Bull. Pure Appl. Sci. Sect. E Math. Stat.
39E(2), 188–192 (2020)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2020.00018.1

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

Graceful labeling on twig diamond graph with pendant edges *

J. Jeba Jesintha[1,†], Subashini K.[2] and Allu Merin Sabu[3]

1,3. P.G. Department of Mathematics, Women's Christian College,
Chennai, Tamil Nadu, India.
2. Department of Mathematics, Jeppiaar Engineering College,
Chennai, Tamil Nadu, India.
2. Research Scholar (Part-Time), P.G. Department of Mathematics, Women's Christian College,
Affiliated to University of Madras, Chennai, Tamil Nadu, India.
1. E-mail: ✉ jjesintha_75@yahoo.com
2. E-mail: k.subashinirajan@gmail.com , 3. E-mail: allusabu003@gmail.com

Abstract A graceful labeling of a graph G with q edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edges are also distinct, where an edge incident with the vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove the graceful labeling of a new family of graphs G called a twig diamond graph with pendant edges.

Key words Graceful labeling, star graph, diamond graph.

2020 Mathematics Subject Classification 05C78.

1 Introduction

The most interesting and famous graph labeling method is the *graceful labeling* of graphs introduced by Rosa [4] in 1967. A graceful labeling of a graph G with q edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edges are also distinct, where an edge incident with the vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. A variety of graphs and families of graphs are known to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [4].
Sethuraman and Jeba Jesintha [5,6] proved that all banana trees and extended banana trees are graceful. Hoede and Kuiper [2] showed that wheels $W_n = C_n + K_1$ are graceful. Rosa [4] showed that the n cycle C_n is graceful if and only if $n = 0$ or $3(\text{mod } 4)$. Kaneria and Makadia [3] proved that a star of a cycle $C_n(n = 0(\text{mod } 4))$ is graceful. Golomb [7] showed that all complete bipartite graphs are graceful. For an exhaustive survey on graceful graphs we refer to the dynamic survey by Gallian [1].
Graceful labeling is actively being used in many research fields such as communication in sensor networks, designing fault tolerant systems, automatic channel allocation, coding theory problems, X-ray, optimal circuit layout and additive number theory. In this paper, we prove the graceful labeling on twig diamond graph attached with pendant edges.

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received May 27, 2019 / Revised October 28, 2020 / Accepted November 11, 2020. Online First Published on December 26, 2020 at https://www.bpasjournals.com/.
†Corresponding author J. Jeba Jesintha, E-mail: jjesintha_75@yahoo.com

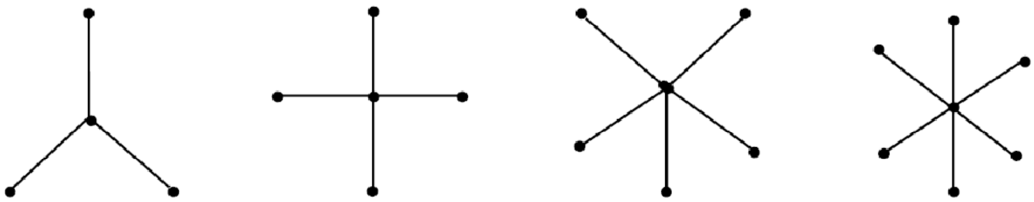

Fig. 1: The star graphs S_4, S_5, S_6 and S_7.

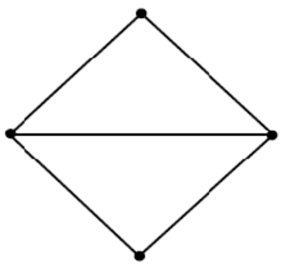

Fig. 2: The diamond graph.

2 Preliminary definitions

We now detail the necessary preliminary definitions to be used by us in this paper.

Definition 2.1. The star graph S_n of order n, sometimes simply known as an *"n star graph"* is a tree on n vertices with one vertex of degree $n-1$ and the other $n-1$ vertices each of degree 1 (see Fig. 1).

Definition 2.2. The *diamond graph* is a planar undirected graph with 4 vertices and 5 edges. It consists of a complete graph K_4 minus one edge (see Fig. 2).

Definition 2.3. The *twig diamond graph* is a planar undirected graph with 8 vertices and 11 edges obtained by the attachment of two diamond graphs by an edge (see Fig. 3).

3 The main result

In this section we prove the main result of this paper in Theorem 3.1 below.

Theorem 3.1. *The twig diamond graph with pendant edges is graceful.*

Proof. Let us consider two diamond graphs D_1 and D_2. The vertices on the diamond graph D_1 are denoted by $u_1, \ldots, u_4$ and the vertices on D_2 are denoted by $u_5, \ldots, u_8$ in the clockwise direction. We connect the two diamond graphs by an edge joining the two vertices u_2 and u_8 and name the resultant graph as twig diamond graph as shown in Fig. 3. Now pendant edges are attached to the remaining vertices of the twig diamond graph, namely, u_1, u_3, u_4 and u_5, u_6, u_7. The vertices attached at u_1 are denoted by $s_1, \ldots, s_n$ in the clockwise direction and the vertices attached at u_3 are denoted by $v_1, \ldots, v_n$ in the clockwise direction. Similarly, the vertices attached at u_4 are denoted by $t_1, \ldots, t_n$ in the clockwise direction, the vertices attached at u_5 are denoted by $w_1, \ldots, w_n$ in the clockwise

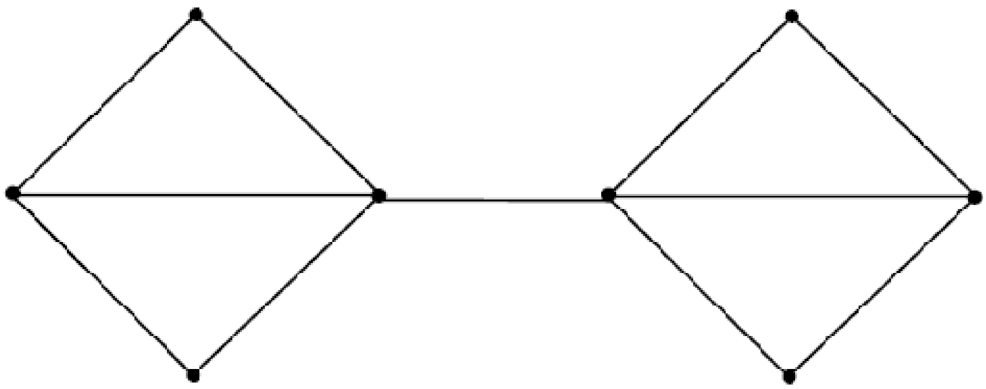

Fig. 3: The twig diamond graph.

direction and the vertices attached at u_6 are denoted by $x_1, \ldots, x_n$ in the clockwise direction. Again, the vertices of the star graph attached at u_7 are denoted by $y_1, \ldots, y_n$ in the clockwise direction as shown in Fig. 4. For reasons of simplicity let us denote the vertices u_1 and u_5 together as u_1^i; u_2 and u_6 together as u_2^i; u_3 and u_7 together as u_3^i and u_4 and u_8 together as u_4^i for $i = 1, 2$, where, $u_1^1 = u_1, u_1^2 = u_5, u_2^1 = u_2, u_2^2 = u_6, u_3^1 = u_3, u_3^2 = u_7, u_4^1 = u_4$ and $u_4^2 = u_8$. The resulting graph has $p = 6n + 8$ vertices and $q = 6n + 11$ edges where n denotes the number of pendant edges as shown in Fig. 4.

The *vertex labels* for the twig diamond graph are as follows:

$$f\left(u_1^i\right) = (i - 1)(n + 3), \text{ for } 1 \leq i \leq 2,$$

$$f\left(u_2^i\right) = (n + 1) + (i - 1)(2n + 8), \text{ for } 1 \leq i \leq 2,$$

$$f\left(u_3^i\right) = i(n + 2), \text{ for } 1 \leq i \leq 2,$$

$$f(u_4^i) = q - n - (i - 1)(n + 2), \text{ for } 1 \leq i \leq 2. \tag{3.1}$$

The *vertex labels* for the pendant edges attached at the vertex u_1 are as given below:

$$f(s_j) = q - (j - 1), \text{ for } 1 \leq j \leq n. \tag{3.2}$$

The *vertex labels* for the pendant edges attached at the vertex u_3 are given by

$$f(v_j) = q - n + (j - 1), \text{ for } 1 \leq j \leq n, \tag{3.3}$$

and the *vertex labels* for the pendant edges attached at the vertex u_4 are given by

$$f(t_j) = n - (j - 1), \text{ for } 1 \leq j \leq n. \tag{3.4}$$

The *vertex labels* for the pendant edges attached at the vertex u_5 are given by

$$f(w_j) = \begin{cases} q - (2n + 1), \text{ for } j = 1, \\ q - 2n - 2 - (j - 1), \text{ for } 2 \leq j \leq n. \end{cases} \tag{3.5}$$

The *vertex labels* for the pendant edges attached at the vertex u_6 are given by

$$f(x_j) = \begin{cases} (n + 4) + j, \text{ for } 1 \leq j \leq n - 1 \\ 2j + 5, \text{ for } j = n \end{cases} \tag{3.6}$$

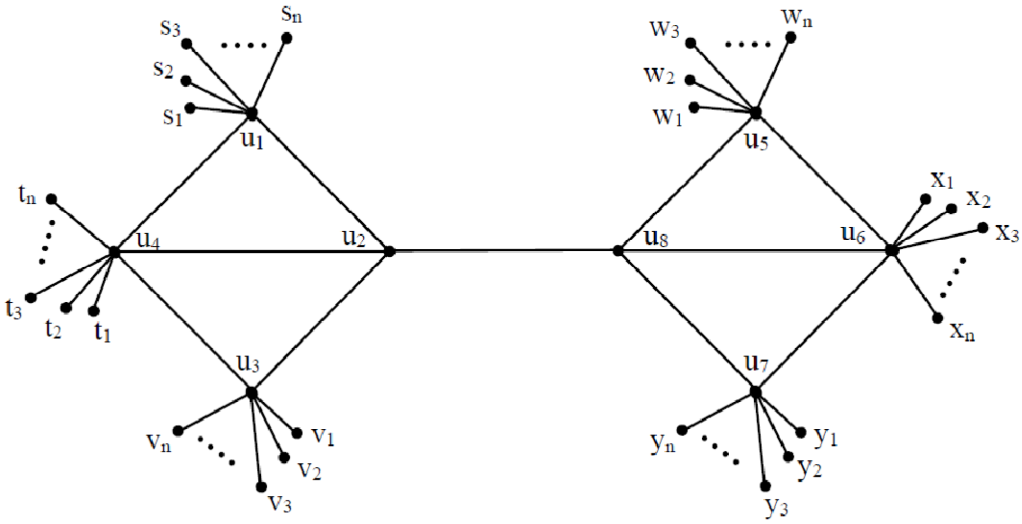

Fig. 4: The twig diamond graph with pendant edges.

The *vertex labels* for the pendant edges attached at the vertex u_7 are given by

$$f(y_j) = \begin{cases} \left\lfloor \frac{q}{2} \right\rfloor + 3 - j, & \text{for } 1 \le j \le 2, \\ \left\lfloor \frac{q}{2} \right\rfloor - j + 1, & \text{for } 3 \le j \le n. \end{cases} \tag{3.7}$$

From equations (3.1) to (3.7) we see that the vertex labels 0, 1, 2, 3, ..., q are distinct.
The *edge labels* for the twig diamond graph are computed as follows:

$$\begin{aligned} \left| f\left(u_1^i\right) - f\left(u_2^i\right) \right| &= \left|4 - i(n+5)\right|, \text{ for } 1 \le i \le 2, \\ \left| f\left(u_2^i\right) - f\left(u_3^i\right) \right| &= \left|(n+6)i - (n+7)\right|, \text{ for } 1 \le i \le 2, \\ \left| f\left(u_3^i\right) - f\left(u_4^i\right) \right| &= \left|(2n+4)i - (q+2)\right|, \text{ for } 1 \le i \le 2, \\ \left| f\left(u_4^i\right) - f(u_1^i) \right| &= \left|(q+n+5) - i(2n+5)\right|, \text{ for } 1 \le i \le 2, \\ \left| f\left(u_2^i\right) - f\left(u_4^i\right) \right| &= \left|(3n+10)i - (n+q+9)\right|, \text{ for } 1 \le i \le 2, \\ \left| f\left(u_2^1\right) - f(u_4^2) \right| &= 4n - 7. \end{aligned} \tag{3.8}$$

The *edge labels* for the pendant edges are given as below:

$$\begin{aligned} \left| f(s_j) - f(u_1) \right| &= q - (j-1), \text{ for } 1 \le j \le n, \\ \left| f(v_j) - f(u_3) \right| &= q - (2m+3) - (j-1), \text{ for } 1 \le j \le n, \\ \left| f(t_j) - f(u_4) \right| &= q - n - j, \text{ for } 1 \le j \le n, \\ \left| f(w_j) - f(u_5) \right| &= \begin{cases} 3j + 7, & \text{for } j = n, \\ q - (3j+5) - j, & \text{for } 1 \le j \le n-1, \end{cases} \\ \left| f(x_j) - f(u_6) \right| &= \begin{cases} 2n + 5 - j & \text{for } 1 \le j \le n-1, \\ j + 4, & \text{for } j = n, \end{cases} \\ \left| f(y_j) - f(u_7) \right| &= \begin{cases} j + 1, & \text{for } 1 \le j \le n-2, \\ j + 3, & \text{for } n-1 \le j \le n. \end{cases} \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) it is obvious that the edge labels $1, \ldots, q$ are distinct. Thus the twig diamond graph is graceful. □

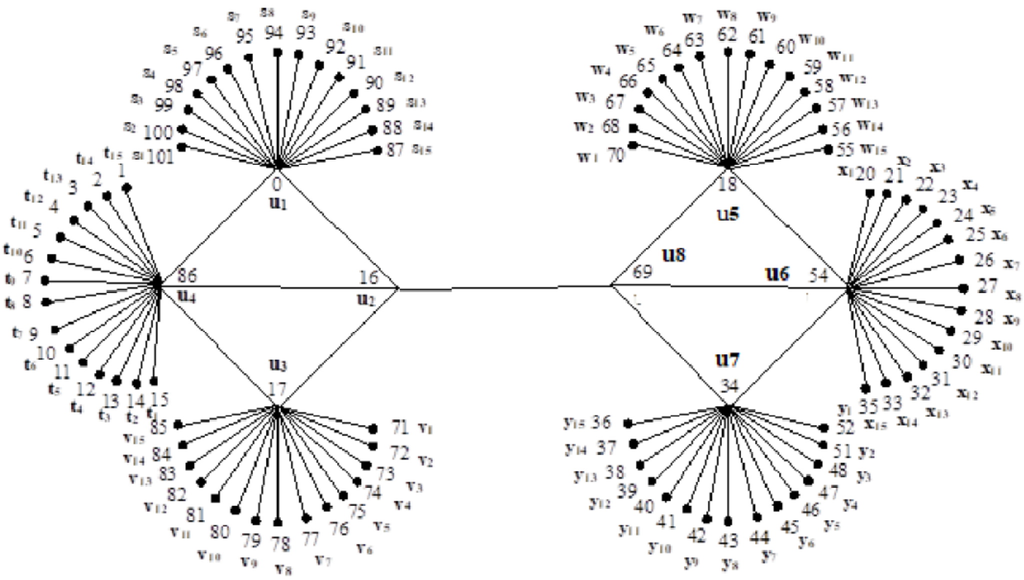

Fig. 5: The twig diamond graph attached with 15 pendant edges.

We illustrate Theorem 3.1 in Fig. 5 for the case when $p = 98$ and $q = 101$.

4 Conclusion

We have shown that the new family of graphs called the twig diamond graph is graceful. Further in our future paper we intend to prove the gracefulness of a new family of graphs obtained by joining pendant edges onto the vertices of the complete graph K_4 and the kite graph.

Acknowledgments The authors are grateful to the referees and the Editor-in-Chief for their comments which have helped them in modifying this paper.

References

[1] Gallian, J.A. (2017). A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 1–415.

[2] Hoede, C. and Kuiper, H. (1987). All wheels are graceful, *Util. Math.*, 14, 311–322.

[3] Kaneria, V.J. and Makadia, H.M. (2012). *Journal of Math. Research*, 4(1),54–57.

[4] Rosa, A. (1967). On certain valuations of the vertices of a graph, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris, 349–355.

[5] Sethuraman, G. and Jeba Jesintha, J. (2009). All banana trees are graceful, *Adv. Appl. Disc. Math.*, 4, 53–64.

[6] Sethuraman, G. and Jeba Jesintha, J. (2009). All extended banana trees are graceful, *Proc. International Conf. Math. Comput. Sci.*, 1, 4–8.

[7] Golomb, S.W. (1972). How to number a graph, in Graph Theory and Computing, R.C. Read, ed., Academic Press, New York, 23–37.