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On the Diophantine Equation $1023^x + 8^y - z^2$

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ABSTRACT
In this paper, we discussed the solution of the Diophantine equation $1023^x + 8^y = z^2$, where x,y and z are non-negative integers. The results showed that this Diophantine equation has exactly three non-negative integer solutions. The solutions are $(x,y,z) = (0,1,3), (1,0,32)$ and $(2,4,1025)$.

Keywords: Diophantine equation, Non-negative integer solutions, Congruence.

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1. INTRODUCTION

Diophantine equations are equations in which the solutions are required to be integers. It is a popular topic in Number theory and has many vital applications. Many mathematicians have studied the Diophantine equation of the form $a^x + b^y = z^2$, where a, b are fixed integers and x, y, z are non-negative integers (see for instance [1,2,3,4,5]).

In 2020, Sudhanshu, A. and Nidhi, S. [1] studied the Diophantine equation $379^x + 397^y = z^2$. They proved that this Diophantine equation has no non-negative integer solution.

In 2024, Manikandan, K. and Venkatraman, R. [4] showed that the Diophantine equation $8^x + 161^y = z^2$ has exactly two non-negative integer solutions. The solutions are $(x,y,z) = (0,1,3)$ and $(1,1,13)$.

2. PRELIMINARIES

Theorem 2.1
(Catalan's conjecture) [6] The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$, has a unique solution $(a, b, x, y) = (3,2,2,3)$.

Lemme 2.2
The Diophantine equation $1 + 8^y = z^2$, where y and z are non-negative integers, has a unique non-negative integer solution $(y, z) = (1,3)$.
Proof. Let y and z be non-negative integers such that $1 + 8^y = z^2$. If $y = 0$, then $z^2 = 2$. It is a contradiction. Thus $y \geq 1$, we obtain $z^2 = 1 + 8^y \geq 1 + 8^1 = 9$, then $z \geq 3$. According to Theorem 2.1, we have $y = 1$. It implies that $z^2 = 9$, then $z = 3$. Thus, $(y, z) = (1,3)$ is a solution of the Diophantine equation $1 + 8^y = z^2$. □

Lemme 2.3
The Diophantine equation $1023^x + 1 = z^2$, where x and z are non-negative integers, has a unique non-negative integer solution $(x, z) = (1,32)$.
Proof. Let x and z be non-negative integers such that $1023^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$. It is a contradiction. Thus $x \geq 1$, we obtain $z^2 = 1023^x + 1 \geq 1023^1 + 1 = 1024$, then $z \geq 32$. According to Theorem 2.1, we have $x = 1$. It implies that $z^2 = 1024$, then $z = 32$. Thus, $(x, z) = (1,32)$ is a solution of the Diophantine equation $1023^x + 1 = z^2$. □

3. RESULTS AND DISCUSSION

Theorem 3.1
The Diophantine equation $1023^x + 8^y = z^2$, where x, y and z are non-negative integers, has exactly three non-negative integer solutions. The solutions are $(x, y, z) = (0,1,3), (1,0,32)$ and $(2,4,1025)$.
Proof. Let x, y and z be non-negative integers such that $1023^x + 8^y = z^2$. We divide the proof into four cases as follows:

Case 1. $x = 0$ and $y = 0$.

We have $z^2 = 2$, which is a contradiction.

Case 2. $x = 0$ and $y > 0$.

We have $1 + 8^y = z^2$. By Lemma 2.2, we obtain $(x, y, z) = (0,1,3)$.

Case 3. $x > 0$ and $y = 0$.

We have $1023^x + 1 = z^2$. By Lemma 2.3, we obtain $(x, y, z) = (1,0,32)$.

Case 4. $x > 0$ and $y > 0$.

Since 1023^x and 8^y are odd and even integers, respectively, thus $z^2 = 1023^x + 8^y$ is an odd integer. It implies that z is an odd integer. We obtain $z^2 \equiv 1 \pmod 4$. Now, since $8^y \equiv 0 \pmod 4$. Thus $1023^x \equiv 1 \pmod 4$. We get that x is even. Let $x = 2k, k \in \mathbb{Z}^+$. We obtain $1023^{2k} + 8^y = z^2$. Then $z^2 - 1023^{2k} = 8^y$. Thus $(z - 1023^k)(z + 1023^k) = 2^{3y}$. There exists a non-negative integer u such that $z - 1023^k = 2^u$ and $z + 1023^k = 2^{3y-u}$, where $3y - u > u, 3y - 2u > 0$. We obtain $2(1023^k) = 2^{3y-u} - 2^u = 2^u(2^{3y-2u} - 1)$. Then we get that $u = 1$. Thus $1023^k = 2^{3y-2} - 1$. If $k = 1$, we have $2^{3y-2} = 1024$. Then $y = 4$. Thus, $(x, y, z) = (2,4,1025)$ is a solution of equation $1023^x + 8^y = z^2$. If $k > 1$, then $2^{3y-2} = 1023^k + 1 > 1023^1 + 1 = 1024 = 2^{10}$. Thus $3y - 2 > 10$. We have $\min\{1023, k, 2, 3y - 2\} = 2 > 1$. According to Theorem 2.1, the equation $2^{3y-2} - 1023^k = 1$ has no non-negative integer solution. □

4. CONCLUSION

In this paper, we showed that the Diophantine equation $1023^x + 8^y = z^2$ has exactly three non-negative integer solutions. The solutions are given by $(x, y, z) = (0,1,3), (1,0,32)$ and $(2,4,1025)$. We can apply it to study the Diophantine equation $1023^x + 2^{ky} = z^2$, where k is a fixed positive integer and x, y, z are non-negative integers.

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