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The non-homogeneous heptic equation with five unknowns $x^4 - y^4 = 41(z^2 - w^2)p^5$ *

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Abstract: The non-homogeneous Heptic equation with five unknowns given by $x^4 - y^4 = 41(z^2 - w^2)p^5$ is considered and analyzed for its non– zero distinct integer solutions. To solve this higher degree Diophantine equation, we have employed techniques such as brute force method and substitution strategy. A few interesting relations between the solutions and special numbers namely, polygonal numbers, pyramidal numbers and centered pyramidal numbers.
Keywords: Non - homogeneous Heptic equation with five unknowns, Integral solutions, Polygonal numbers, Pyramidal numbers.

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NOTATIONS USED:

- Regular Polygonal Number of rank n with sides m : $t_{m,n} = n[1 + \frac{(n-1)(m-1)}{2}]$
- Pyramidal Number of rank n with sides m : $p_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$
- Centered Polygonal Number of rank n with sides m : $ct_{m,n} = \frac{1}{2}[mn(n+1)] + 1$
- Pronic Number of rank n : $pr_n = n(n+1)$
- Pentatope Number of rank n : $pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$
- Star Number of rank n : $S_n = 6n(n-1) + 1$

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems and has a wealth of historical significance. In particular, Heptic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-2]. For illustration, one may refer [3-5] for Heptic equation with three unknowns and [6-16] for Heptic equation with five unknowns. This paper concerns with the problem of obtaining distinct integer solutions to the non-homogeneous Heptic equation with five unknowns given

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by $x^4 - y^4 = 41(z^2 - w^2)p^5$. A few interesting relations between the solutions and special numbers, namely, polygonal numbers ,pyramidal numbers, Star number and centered pyramidal numbers.

2. Method of Analysis

The non-homogeneous Heptic equation with 5 unknowns to be solved is given by

$$x^4 - y^4 = 41(z^2 - w^2)p^5 \tag{1}$$

Introducing the linear transformations

$$x = u + v, y = u - v, z = 2u + v \text{ and } w = 2u - v \tag{2}$$

in (1), it leads to

$$u^2 + v^2 = 41p^5 \tag{3}$$

Pattern-1

Assume $p = a^2 + b^2$ (4)

where a and b are non-zero distinct integers.

Write 41 as $41 = (4+5i)(4-5i)$ (5)

Using (4) & (5) in (3) and employing the method of factorization, define

$$u + iv = (4+5i)(a+ib)^5 \tag{6}$$

Equating the real and imaginary parts of (6), we get

$$\begin{aligned} u &= 4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5 \\ v &= 5a^5 + 20a^4b - 50a^3b^2 - 40a^2b^3 + 25ab^4 + 4b^5 \end{aligned} \tag{7}$$

Substituting (7) in (2), the integral solutions of (1) are given by

$$\begin{aligned} x(a,b) &= 9a^5 - 5a^4b - 90a^3b^2 + 10a^2b^3 + 45ab^4 - b^5 \\ y(a,b) &= -a^5 - 45a^4b + 10a^3b^2 + 90a^2b^3 - 5ab^4 - 9b^5 \\ z(a,b) &= 13a^5 - 30a^4b - 130a^3b^2 + 60a^2b^3 + 65ab^4 - 6b^5 \\ w(a,b) &= 3a^5 - 70a^4b - 30a^3b^2 + 140a^2b^3 + 15ab^4 - 14b^5 \\ p(a,b) &= a^2 + b^2 \end{aligned} \tag{8}$$

Properties:

- $a[x(a,a) + y(a,a) + z(a,a) + w(a,a)]$ is a Nasty number.
- $x(a,1) + y(a,1,) - 100t_{3,a^2} + 150t_{4,a} \equiv -10 \pmod 8$
- $-x(a,a) + y(a,a) + z(a,a) - w(a,a) + 2p(a,a)$ is a perfect square.

Note 1:

In (2), the representations of *z* and *w* may be taken as

$$z = 2uv + 1, w = 2uv - 1 \tag{9}$$

In this case, the values of z and w are given by

$$\begin{aligned} z(a,b) &= 2(4a^5-25a^4b-40a^3b^2+50a^2b^3+20ab^4-5b^5) \\ &\quad (5a^5+20a^4b-50a^3b^2+40a^2b^3+25ab^4+4b^5)+1 \\ w(a,b) &= 2(4a^5-25a^4b-40a^3b^2+50a^2b^3+20ab^4-5b^5) \\ &\quad (5a^5+20a^4b-50a^3b^2+40a^2b^3+25ab^4+4b^5)-1 \end{aligned} \tag{10}$$

Thus (8) and (10) represent a different set of solutions to (1)

Note 2:

Observe that z and w in (2) may also be considered as

$$z = uv+2, w = uv-2 \tag{11}$$

For this choice, the corresponding values of z and w are obtained as

$$\begin{aligned} z(a,b) &= (4a^5-25a^4b-40a^3b^2+50a^2b^3+20ab^4-5b^5) \\ &\quad (5a^5+20a^4b-50a^3b^2+40a^2b^3+25ab^4+4b^5)+2 \\ w(a,b) &= (4a^5-25a^4b-40a^3b^2+50a^2b^3+20ab^4-5b^5) \\ &\quad (5a^5+20a^4b-50a^3b^2+40a^2b^3+25ab^4+4b^5)-2 \end{aligned} \tag{12}$$

Thus (8) and (12) represent another set of integer solutions to (1).

Pattern-2:

Instead of (5), 41 can also be written as $41 = (5+4i)(5-4i)$ (13)

By using (4) and (13) in (3) and applying the same procedure in pattern - 1, the corresponding integer solutions to (1) are found to be

$$\begin{aligned} x(a,b) &= 9a^5+5a^4b-90a^3b^2-10a^2b^3+45ab^4+b^5 \\ y(a,b) &= a^5-45a^4b-10a^3b^2+90a^2b^3+5ab^4-9b^5 \\ z(a,b) &= 14a^5-15a^4b-140a^3b^2+30a^2b^3+70ab^4-3b^5 \\ w(a,b) &= 6a^5-65a^4b-60a^3b^2+130a^2b^3+30ab^4-13b^5 \\ p(a,b) &= a^2+b^2 \end{aligned} \tag{14}$$

Properties:

- $324[x(a,a)+y(a,a)+z(a,a)+w(a,a)]$ is a quintic integer.
- $x(a,b)+y(a,b)-10(pr_a)^5 \equiv 2(\bmod 10)$
- $-x(a,a)+y(a,a)+z(a,a)-w(a,a)=0$.

Note 3:

For this choice of z and w given by (9) and (11), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = 2(5a^5-20a^4b-50a^3b^2+40a^2b^3+25ab^4-4b^5)(4a^5+25a^4b-40a^3b^2-50a^2b^3+20ab^4+5b^5)+1$$

$$w(a,b) = 2(5a^5-20a^4b-50a^3b^2+40a^2b^3+25ab^4-4b^5)(4a^5+25a^4b-40a^3b^2-50a^2b^3+20ab^4+5b^5)-1$$

SET II:

$$z(a,b) = (5a^5-20a^4b-50a^3b^2+40a^2b^3+25ab^4-4b^5)(4a^5+25a^4b-40a^3b^2-50a^2b^3+20ab^4+5b^5)+2$$

$$w(a,b) = (5a^5-20a^4b-50a^3b^2+40a^2b^3+25ab^4-4b^5)(4a^5+25a^4b-40a^3b^2-50a^2b^3+20ab^4+5b^5)-2$$

Considering (12) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-3:

One may write (3) as $u^2+v^2=41p^5*1$ (15)

Write 1 as $1=\frac{(4+3i)(4-3i)}{25}$ (16)

Using (4), (5) & (16) in (15) and applying the method of factorization, define

$$u+iv=\frac{(4+3i)(4+5i)}{5}(a+ib)^5$$

Equating the real and imaginary parts, we get

$$u=\frac{(a^5-160a^4b-10a^3b^2+320a^2b^3+5ab^4-32b^5)}{5} \quad (17)$$

$$v=\frac{(32a^5+5a^4b-320a^3b^2-10a^2b^3+160ab^4+b^5)}{5}$$

As our interest is on finding integer solutions, choose $a=5A, b=5B$ in (17), we get the corresponding integral solutions of (1) are given by

$$x(A,B)=20625A^5-96875A^4B-206250A^3B^2+193750A^2B^3+103125AB^4-19375B^5$$

$$y(A,B)=-1937A^5-103125A^4B+193750A^3B^2+206250A^2B^3-96875AB^4+20025B^5$$

$$z(A,B)=21250A^5-203125A^4B-215200A^3B^2+393750A^2B^3+106250AB^4-39375B^5 \quad (18)$$

$$w(A,B)=-18750A^5-203125A^4B-187500A^3B^2+406250A^2B^3-93750AB^4-40625B^5$$

$$p(A,B)=25A^2+25B^2$$

Note 4:

For this choice of z and w given by (9) and (11), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(A,B)=2(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)+1$$

$$w(A,B)=2(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)-1$$

SET II:

$$z(A,B)=(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)+2$$

$$w(A,B)=(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)-2$$

Considering (18) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-4:

Introduction of the transformations

$$x=ky\,,z=kw\,,k>1 \quad (19)$$

In (1) leads to

$$(k^2+1)\,y^4=41w^2P^5 \quad (20)$$

Which is satisfied by

$$y = 41^2 (k^2+1)^s \alpha^{3\beta},$$
$$w = 41(k^2+1)^{2s-2} \alpha^{\beta}, \quad (21)$$
$$P = 41(k^2+1)\alpha^{2\beta}, \alpha > 1, s \geq 1$$

In view of (19), we have

$$x = 41^2 k(k^2+1)^s \alpha^{3\beta},$$
$$z = 41k(k^2+1)^{2s-2}\alpha^{\beta} \quad (22)$$

Thus, (21) and (22) represent the integer solutions to (1).

3. Remarkable Observations

Employing the integral solutions of (1) and (3) the following expressions among the special polygonal & pyramidal numbers are given below

1. $\left[\dfrac{3P_x^3}{t_{3,x+1}}\right]^2 + \left[\dfrac{12p_y^5}{s_{y+1}-1}\right]^2 \equiv 0 \pmod{41}$
2. $\left[\dfrac{4Pt_{x-3}}{P_{x-3}^3}\right]^4 - 41\left[\left[\dfrac{3P_z^3}{t_{3,z+1}}\right]^2 - \left[\dfrac{P_w^5}{t3,w}\right]^2\right]\left[\dfrac{4P_p^5}{ct_{4,p}-1}\right]^5$ is a bi quadratic integer.
3. $41\left[\left[\dfrac{6P_{z-2}^3}{\Pr_{z-2}}\right]^2 - \left[\dfrac{4Pt_{w-3}}{P_{w-3}^3}\right]^2\right][4\Pr_p+1]^2 + \left[\dfrac{P_y^5}{t_{3,y}}\right]^4$ is a perfect square.
4. $\left[\dfrac{4P_u^5}{Ct_{4,u}-1}\right]^2 - \left[\dfrac{12p_v^5}{s_{v+1}-1}\right]^2 \equiv 0 \pmod{41}$
5. $41^4\left\{\left[\dfrac{3P_{u-2}^3}{t_{3,u-2}}\right]^2 - \left[\dfrac{P_v^5}{\Pr_v}\right]^2\right\}$ is a quintic integer

4. Conclusion

In this paper, we have illustrated different methods of obtaining non-zero integer solutions to the Heptic equation with 5 unknowns given by $x^4 - y^4 = 41(z^2 - w^2)p^5$. As the Heptic Diophantine equation are rich in variety, one may consider other forms of Heptic equation with variable ≥ 5 and search for their corresponding integer solutions along with the corresponding properties.

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