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Cosmological solution in Brans-Dicke theory involving particle creation in relativistic viscous fluid universe *

Anita Sorokhaibam[1,†]

1. Department of Humanities and Basic Sciences, Manipur Institute of Technology, Imphal, Manipur- 795004, India.

1. E-mail: sorokhaibi@gmail.com

Abstract The problem of the particle creation in the viscous fluid universe in the Brans-Dicke field theory is studied in this paper. The solutions obtained are investigated with respect to the Dirac's hypothesis along with the discussion of the gravitational parameters as well as the variable gravitational constants. By adopting a particular method of integration, exact solutions are obtained corresponding to the three values of curvature index and physical interpretations are presented.

Key words Viscosity, Radius of Curvature, Tensor, Correction factor.

2020 Mathematics Subject Classification 83C99, 83E99 83F05, 83F99.

1 Introduction

The attention of a number of authors had earlier been drawn to consider the Brans-Dicke scalar field interacting with perfect fluid distribution in the study of particle creation in the relativistic cosmological models. Deo [1, Chapter VIII, p.112] studied the problem of particle creation in Brans-Dicke theory involving perfect fluid distribution by considering the radius of the universe to be a linear function of time. It was observed that though the first Dirac's hypothesis [4] is satisfied, the value of the gravitational variable differs from its value obtained in the case of the closed universe because of the variation in the value of ω while the value of mass density in both the cases is the same at any instant of time.

Chand et al. [2] obtained an exact cosmological solution by considering spin-half generated particles as matter sources in the Jordan-Brans-Dicke theory. It was observed that the gravitational variable decreased linearly with time and the mass of the universe increases proportionally to the square of the age while the density decreases with time.

Singh [6] obtained the dynamics of the particle creation in the slowly rotating Robertson-Walker universe, using the Brans-Dicke theory. Along with the discussion of the nature of the gravitational constant and other physical parameters in this model universe, the behavior of these solutions with respect to the Dirac's hypothesis [4] was obtained. Except for the case of a flat universe, the metric rotation as well as the matter rotation is found to have a dampening effect on the creation of particles.

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†Corresponding author Anita Sorokhaibam, E-mail: sorokhaibi@gmail.com

Ibochouba and Ali [5] investigated the problem of dynamics of particle creation in the relativistic viscous fluid universe in the Brans-Dicke scalar field theory and obtained the exact solution on the assumption that φ is a function of the first derivative of the radius of the universe.
In the present paper, a study of particle creation in the cosmological viscous fluid universe in the Brans-Dicke scalar theory is done on the assumption that the Brans-Dicke scalar field varies directly as the radius of the universe. Exact solutions corresponding to the three values of the curvature index are obtained. The results obtained in different cases are compared.

2 Field equations and their solutions

The line element considered for this problem is the Robertson-Walker metric

$$ds^2 = c^2dt^2 - R^2\left[\left(1-Kr^2\right)^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2\right] \tag{2.1}$$

where, as usual, $K = -1,\ 0,\ 1$ for the open, the flat and the closed universe respectively. We take up here a viscous fluid distribution whose energy momentum tensor T_{ij} given by

$$T_{ij} = \rho U_iU_j + \left(\frac{P}{c^2}\xi\theta\right)H_{ij} - 2\eta\sigma_{ij} \tag{2.2}$$

which satisfies the conservation equation

$$T^{ij}_{;j} = 0 \tag{2.3}$$

where P is the isotropic pressure, ρ is the fluid density, η and ξ are the coefficients of the shear and the bulk viscosities respectively and U_i the $4-$ flow vector.
The H_{ij} are the projection tensors defined by
$H_{ij} \equiv U_{ij} - g_{ij}$ while, $\sigma_{ij} \equiv \frac{1}{2}\left(U_{i;\beta}H^{\beta}_{j} + H_{j;\beta}H^{\beta}_{i}\right) - \frac{1}{3}\theta H_{ij}$ is the shear tensor where, $\theta = U^{\alpha}_{,\alpha}$ is the expansion factor.
Now the Brans-Dicke field equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\left[\frac{8\pi}{c^4\varphi}\right]T_{ij} - \frac{\omega}{\varphi^2}\left[\varphi_{,i}\varphi_{,j} - g_{ij}\varphi_{,k}\varphi^{,k}\right] - \frac{1}{\varphi}\left[\varphi_{ij} - g_{ij}\varphi^{m}_{;m}\right] \tag{2.4}$$

yields the equations

$$\frac{d}{dt}\left(\dot{\varphi}R^3\right) = \left(\frac{8\pi}{3+2\omega}\right)\left[\rho - \frac{3}{c^2}\left(P - \xi\theta\right)\right]R^2 \tag{2.5}$$

and

$$\frac{1}{c^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{K}{R^2} = \frac{8\pi\rho}{3c^2\varphi} - \frac{1}{c^2}\left(\frac{\dot{\varphi}}{\varphi}\right)\left(\frac{\dot{R}}{R}\right) + \frac{\omega}{6c^2}\left(\frac{\dot{\varphi}}{\varphi}\right)^2 \tag{2.6}$$

where the wave equation for the scalar field is given by

$$\Box\varphi \equiv \varphi^{m}_{;m} = \frac{8\pi T}{\left(3+2\omega\right)c^4}. \tag{2.7}$$

Also,

$$\varphi = G^{-1}\left[\frac{4+2\omega}{3+2\omega}\right] \tag{2.8}$$

where it may be noted that $G = G_0$ at the present time and G_0 is the Newton's gravitational constant.
For the density of the material substratum under consideration (Schafer and Dehnen [3]), we get

$$\rho = \frac{Nm_o}{2\pi^2R^3} \tag{2.9}$$

where N is the variable number of the created particles and m_o is their rest mass. In view of Heisenberg's uncertainty relation, the differential equation for the particle number N is given by

$$\frac{dN}{dt} = \left(\frac{\pi}{16}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2\dot{R}^2R \tag{2.10}$$

where $a \approx 0.22$ is the correction factor found by Schafer Dehnen [3].
From (2.3) we have

$$\dot{\rho} = -3\left(\frac{\dot{R}}{R}\right)\left[\rho + \frac{1}{c^2}\left(P - \xi\theta\right)\right]. \tag{2.11}$$

Now (2.9) and (2.10) yield

$$\frac{dN}{dt} = -\left(\frac{6\pi^2}{m_0c^2}\right)R^2\dot{R}\left(P - \xi\theta\right) \tag{2.12}$$

From (2.10) and (2.11), we obtain

$$P = -\left[\left(am_0^3c^3/96\pi h^2\right) - 3\xi\right]\left(\frac{\dot{R}}{R}\right). \tag{2.13}$$

Since

$$\theta = 3\dot{R}/R \tag{2.14}$$

which is the negative pressure produced by the created particles, and as $\left(\frac{\dot{R}}{R}\right)$ is the Hubble's parameter, the pressure is directly proportional to the Hubble's parameter.

3 The method of integration

To obtain the exact solutions of the highly non-linear differential equation we make the assumption that the radius of curvature $R(t)$ is a function of time.
Let,

$$\varphi = bR(t). \tag{3.1}$$

By making use of (3.1) in (2.6), we obtain

$$\rho = a_1\frac{R^2}{R} + a_2\frac{K}{R} \tag{3.2}$$

where, $a_1 = \frac{3b}{8\pi}\left(2 - \frac{1}{6}\omega\right)$, $a_2 = \frac{3bc^2}{8\pi}$.
Using (2.11), (2.13) and (2.14), we have

$$\dot{\rho} + 3\left(\frac{\dot{R}}{R}\right)\rho = A\left(\frac{\dot{R}}{R}\right)^2 \tag{3.3}$$

where $A = \frac{3}{c^2}\left(\frac{am_0^3c^3}{32\pi h^2}\right)$.
The following three cases arise:
Case I. When the curvature index $K = 0$, we get from (3.2)

$$\rho = a_1\frac{\dot{R}^2}{R} \tag{3.4}$$

From (3.3) and (3.4), we get

$$2a_1\ddot{R}\,R + 2a_1\dot{R}^2 = A\dot{R}. \tag{3.5}$$

A particular solution of (3.5) is

$$R = s_1t \tag{3.6}$$

where

$$s_1 = A/2a_1. \tag{3.7}$$

To obtain a physically realistic solution, we obtain

$$6b\left(2 - \frac{1}{6}\omega\right) > 0 \;\; i.e. 12 > \omega,\; b > 0 \tag{3.8}$$

Making use of (3.6) in (3.1), we obtain

$$\varphi = bs_1t \tag{3.9}$$

Using (3.6) in (3.4), we get

$$\rho = \left(\frac{A}{2}\right)\frac{1}{t} \tag{3.10}$$

Using (3.6) in (2.13) , we obtain

$$P = -\left[\left(am_0^3c^2/96\pi h^2\right) - 3\xi\right]\frac{1}{t} \tag{3.11}$$

Substituting the value of $R,\ P$ and ρ in (2.5) we obtain the relation

$$b^2 = \frac{(3+2\omega)}{8(12-\omega)^3}\left(\frac{am_0^3c}{h^2}\right)^2 \tag{3.12}$$

Using (3.6) in (2.10), we get

$$\frac{dN}{dt} = \left(\frac{\pi}{16}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_1^3t^2 \tag{3.13}$$

which on integration yields the solution

$$N = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_1^3t^2 + C_1 \tag{3.14}$$

where C_1 is an arbitrary constant of integration. If we take $C_1 = 0$ in (3.14), we get

$$N = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_1^3t^2 \tag{3.15}$$

By making use of (3.9) in (2.8), we get

$$G = \left(\frac{4+2\omega}{3+2\omega}\right)\left(\frac{1}{bs_1}\right)\frac{1}{t} \tag{3.16}$$

From the above solution we have observed that the radius of curvature of the universe is a function of time. The pressure ρ is inversely proportional to the age of the universe. The gravitational variable G varies inversely as the age of the universe. From (3.16) we have seen that the first Dirac's hypothesis [4] is satisfied. The total mass M of the universe is given by

$$M = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_1^3t^2 \tag{3.17}$$

and the present value of the radius of curvature of the universe is given by

$$R = s_1/H \tag{3.18}$$

where $H = \left(\dot{R}/R\right)_0 = 55$ kms^{-1} which is in agreement with the non-cosmological determination by Dicke [7]. Since the density $\rho > 0$, we get from (3.4) that

$$a_1 = \frac{3b}{8\pi}\left(2 - \frac{\omega}{6}\right) > 0$$

Furthermore, to have an expanding model of the universe $A > 0$, so that, $s_1 > 0$.
Case II When $K = 1$ we get from (3.2)

$$\rho = a_1\frac{\dot{R}^2}{R} + \frac{a_2}{R} \tag{3.19}$$

From (3.3) and (3.19), we get

$$2a_1\ddot{R}R + 2a_1\dot{R}^2 + 2a_2 = A\dot{R} \tag{3.20}$$

A particular solution of (3.20) is given by

$$R = s_2t \tag{3.21}$$

where,

$$s_2 = \frac{A \pm \sqrt{A^2 - 16a_1a_2}}{4a_1} \tag{3.22}$$

To obtain physically realistic solution, we have from (3.22),

$$A^2 > 16a_1a_2.$$

In order to have an always expanding model, we consider the positive sign before the radical sign in (3.22). Then, we write

$$s_2 = \frac{A + \sqrt{A^2 - 16a_1a_2}}{4a_1}$$

Using (3.21) in (3.1), we obtain

$$\varphi = bs_2t. \tag{3.23}$$

By making use of (3.21) in (3.19), we get

$$\rho = \left(a_1s_2 + \frac{a_2}{s_2}\right)\frac{1}{t}. \tag{3.24}$$

Using (3.21) in (2.13), we get

$$P = -\left[\left(am_0^3c^3/96\pi h^2\right) - 3\xi\right]\frac{1}{t}. \tag{3.25}$$

Substituting the value of $R,\ P$ and ρ in (2.5), we get

$$a_1s_2^2 - \frac{3\,(3+2\omega)\,bs_2^3}{8\pi} + \frac{am_0^3s_2}{32\pi h^2} + a_2 = 0 \tag{3.26}$$

which is a relation between constants.
Using (3.21) in (2.10), we get

$$\frac{dN}{dt} = \left(\frac{\pi}{16}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_2^3t \tag{3.27}$$

Integrating (3.27) we get

$$N = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_2^3t^2 \tag{3.28}$$

where the arbitrary constant of integration is taken to be zero.
By making use of(3.23) in (2.8), we get

$$G = \left[\frac{4+2\omega}{3+2\omega}\right]\frac{1}{bs_2t} \tag{3.29}$$

We observe from (3.21), (3.25) and (3.28) that the radius of curvature of the universe increases with time, the pressure is inversely proportional to the age of the universe and the number of particles created at any time varies directly as the square of the age of the universe. From (3.29) we observe that the gravitational variable G varies inversely as the age of the universe. The density and pressure are to be decreasing functions of time and tend to zero as t tends to infinity. For a closed universe the first Dirac's hypothesis [4] is satisfied by (3.29).
The total mass of the universe at any time is given by

$$M = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_2^3t^2 \tag{3.30}$$

and the present value of the radius of curvature of the closed universe is given by
$R = s_2/H$, where $H = \left(\dot{R}/R\right)_0 = 55\ \text{kms}^{-1}$
Case III Taking the curvature index $K = -1$ in (3.2), we obtain

$$\rho = a_1\frac{\dot{R}^2}{R} - \frac{a_2}{R} \tag{3.31}$$

From (3.3) and (3.31), we obtain

$$2a_1\ddot{R}\ R + 2a_1\dot{R}^2 - 2a_2 = A\dot{R} \tag{3.32}$$

A particular solution of (3.32) is given by

$$R = s_3t \tag{3.33}$$

where

$$s_3 = \frac{A + \sqrt{A^2 + 16a_1a_2}}{4a_1} \tag{3.34}$$

Using (3.33) in (3.1), we obtain

$$\varphi = bs_3t. \tag{3.35}$$

Using (3.33) in (2.13), we obtain

$$P = -\left[\left(am_0^3c^3/96\pi h^2\right) - 3\xi\right]\frac{1}{t}. \tag{3.36}$$

By making use of (3.33) in (3.31), we obtain

$$\rho = \left(a_1s_3 - \frac{a_2}{s_3}\right)\frac{1}{t}. \tag{3.37}$$

Substituting the value of R, P and ρ in (2.6), we obtain

$$3bs_3 = \left(\frac{8\pi}{3+2\omega}\right)\left[a_1s_3 - \frac{a_2}{s_3} + \frac{am_0^3c}{32\pi h^2}\right] \tag{3.38}$$

which is a relation between constants.
Using (3.33) in (2.10), we get

$$\frac{dN}{dt} = \left(\frac{\pi}{16}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_3^3t \tag{3.39}$$

which on integrating yield

$$N = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_3^3t^2 \tag{3.40}$$

where the arbitrary constant of integration is taken to be zero.
Using (3.35) in (2.8), we get

$$G = \left[\frac{4+2\omega}{3+2\omega}\right]\left(\frac{1}{bs_3}\right)\frac{1}{t} \tag{3.41}$$

The total mass M of the universe is given by

$$M = \left(\frac{\pi}{32}\right)\left(\frac{a}{c}\right)\left(\frac{m_0c}{h}\right)^2 s_3^3t^2 \tag{3.42}$$

In the Case III also we have seen that the radius of curvature of the universe is an increasing function of time whereas the mass density ρ and the pressure P both are decreasing functions of time. The gravitational variable G varies inversely as the age of the universe and the first Dirac's hypothesis [4] is satisfied. From (3.42) we find that the total mass of particles at any instant varies directly as the square of its age thereby showing that the mass of the universe increases proportionally to the square of the age of the universe.

4 General conclusion

In all the three cases the total mass number of particles at any instant of time t increases proportionally with the square of the age of the universe thereby satisfying the second Dirac's hypothesis [4] and the gravitational variable varies inversely as the age of the universe thereby satisfying the first Dirac's hypothesis [4]. The pressure is found to be negative in all the three cases due to the pressure generated by the created particles.

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References

[1] Deo, Shridhar (1986). Some cosmological problems of physical distribution in Robertson-Walker universe, Ph.D. Thesis, Manipur University, Manipur, India.

[2] Singh, R.K. Tarachand and Devi, Ratnaphabha A. (1989). Cosmological solution in B-D Theory involving particle creation, *Astrophys. Space Sci.*, 155, 233–240.

[3] Schafer, G. and Dehnen, H.(1977). On the origin of matter in the universe, *Astronomy and Astrophysics*, 54, 823–836.

[4] Dirac, P.A.M. (1938). A new basis of cosmology, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 165 (921), 199–208.

[5] Ibochouba, L. and Ali, Mohammed Ashraf (1996). Dynamics of particle creation in relativistic viscous fluid universe in B-D theory, *Astrophys. Space Sci.*, 241, 227–235.

[6] Singh, K. Manihar (1991). Particle creation in slowly rotating Robertson-Walker universes, *Astophys. Space Sci.*, 182, 141–153.

[7] Dicke, R.H. (1962). Mach's principle and invariance under transformation of units, *Phys. Rev.*, 125, 2163.