V_{k-1}) \in MLH\} + P\{(U_k,V_k) \in MHGS, (U_{k-1},V_{k-1}) \in LTH\}\\ &= A_k + B_k, \qquad \text{(say)}.\end{aligned}$$

Now,

$$\begin{aligned}&A_k = \frac{(k-1)(k-2)}{(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)}\int_u^{b(u/a-c)} \left[((v/b+c)a - a\theta) + (b\theta - bd - (u/a-c)b)\right](v-u)^{k-3}dvdu\\ &= \frac{(k-1)(k-2)}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)}\int_u^{b(u/a-c)} \left[av + \left(b^2-ab\right)\theta + \left(ab+b^2\right)c - b^2\left(d+u/a\right)\right](v-u)^{k-3}dvdu\\ &= \frac{(k-1)}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)} \left[(av + (b^2-ab)\theta + (ab+b^2)c - b^2(d+u/a))(v-u)^{k-2}\right]_u^{b(u/a-c)} du\\ &-\frac{1}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)} a\left[(v-u)^{k-1}\right]_u^{b(u/a-c)} du\\ &= \frac{(k-1)}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)} \left[\left((b^2-ab)(\theta-u/a) + b^2(c-d)\right)((b-a)u/a - bc)^{k-2}\right] du\\ &-\frac{a}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta+c-d)} ((b-a)u/a - bc)^{k-1} du\\ &= \frac{1}{b(b-a)^{k+1}\theta^k}\left[\left((b^2-ab)(\theta-u/a) + b^2(c-d)\right)((b-a)u/a - bc)^{k-1}\right]_{abc/(b-a)}^{a(\theta-d+c)}\\ &+\frac{1}{(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta-d+c)} ((b-a)u/a - bc)^{k-1} du - \frac{a}{b(b-a)^k\theta^k}\int_{abc/(b-a)}^{a(\theta-d+c)} ((b-a)u/a - bc)^{k-1} du\end{aligned}$$

$$= \tfrac{a}{b(b-a)^{k+1}\theta^k}\left[(b^2-ab)(d-c)+b^2(c-d)\right]\left[(b-a)(\theta-d)-ac\right]^{k-1}$$
$$+\tfrac{a}{b(b-a)^k\theta^k k}\left[(b-a)(\theta-d)-ac\right]^k$$
$$= \tfrac{a^2(c-d)}{(b-a)^{k+1}\theta^k}\left[(b-a)(\theta-d)-ac\right]^{k-1}+\tfrac{a}{b(b-a)^k\theta^k k}\left[(b-a)(\theta-d)-ac\right]^k.$$

Further

$$B_k=\frac{(k-1)(k-2)}{(b-a)^k\theta^k}\int_{a(\theta+c-d)}^{b\theta-bd}\int_u^{b\theta-bd}\left[(v/b+c)a-a\theta\right](v-u)^{k-3}dvdu$$
$$= \tfrac{(k-1)}{b(b-a)^k\theta^k}\int_{a(\theta+c-d)}^{b\theta-bd}\left[(av+abc-ab\theta)(v-u)^{k-2}\right]_u^{b\theta-bd}du-\tfrac{a}{b(b-a)^k\theta^k}\int_{a(\theta+c-d)}^{b\theta-bd}\left[(v-u)^{k-1}\right]_u^{b\theta-bd}du$$
$$= \tfrac{ab(c-d)(k-1)}{b(b-a)^k\theta^k}\int_{a(\theta+c-d)}^{b\theta-bd}\left[b\theta-bd-u\right]^{k-2}du-\tfrac{a}{b(b-a)^k\theta^k}\int_{a(\theta+c-d)}^{b\theta-bd}\left[b\theta-bd-u\right]^{k-1}du$$
$$= \tfrac{a(c-d)}{(b-a)^k\theta^k}\left[(b-a)(\theta-d)-ac\right]^{k-1}-\tfrac{a}{b(b-a)^k\theta^k k}\left[(b-a)(\theta-d)-ac\right]^k.$$

Thus, for $ac/(b-a)+d\le a\theta\le abc/(b-a)$, the non-coverage probability will be

$$q(\theta,d,c)=\sum_{k=1}^{\infty}q(\theta,d,c,k)$$
$$=1-\tfrac{bd}{(b-a)\theta}+\tfrac{b[(b-a)(\theta-d)-ac]}{2a(b-a)^3\theta}\left[b^2((b-a)(\theta-d)+ac)-2a(b-a)(bc+a\theta)\right]$$
$$+\tfrac{ab(c-d)}{(b-a)^2\theta}\sum_{k=3}^{\infty}\left[1-\tfrac{(b-a)d+ac}{(b-a)\theta}\right]^{k-1}$$
$$=1-\tfrac{bd}{(b-a)\theta}+\tfrac{b[(b-a)(\theta-d)-ac]}{2a(b-a)^3\theta}\left[b^2((b-a)(\theta-d)+ac)-2a(b-a)(bc+a\theta)\right]$$
$$+\tfrac{ab(c-d)}{\left[(b-a)^2d+ac(b-a)\right]}\left[1-\tfrac{(b-a)d+ac}{(b-a)\theta}\right]^2.$$

Since $ac/(b-a)+d\le a\theta<abc/(b-a)$, $q(\theta,d,c)$ is uniformly bounded above by

$$1-\frac{d}{c}+\frac{b^2(c-d)^2}{2ac}+\frac{ab(c-d)}{[(b-a)^2d+ac(b-a)]}.$$

Thus, we have

$$q(\theta,d,c)<1-\frac{d}{c}+\frac{b^2(c-d)^2}{2ac}+\frac{ab(c-d)}{[(b-a)^2d+ac(b-a)]}=h_1(d,c) \tag{2.4}$$

Therefore, in this case, we take

$$q(\theta,d,c)<\max\left(1-d/c,h_1(d,c)\right). \tag{2.5}$$

(iii) $abc/(b-a)\le a\theta$

In this case for $k=1$, we have

$$q(\theta,d,c,1)=P\{X_1<b(\theta-d),X_1<abc/(b-a)\}<P\{X_1<abc/(b-a)\}<P\{X_1<a\theta\}=0.$$

For $k=2$, $q(\theta,d,c,2)=P\{V_2\le b\theta-bd,N_c=2\}$. The continuity point leading to non-coverage region are divided into three parts I_1 (leading from down), I_2 (leading from down and left) and I_3 (leading from left) as shown in the Figure 3. Thus we have,

$$q(\theta,d,c,2)=P\{(U_2,V_2)\in \text{ non}-\text{coverage}, (U_1,V_1) \text{ is on the line } u=v\}$$
$$=\int_{a\theta}^{a(\theta-d+c)}\tfrac{b\theta-bd-b(u/a-c)}{(b-a)\theta}du+\int_{b(\theta-c)}^{b\theta-bd}\tfrac{a(u+bc)/b-a\theta}{(b-a)\theta}du$$
$$=\int_{a\theta}^{a(\theta-d+c)}\tfrac{ab\theta-abd-bu+abc}{a(b-a)\theta}du+\int_{b(\theta-c)}^{b\theta-bd}\tfrac{au+abc-ab\theta}{b(b-a)\theta}du$$
$$=\tfrac{ab(\theta-d+c)[c-d]}{(b-a)\theta}-\tfrac{b[a^2(\theta-d+c)^2-a^2\theta^2]}{2a(b-a)\theta}+\tfrac{b(abc-ab\theta)[c-d]}{b(b-a)\theta}+\tfrac{ab^2[(\theta-d)^2-(\theta-c)^2]}{2b(b-a)\theta}$$
$$=\tfrac{ab[2c\theta-2cd+2c^2-2d\theta+2d^2-2dc-d^2+2d\theta-c^2-2c\theta+2cd]}{2(b-a)\theta}$$
$$+\tfrac{ab\left[2c^2-2c\theta-2cd+2d\theta+d^2-2d\theta-c^2+2c\theta\right]}{2(b-a)\theta}$$
$$=\tfrac{ab[c^2-2dc+d^2]}{2(b-a)\theta}+\tfrac{ab\left[c^2+d^2-2cd\right]}{2(b-a)\theta}=\tfrac{ab[c-d]^2}{2(b-a)\theta}+\tfrac{ab[c-d]^2}{2(b-a)\theta}=\tfrac{ab[c-d]^2}{(b-a)\theta}.$$

Lastly, for $k\ge 3$, we have

$$\begin{aligned}q(\theta,d,c,k)&=P\{(U_k,V_k)\in \text{non}-\text{coverage},(U_{k-1},V_{k-1})\in I_1\cup I_2\}+\\&\quad P\{(U_k,V_k)\in \text{non}-\text{coverage},(U_{k-1},V_{k-1})\in I_2\cup I_3\}\\&=A_k+B_k,\quad \text{(say)}.\end{aligned}$$

Now,

$$\begin{aligned}A_k &= \tfrac{(k-1)(k-2)}{(b-a)^k\theta^k}\int_{a\theta}^{a(\theta+c-d)}\int_u^{b(u-ac)/a}\left[b\theta-bd-(b(u-ac))/a\right](v-u)^{k-3}dvdu\\ &= \tfrac{(k-1)}{(b-a)^k\theta^k}\int_{a\theta}^{a(\theta+c-d)}\left[b\theta-bd-(b(u-ac))/a\right]\left(\tfrac{(b-a)u}{a}-bc\right)^{k-2}du\\ &= \tfrac{a}{(b-a)^{k+1}\theta^k}\left[(b\theta-bd-b(u-ac)/a)\left(((b-a)u)/a-bc\right)^{k-1}\right]_{a\theta}^{a(\theta-d+c)}\\ &+\tfrac{ab}{(b-a)^{k+2}\theta^k k}\left[\left(((b-a)u)/a-bc\right)^k\right]_{a\theta}^{a(\theta-d+c)}\\ &= \tfrac{ab(d-c)}{(b-a)^{k+1}\theta^k}\left[(b-a)\theta-bc\right]^{k-1}+\tfrac{ab}{(b-a)^{k+2}\theta^k k}\{[(b-a)(\theta-d)-ac]^k-[(b-a)\theta-bc]^k\}.\end{aligned}$$

Further,

$$\begin{aligned}B_k &= \tfrac{(k-1)(k-2)}{b(b-a)^k\theta^k}\int_{b\theta-bc}^{b\theta-bd}\int_{(v/b+c)a}^{v}\left[av+abc-ab\theta\right](v-u)^{k-3}dvdu\\ &= -\tfrac{a(k-1)}{b(b-a)^k\theta^k}\int_{b\theta-bc}^{b\theta-bd}(v+bc-b\theta)\left[(v-u)^{k-2}\right]_{(v/b+c)/a}^{v}dv\\ &= \tfrac{a(k-1)}{b(b-a)^k\theta^k}\int_{b\theta-bc}^{b\theta-bd}(v+bc-b\theta)\left[(b-a)v/b-ac\right]^{k-2}dv\\ &= \tfrac{a}{(b-a)^{k+1}\theta^k}\left[(v+bc-b\theta)((b-a)v/b-ac)^{k-1}\right]_{b\theta-bc}^{b\theta-bd}-\tfrac{ab}{(b-a)^{k+2}\theta^k k}\left[((b-a)v/b-ac)^k\right]_{b\theta-bc}^{b\theta-bd}\\ &= \tfrac{ab(c-d)}{(b-a)^{k+1}\theta^k}\left[(b-a)(\theta-d)-ac\right]^{k-1}-\tfrac{ab}{(b-a)^{k+2}\theta^k k}\left\{\left[(b-a)(\theta-d)-ac\right]^k-\left[(b-a)\theta-bc\right]^k\right\}.\end{aligned}$$

Thus for $a\theta \geq abc/(b-a)$, the non-coverage probability will be

$$\begin{aligned}q(\theta,d,c) &= \textstyle\sum_{k=1}^{\infty} q(\theta,d,c,k)\\ &= \tfrac{ab(c-d)^2}{(b-a)\theta}+\tfrac{ab(d-c)}{(b-a)^{k+1}\theta^k}\left[(b-a)\theta-bc\right]^{k-1}+\tfrac{ab(c-d)}{(b-a)^{k+1}\theta^k}\left[(b-a)\theta-(b-a)d-ac\right]^{k-1}\\ &= \tfrac{ab(c-d)^2}{(b-a)\theta}+\tfrac{ab(d-c)}{(b-a)^2\theta}\textstyle\sum_{k=3}^{\infty}\left[1-\tfrac{bc}{(b-a)\theta}\right]^{k-1}+\tfrac{ab(c-d)}{(b-a)^2\theta}\textstyle\sum_{k=3}^{\infty}\left[1-\tfrac{(b-a)d+ac}{(b-a)\theta}\right]^{k-1}\\ &= \tfrac{ab(c-d)^2}{(b-a)\theta}+\tfrac{a(d-c)}{(b-a)c}\left[1-\tfrac{bc}{(b-a)\theta}\right]^2+\tfrac{ab(c-d)}{(b-a)^2d+(b-a)ac}\left[1-\tfrac{(b-a)d+ac}{(b-a)\theta}\right]^2.\end{aligned}$$

Since $abc/(b-a) \leq a\theta$ and $ac/(b-a)+d < abc/(b-a) \leq a\theta, q(\theta,d,c)$ is uniformly bounded above by

$$a(c-d)^2/c - a(c-d)/[(b-a)c] + ab(c-d)/[(b-a)^2d+(b-a)ac].$$

Thus we have

$$\begin{aligned}q(\theta,d,c) &< a(c-d)^2/c - a(c-d)/[(b-a)c] + ab(c-d)/[(b-a)^2d+(b-a)ac]\\ &= h_2(d,c).\end{aligned} \tag{2.6}$$

From (2.2) to (2.6), for given a, b, c and d, an upper bound of the function $q(\theta,d,c), \theta > 0$, is given by

$$q(\theta,d,c) < \max\left\{[1-bd/c]^+, 1-d/c, h_1(d,c), h_2(d,c)\right\} = Q(d,c), \quad \text{(say)} \tag{2.7}$$

where $[t]^+ = \max[0,t]$.
Note that as c decreases to d, each term considered in the right hand side of (2.7) decreases to 0, hence $Q(d,c)$ and $q(\theta,d,c)$ decrease to 0. For given a, b, d and α, any $c > d$ for which $Q(d,c) \leq \alpha$, the rule N_c yields a $(1-\alpha)$ level fixed width CI $(V_{N_c}/b, V_{N_c}/b+d)$. In order to reduce the sample size it is desirable to take such a c as large as possible.

3 Average Sample Number (ASN) Function

The joint distribution of (U_k, V_k) is given by

$$f(u,v) = \frac{k(k-1)}{(b-a)^2\theta^2}(v-u)^{k-2}, \quad a\theta < u < v < b\theta.$$

Hence for $k \geq 2$ we have,

$$P(N_c > k) = \frac{k(k-1)}{(b-a)^k\theta^k}\iint\limits_{u/a-v/b \geqslant c}(v-u)^{k-2}du\,dv \tag{3.1}$$

Further the ASN function of rule N_c is given by

$$E(N_c) = 1 + P(N_c > 1) + \sum_{k=2}^{\infty} P(N_c > k)$$

$$= 1 + P(N_c > 1) + \frac{1}{(b-a)^2\theta^2} \sum_{k=2}^{\infty} \iint\limits_{u/a - v/b \geqslant c} k(k-1) \left[\frac{(v-u)}{(b-a)\theta}\right]^{k-2} du\, dv$$

$$= 1 + P(N_c > 1) + \frac{2}{(b-a)^2\theta^2} \iint\limits_{u/a - v/b \geqslant c} \left[1 - \frac{(v-u)}{(b-a)\theta}\right]^{-3} du\, dv \tag{3.2}$$

Here we obtain ASN function $E(N_c)$ for this model for any $c > 0$. Consider a single observation X_1. Then according to the stopping rule N_c, stop if $X_1/a - X_1/b < c$, that is, if $X_1 < abc/(b-a)$ and the probability of stopping is

$$P(N=1) = P\{X_1 < abc/(b-a)\} = \begin{cases} 1, \text{ if } 0 < a\theta < ac/(b-a) \\ a/(b-a)\left[bc/(b-a)\theta - 1\right], \text{ if } ac/(b-a) < a\theta < abc/(b-a) \\ 0, \text{ if } a\theta \geqslant abc/(b-a). \end{cases}$$

For $0 \leq a\theta \leq ac/(b-a)$, $P(N=1) = 1$, and the 100% CI for θ is $(X_1/b, X_1/a)$. In the following we consider the cases: (i) $ac/(b-a) < a\theta < abc/(b-a)$ and (ii) $a\theta > abc/(b-a)$.

Case (i) $ac/(b-a) < a\theta < abc/(b-a)$

In this case

$$P(N_c > 1) = \frac{b}{(b-a)}\left[1 - \frac{ac}{(b-a)\theta}\right]$$

and for $k \geq 2$ from (3.1) we have

$$P(N_c > k) = \frac{k(k-1)}{(b-a)^k\theta^k} \iint\limits_{u/a - v/b \geqslant c} (v-u)^{k-2} du dv.$$

Also from (3.2) we have,

$$E(N_c) = 1 + \frac{b}{(b-a)}\left[1 - \frac{ac}{(b-a)\theta}\right] + \frac{2}{(b-a)^2\theta^2} \iint\limits_{u/a - v/b \geqslant c} \left[1 - \frac{v-u}{(b-a)\theta}\right]^{-3} du dv.$$

Since $a\theta < u < b\theta$ and $ac/(b-a) < a\theta < abc/(b-a)$ and the domain of integration is $u/a - v/b \geq c$, so from the Figure 4 we have,

$$\begin{aligned}
E(N_c) &= 1 + \tfrac{b}{(b-a)}\left[1 - \tfrac{ac}{(b-a)\theta}\right] + \tfrac{2}{(b-a)^2\theta^2} \int_{abc/(b-a)}^{a(\theta+c)} \int_u^{b(u/a-c)} \left[1 - \tfrac{v-u}{(b-a)\theta}\right]^{-3} dv du \\
&\quad + \tfrac{2}{(b-a)^2\theta^2} \int_{a(\theta+c)}^{b\theta} \int_u^{b\theta} \left[1 - \tfrac{v-u}{(b-a)\theta}\right]^{-3} dv du \\
&= 1 + \tfrac{b}{(b-a)}\left[1 - \tfrac{ac}{(b-a)\theta}\right] + \tfrac{1}{(b-a)\theta} \int_{abc/(b-a)}^{a(\theta+c)} \left\{\left[1 - \tfrac{(b-a)u/a - bc}{(b-a)\theta}\right]^{-2} - 1\right\} du \\
&\quad + \tfrac{1}{(b-a)\theta} \int_{a(\theta+c)}^{b\theta} \left\{\left[1 - \tfrac{b\theta - u}{(b-a)\theta}\right]^{-2} - 1\right\} du \\
&= 1 + \tfrac{b}{(b-a)}\left[1 - \tfrac{ac}{(b-a)\theta}\right] + \tfrac{a}{(b-a)} \left\{\left[1 - \tfrac{(b-a)(\theta+d) - bc}{(b-a)\theta}\right]^{-1} - 1\right\} - \tfrac{a[(b-a)(\theta+c) - bc]}{(b-a)^2\theta} \\
&\quad + \left\{\left[1 - \tfrac{(b-a)\theta - ac}{(b-a)\theta}\right]^{-1} - 1\right\} - 1 + \tfrac{ac}{(b-a)\theta} \\
E(N_c) &= \tfrac{b}{(b-a)} - \tfrac{ac}{(b-a)\theta} + \tfrac{\theta}{c} - \tfrac{2a}{(b-a)} + \tfrac{(b-a)\theta}{ac} - 1 + \tfrac{ac}{(b-a)\theta}
\end{aligned}$$

$$E(N_c) = \frac{b\theta}{ac} - \frac{a}{(b-a)} \tag{3.3}$$

Case (ii) $a\theta > abc/(b-a)$

In this case we have $P(N_c > 1) = 1$ and for $k \geq 2$ from (3.1) we have,

$$P(N_c > k) = \frac{k(k-1)}{(b-a)^k \theta^k} \iint\limits_{u/a - v/b \geqslant c} (v-u)^{k-2} du dv.$$

and from (3.2) we have,

$$E(N_c) = 1 + 1 + \frac{2}{(b-a)^2\theta^2} \iint\limits_{u/a - v/b \geqslant c} \left[1 - \frac{v-u}{(b-a)\theta}\right]^{-3} du dv.$$

Since $a\theta < u < b\theta$ and $a\theta \geq abc/(b-a)$ and the domain of integration is $u/a - v/b \geq c$, so from the Figure 5, we have,

$$\begin{aligned} E(N_c) &= 2 + \tfrac{2}{(b-a)^2\theta^2} \int_{a\theta}^{a(\theta+c)} \int_u^{b(u/a-c)} \left[1 - \tfrac{v-u}{(b-a)\theta}\right]^{-3} dv du + \tfrac{2}{(b-a)^2\theta^2} \int_{a(\theta+c)}^{b\theta} \int_u^{b\theta} \left[1 - \tfrac{v-u}{(b-a)\theta}\right]^{-3} dv du \\ &= 2 + \tfrac{1}{(b-a)\theta} \int_{a\theta}^{a(\theta+c)} \left\{\left[1 - \tfrac{(b-a)u/a - bc}{(b-a)\theta}\right]^{-2} - 1\right\} du + \tfrac{1}{(b-a)\theta} \int_{a(\theta+c)}^{b\theta} \left\{\left[1 - \tfrac{b\theta - u}{(b-a)\theta}\right]^{-2} - 1\right\} du \\ &= 2 + \tfrac{a}{(b-a)} \left\{\left[1 - \tfrac{(b-a)(\theta+c) - bc}{(b-a)\theta}\right]^{-1} - \left[1 - \tfrac{(b-a)\theta - bc}{(b-a)\theta}\right]^{-1}\right\} - \tfrac{ac}{(b-a)\theta} \\ &\quad + \left\{\left[1 - \tfrac{(b-a)\theta - ac}{(b-a)\theta}\right]^{-1} - 1\right\} - \tfrac{(b-a)\theta - ac}{(b-a)\theta} \end{aligned}$$

$$E(N_c) = \frac{(b^2 - a^2)\theta}{abc} \tag{3.4}$$

4 Numerical Evaluation

In this section, we compute c and ASN by using a C- program. To study the performance of proposed procedure a simulation study with 30,000 iterations is carried out. Simulated coverage (S-COV) and simulated ASN (S-ASN) for various values of a, b, θ, α and d are tabulated as below in the Tables 1–6.

Remark 4.1. Form the above Tables 1–6, it is clear that the proposed procedure attains the desired coverage for all a, b, θ, α and d. Also the simulated ASN and exact ASN are almost equal. Moreover as attained confidence level is far above than the desired $(1-\alpha)$, it appears that there is further scope to increase the value of c which in turn will decrease the ASN.

Remark 4.2. To have further reduction in the length of CI, one can consider the interval $(V_{N_t}/b,$ $\min(V_{N_t}/b + d, U_{N_t}))$.

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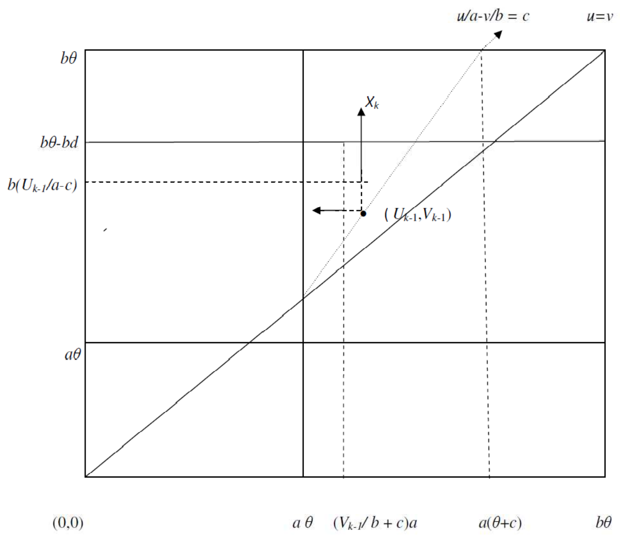

Fig. 1: Continuity region and position of X_k for $k \geq 3$.

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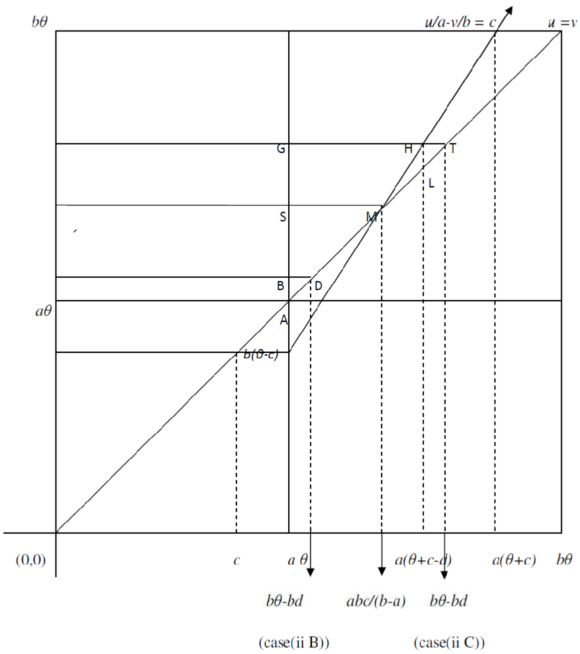

Fig. 2: Non-coverage regions and continuity region for sub-cases of case (ii).

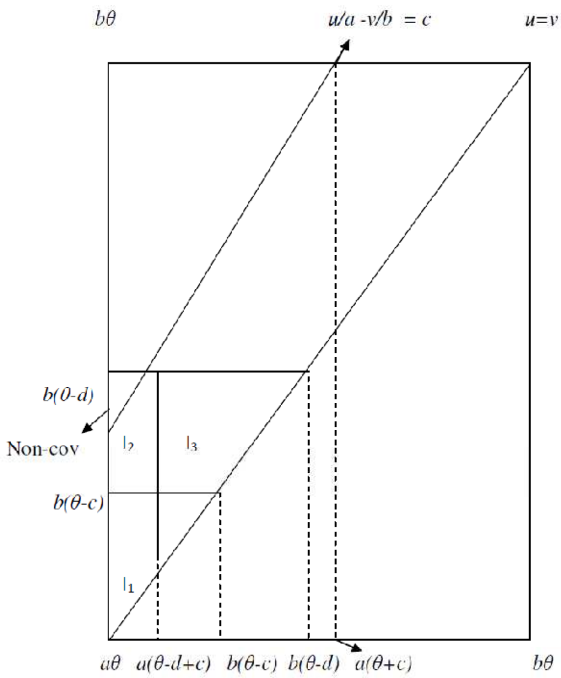

Fig. 3: Non-coverage region and continuity points for $a\theta \geq abc/\,(b-a)$.

Table 1: Computation of c, ASN, S-ASN and S-COV for $\alpha = 1$ and $b = 2$ (Continued as Table 2).

θ	α	d	c	ASN	S-ASN	S-COV
5	0.05	0.5	0.512578	14.63	14.6	0.999667
10				29.26	29.17	0.999733
30				87.79	87.76	0.999567
100				292.64	290.46	0.999533
200				585.28	581.04	0.999267
5		1	1.024851	7.32	7.39	0.999600
10				14.64	14.56	0.999833
30				43.91	43.77	0.999667
100				146.36	144.76	0.999467
200				292.73	289.3	0.9994
5		5	5.114174	1	1	1
10				2.91	2.9	0.999967
30				8.8	8.86	0.999833
100				29.33	29.29	0.999700
200				58.66	58.87	0.999500
5	0.01	0.5	0.502503	14.93	14.93	1
100				146.36	144.76	0.999467
10				29.85	29.96	0.999967
30				89.55	89.1	0.999967
100				298.51	295.86	0.999967
200				597.01	591.42	0.999800
5		1	1.004993	7.46	7.47	0.999967
10				14.93	14.97	1.000000
30				44.78	44.84	0.999967
100				149.25	148.08	0.999967
200				298.51	296.87	0.999933

Table 2: Computation of c, ASN, S-ASN and S-COV for $\alpha = 1$ and $b = 2$ (Continuation of the Table 1).

θ	α	d	c	ASN	S-ASN	S-COV
5		5	5.024492	1	1	1
10				2.98	2.98	0.999933
30				8.96	8.94	1
100				29.85	29.82	1
200				59.71	59.96	1
5	0.1	0.5	0.525312	14.28	14.3	0.998533
10				28.55	28.41	0.998733
30				85.66	85.64	0.998600
100				285.54	283.18	0.998367
200				571.09	561.5	0.997967
5		1	1.049432	7.15	7.18	0.998800
10				14.29	14.32	0.998667
30				42.88	42.99	0.998600
100				142.93	143.5	0.998733
200				285.87	285.03	0.998667
5		5	5.213042	1	1	1.000000
10				2.84	2.83	0.999967
30				8.63	8.66	0.999067
100				28.77	28.89	0.999067
200				57.55	57.77	0.999100

Table 3: Computation of c, ASN, S-ASN and S-COV for $\alpha = 1$ and $b = 3$ (Continued as Table 4).

θ	α	d	c	ASN	S-ASN	S-COV
5	0.05	0.5	0.516296	25.83	25.8	0.999667
10				51.65	51.51	0.999500
30				154.95	153.61	0.999400
100				516.5	510.51	0.999100
200				1033	999.77	0.998967
5		1	1.031233	12.93	12.88	0.999400
10				25.86	25.7	0.999700
30				77.58	77.14	0.999533
100				258.59	255.86	0.999700
200				517.18	509.96	0.999267

Table 4: Computation of c, ASN, S-ASN and S-COV for $\alpha = 1$ and $b = 3$ (Continuation of the Table 3).

θ	α	d	c	ASN	S-ASN	S-COV
5		5	5.123996	2.43	2.43	1
10				5.2	5.23	0.999667
30				15.61	15.68	0.999767
100				52.04	51.78	0.999800
200				104.09	104.3	0.999767
5	0.01	0.5	0.503317	26.49	26.56	1
10				52.98	52.67	1
30				158.95	158.6	0.999967
100				529.82	525.31	0.999933
200				1059.64	1035.61	0.999733
5		1	1.006571	13.25	13.24	1
10				26.49	26.58	0.999967
30				79.48	79.3	0.999967
100				264.93	261.72	0.9999
200				529.85	520.83	0.999967
5		5	5.030673	2.48	2.46	1
10				5.3	5.28	1
30				15.9	16.05	0.999967
100				53.01	52.38	0.999933
200				106.02	105.52	1
5	0.1	0.5	0.531956	25.06	25.2	0.998767
10				50.13	50	0.998967
30				150.39	149.78	0.998500
100				501.29	496.45	0.997500
200				1002.59	980.39	0.997000
5		1	1.059302	12.59	12.59	0.999133
10				25.17	25.25	0.999300
30				75.52	75.56	0.999033
100				251.74	249.44	0.998567
200				503.48	494.12	0.998600
5		5	5.211399	2.38	2.38	1
10				5.12	5.1	0.999433
30				15.35	15.42	0.999167
100				51.17	51.49	0.999400
200				102.34	103.12	0.999267

Table 5: Computation of c, ASN, S-ASN, S-COV for $\alpha = 2$ and $b = 5$ (Continued as Table 6).

θ	α	d	c	ASN	S-ASN	S-COV
5	0.05	0.5	0.514543	20.41	20.54	0.999767
10				40.81	40.68	0.999600
30				122.44	121.94	0.999367
100				408.13	405.93	0.999600
200				816.26	799.99	0.998800
5		1	1.02776	10.22	10.23	0.999833
10				20.43	20.46	0.999667
30				61.3	61.53	0.999500
100				204.33	202.27	0.999767
200				408.66	404.64	0.999433
5		5	5.108527	1.78	1.79	1
10				4.11	4.12	0.999900
30				12.33	12.38	0.999800
100				41.11	40.89	0.999900
200				82.22	82.09	0.999733
5	0.01	0.5	0.502980	20.88	20.65	0.999967
10				41.75	41.59	0.999967
30				125.25	124.61	0.999967
100				417.51	410.56	0.999900
200				835.02	814.4	0.999867
5		1	1.005897	10.44	10.43	1
10				20.88	21.06	1
30				62.63	62.28	1
100				208.77	209.37	0.999967
200				417.54	415.12	0.999933
5		5	5.027328	1.82	1.82	1
10				4.18	4.17	1
30				12.53	12.6	1
100				41.77	41.8	1
200				83.54	83.17	0.999967
5	0.1	0.5	0.528315	19.87	19.87	0.999100
10				39.75	39.83	0.999000
30				119.25	118.66	0.998667
100				397.49	393.93	0.997667
200				794.98	783.82	0.997667
5		1	1.052254	9.98	10.01	0.998867
10				19.96	19.95	0.998633
30				59.87	59.99	0.999267
100				199.57	198.32	0.998800
200				399.14	394.36	0.998167

Table 6: Computation of c, ASN, S-ASN, S-COV for $\alpha = 2$ and $b = 5$ (Continuation of the Table 5).

θ	α	d	c	ASN	S-ASN	S-COV
5		5	5.183348	1.74	1.75	1
10				4.05	4.05	0.999433
30				12.15	12.12	0.999633
100				40.51	40.59	0.999367
200				81.03	80.52	0.999533

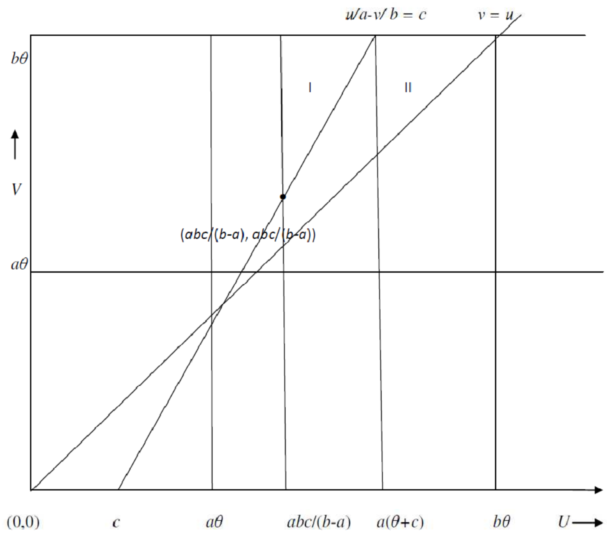

Fig. 4: Domain of integration for $ac/(b-a) < a\theta < abc/(b-a)$.

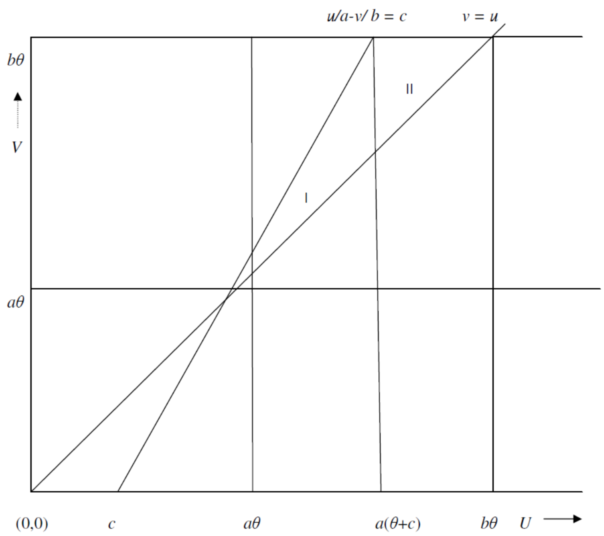

Fig. 5: Domain of integration for $a\theta \geq bc/(b-a)$.