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Some results on convergence closed function spaces *

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Abstract P. Dienes (see, P. Dienes, The Taylor Series, Oxford, 1931) studied sequence and sequence space and contributed the notion of convergence closed sequence space. That is under what circumstances a sequence space will be called convergence closed. In fact he gave a set of conditions dealing with parametric limit and projective limit under the umbrella of which a sequence space will be convergence closed under a definition of convergence in sequence space. Later on he established a few results using the notion of convergence closed sequence space. Through these established results efforts has been made by him to exhibit the set of different conditions under which different sequence spaces can get the title of being convergence closed sequence space. Later on Sharan (see, L. K. Sharan, Some contributions to the theory of function spaces, Ph.D. Thesis, Magadh University, Bodh Gaya, Bihar, India, 1986) extended the notion of convergence closed for function space (or spaces). He investigated that a few function spaces suitably defined are convergence closed function spaces. In this paper our aim is to extend some results on convergence closed function space. In the course of the study of convergence closed function spaces the notions of parametric limit, projective limit, projective convergence and the dual space of a function space are used, but it is only for function spaces that we have also used the notion of the section of a function. Efforts are made by us here to establish a few results which show the fact that there are some function spaces which are convergence closed, while establishing these results the vital role played by the dual space of a function space is also discussed.

Key words Function Space, Dual Function Space, Perfect Function Space, Regular Function Space, Convergence Closed Function Space, Parametric Convergent, Parametric Limit, Projective Convergent and Projective Limit, Section of Function.

2020 Mathematics Subject Classification 46A45, 42B35, 46E99, 46Exx.

1 Introduction

Dienes [5] studied the sequence and sequence space and contributed to the notion of convergence closed sequence space. That is under what circumstances a sequence space will be called convergence closed. In fact he gave a set of conditions dealing with parametric limit and projective limit under the umbrella of which a sequence space will be convergence closed under a definition of convergence in sequence space.

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Later on he established a few results using the notion of convergence closed sequence space. Through these established results efforts were made by him to exhibit the set of different conditions under which the different sequence spaces become convergence closed sequence spaces. Later on Sharan [18] extended the notion of convergence closed for function spaces (or spaces). He investigated that a few function spaces suitably defined are convergence closed function spaces. The first author has earlier also studied the concept of function spaces [19–21]. Some other relevant references in this field are [1–4, 6–17].

2 Preliminaries

Now we give below some of the relevant definitions by making the use of which some of the results will be established in the next section, in order to serve as a ready reference.

Definition 2.1. Sequence Space: A linear space whose elements are sequences is called a sequence space.

Definition 2.2. Function Space: A linear space whose elements are functions is called a function space.

In this paper we consider only real functions of real variables. Some definitions of special function spaces are being given below making the use of which some outcomes are established. The integration is taken in the Lebesgue sense in the interval $[0, \infty)$, which is denoted by us by the symbol E. Some notations that we use in this paper are listed below:

1. L_∞- Space of all bounded functions.
2. Γ- Space of all convergent and bounded unctions. Clearly, $\Gamma < L_\infty$, which means that the space of all convergent and bounded functions is space of all bounded functions.
3. L_1- Space of all integrable functions.
4. L_2- Space of all twice integrable functions.
5. ζ- Space of all continuous and bounded functions in E. Clearly, $\zeta < L_\infty$.
6. ϕ- Space of all functions f such that $f(x)$ is finite.

Definition 2.3. Dual Function Space: If we take α to be a function space then the dual function space of the function space α is denoted by α^* and it is defined to be the space of all functions f such that $\int_E |f(x)g(x)|dx < \infty$ for every function $g(x)$ in α.

Clearly α^* is also a function space. Also $\Gamma^* = L_1$, $L_\infty^* = L_1$, $\zeta^* = L_1$, $L_1^* = L_\infty$.

Definition 2.4. Perfect Space: A function space α is called perfect if its dual of dual is itself.

Clearly, L_1 and L_∞ are perfect. Since $L_1^{**} = L_1$ and $L_\infty^{**} = L_\infty$.

Definition 2.5. Parametric Convergent(or t-convergent): Let $f_t(x)$ be a family of functions of x, defined $\forall t \in E$, where t is a parameter. If for every given $\varepsilon > 0$, $\exists$ a positive number $T(\varepsilon)$, such that, $\forall x \geq 0, |f_t(x) - f_t^1(x)| \leq \varepsilon, \forall t, t^1 \geq T(\varepsilon)$, then $f_t(x)$ is called parametric convergent [11].

Definition 2.6. Parametric Limit (or t- limit): If for every given $\varepsilon > 0$, $\exists$ a positive number $T(\varepsilon)$, such that, $\forall x \geq 0, |f_t(x) - \psi(x)| \leq \varepsilon$, $\forall t \geq T(\varepsilon)$, then $\psi(x)$ is said to be the parametric limit of $f_t(x)$ and we write the parametric limit of $f_t(x)$ as $\psi(x)$.

Here we observe that any function equal to ψ(x) , $\forall x \geq 0$, is also a t-limit of $f_t(x)$.
We mean that $\psi(x)$ is a t-limit of $f_t(x)$ and all functions equivalent to $\psi(x)$ in E are t-limits of $f_t(x)$.
A function θ is said to be equivalent to $\psi(x)$ in E when $\theta(x) = \psi(x)$ almost everywhere in E.

Definition 2.7. Projective Convergence (or $\alpha\beta$-convergence or p-convergence): Let α and β be two function spaces such that $\alpha^* \supseteq \beta$ and $F_g(t) = \int_E f_t(x)\, g(x)dx$, where $f_t(x) \in \alpha$ and $g(x) \in \beta$ then if $F_g(t) \to$ a definite finite limit as $t \to \infty \forall g(x) \in \beta$ then we say that $f_t(x)$ is projective convergent relative to β, and $f_t(x)$ is simply called projective convergent in α when $\beta = \alpha^*$.

Thus we can get the necessary and sufficient condition for $\alpha\beta$- convergence of $f_t\,(x)$ is that $\forall g \in \beta$ and to every $\varepsilon > 0$, there exists a $T(\varepsilon, g) > 0$ such that $|\int_E g\,(x)\,\{f_t\,(x) - f_{t^1}\,(x)\}dx| \leq \varepsilon, \forall t, t^1 \geq T(\varepsilon, g)$.

Definition 2.8. Projective Limit (p- limit or $\alpha\beta$ - limit): Let α and β be two function spaces and $f_t\,(x)$ be family of functions belongs to α. A function ψ is called projective limit of $f_t\,(x)$ in α relative to β if there exist a function $g(x)$ belonging to β such that $\int_E |g\,(x)\,\psi\,(x)\,|dx < \infty$, and $\lim_{t\to\infty} \int_E f_t\,(x)\,g(x)dx = \int_E \psi\,(x)\,g(x)dx$. Symbolically, we write the projective limit of $f_t\,(x)$ in α relative to β as the p-limit of $f_t\,(x)$ in α relative to β or the $\alpha\beta$-limit of $f_t\,(x)$, where the function ψ may lie in α or outside α.

When $\beta = \alpha^*$, where α^* is the dual space of α, then we understand that the projective limit or p-limit of $f_t\,(x)$ in α relative to α^* by α-limit of $f_t\,(x)$ and we write α-limit of $f_t\,(x) = \psi$.

Definition 2.9. Section: If for a fixed $\varepsilon > 0$, $f_t\,(x) = \begin{cases} \psi\,(x)\,, \text{ for } 0 \leq x \ \leq t, \\ 0\ , \text{ for } x>t \end{cases}$ then $f_t\,(x)$ is called a section of $\psi(x)$. Thus for $t > 0$, we get the sections $f_t\,(x)$ of any given function $\psi(x)$.

Definition 2.10. Convergence Closed Function Spaces: Let α and β be any two function spaces such that $\alpha^* \supseteq \beta$ and we define $F_g(t) = \int_E f_t\,(x)\,g(x)dx$ where $f_t\,(x)$ is in α and $g(x)$ is in β, if $\lim_{t\to\infty} F_g(t)$ = definite finite for every $g(x)$ in β then we say that $f_t\,(x)$ is projective convergent (or p-convergent) relative to β. Suppose α be a function space such that every family of function $f_t\,(x)$ is p-convergent relative to the function space β and if for any given $\varepsilon > 0$, there corresponds a number $T(\varepsilon)$, independent of x, such that for almost all $x \geq 0$, $|f_t(x) - \psi(x)| \leq \varepsilon$ for all $t \geq T(\varepsilon)$, then $\psi(x)$ is called the parametric limit (t-limit) of $f_t(x)$ and we write that the t-limit of $f_t(x)$ is $\psi(x)$. If $\psi(x) \in \alpha$ then we say α is a *convergence closed function space* under the definition of p-convergence. Hence with a projective convergence, the parametric limit of every convergent family in a function space α is itself in α. If $\psi(x) \notin \alpha$ then we say that α is a *non convergence closed function space* under definition of p-convergence.

3 Results

In this section we establish some results on convergence closed function spaces. We also note that the forms of the results for sequence spaces are the same as those for the function spaces.

Theorem 3.1. *If $\alpha > \phi$ (which means that α (which denotes a function space) is a function space such that it contains the space ϕ of all convergent functions) then α is convergence closed under $\alpha\beta$-convergence provided that (i) $\alpha = L_\infty$, (ii) $\beta \subseteq L^*_\infty$.*

Proof. Let $\psi(x) \in L_\infty$. Also let for a fixed $\varepsilon > 0, f_t\,(x) = \begin{cases} \psi\,(x)\,, \text{ for } 0 \leq x \ \leq t \\ 0\ , \text{ for } x>t \end{cases}$ where t is in E . Then $f_t\,(x)$ is a section of $\psi(x)$. Now $f_t\,(x) = \psi(x) \in L_\infty$. Therefore, $f_t\,(x)$ is bounded in $[0, t]$ and $f_t\,(x) = 0$, for every $x > t$. Thus $f_t\,(x) \in \phi$ which implies that $f_t(x)$ is in α as $\alpha \supset \phi$. Thus $\psi(x) \in L_\infty \Longrightarrow \psi(x) \in \alpha$ as $\alpha = L_\infty$. Let $\psi(x)$ be the parametric limit of $f_t\,(x)$.
Then for each given $\varepsilon > 0$, there exists a positive number $T(\varepsilon)$, independent of x, such that $\forall x = 0$,

$$f_t(x) - \psi(x) {=} \varepsilon \ \ \forall t \geq T(\varepsilon). \tag{3.1}$$

Again let $E = [0, \infty)$ and let $g(x) \in \beta$. Hence

$$\int_E |g(x)|dx = \ \text{ a definite finite limit.} \tag{3.2}$$

Now for $g(x)$ in β and $\varepsilon > 0, t \geq T(\varepsilon)$ from (3.1) and (3.2), we get

$$\begin{aligned} \int_E |g(x)\psi(x)|dx &= \int_E |\ g\,(x)|\,|\psi\,(x) - f_t\,(x) + f_t\,(x)\,|dx \\ &\leqslant \int_E |g\,(x)|\,|\psi\,(x) - f_t\,(x)\,|dx + \int_E |g(x) f_t\,(x)\,|dx \\ &< \varepsilon K(g) + \int_E |g(x) f_t\,(x)\,|dx \end{aligned} \tag{3.3}$$

where $K(g)$ is a constant depending on g but independent of t. Also $f_t(x)$ is in $\alpha = L_\infty$ and g is in β. Hence

$$\int_E |g(x)f_t(x)|dx = \text{ a definite finite limit.} \tag{3.4}$$

Thus in the light of the relations (3.3) and (3.4) we get

$$\int_E |g(x)\psi(x)|dx = \text{ a definite finite limit.} \tag{3.5}$$

Again from (3.1) and (3.2) we have

$$\int_E f_t(x)\, g(x)dx - \int_E \psi(x)\, g(x)dx \leq \int_E |g(x)||f_t(x) - \psi(x)|dx < \varepsilon K(g).$$

Therefore

$$\lim_{t\to\infty} \int_E f_t(x)\, g(x)dx = \int_E \psi(x)\, g(x)dx. \tag{3.6}$$

Thus from (3.5) and (3.6) it follows that $\psi(x)$ is an $\alpha\beta$-limit of $f_t(x)$ and also $\psi(x) \in \alpha$. Hence α is convergence closed. Thus the theorem is established. □

Theorem 3.2. *Let $\alpha > \phi$ then α is convergence closed under $\alpha\beta$-convergence provided that (i) $\alpha = \Gamma$ (ii) $\beta = \Gamma^*$.*

Proof. Let $\psi(x) \in \Gamma$. Also let $f_t(x) = \begin{cases} \psi(x), & \text{for } 0 \leq x \leq t \\ 0, & \text{for } x > t \end{cases}$, where $t \in E$.
Then $f_t(x)$ is a section of $\psi(x)$. Also $f_t(x) = \psi(x) \in \Gamma \subseteq L_\infty$. But $f_t(x) = \psi(x) \in L_\infty$. Therefore, $f_t(x)$ is bounded in $[0, t]$ and $f_t(x) = 0, \forall x > t$ thus $f_t(x) \in \phi$, but by hypothesis $\alpha > \phi$. Hence $f_t(x) \in \alpha$, since $\psi(x) \in \Gamma$. But $\psi(x) = f_t(x) \in \alpha \Rightarrow \psi(x) \in \alpha$ as $\alpha = \Gamma$.
Now if $\psi(x)$ be the parametric limit of $f_t(x)$. Then for every given $\varepsilon > 0$, $\exists\, T(\varepsilon) > 0$ such that $\forall x \geq 0$, we have

$$|f_t(x) - \psi(x)| \leq \varepsilon, \forall t \geq T(\varepsilon). \tag{3.7}$$

Let $E = [0, \infty)$ and $g(x) \in \beta$, but $\beta \subseteq \Gamma^*$, hence $g(x)$ is an integrable function. Hence $g(x) \in L_1$. But then

$$\int_E |g(x)|dx \text{ is finite.} \tag{3.8}$$

Now for every $g(x) \in \beta = \Gamma^*$, $\varepsilon > 0$, $t \geq T(\varepsilon)$, we get

$$\begin{aligned} \int_E |g(x)\psi(x)|dx &= \int_E |g(x)|\, |\psi(x) - f_t(x) + f_t(x)|\, dx \\ &\leqslant \int_E |g(x)|\, |\psi(x) - f_t(x)|\, dx + \int_E |g(x) f_t(x)|\, dx \\ &\leqslant \varepsilon.\, k(g) + \int_E |g(x) f_t(x)|\, dx, \end{aligned} \tag{3.9}$$

which gives,

$$\int_E |g(x)\, f_t(x)|\, dx < \infty, \tag{3.10}$$

for $g(x) \in \beta$, $f_t(x) \in \alpha$.
Thus from (3.9) and (3.10)

$$\int_E |g(x)\psi(x)|dx < \infty. \tag{3.11}$$

Also,

$$\begin{aligned} &\left| \int_E f_t(x)\, g(x)dx - \int_E \psi(x)\, g(x)dx \right| \\ &\leqslant \int_E |g(x)|\, |f_t(x) - \psi(x)|\, dx \\ &\leqslant \varepsilon.\, k(g), \text{ since } \lim_{t\to\infty} \int_E f_t(x)\, g(x)dx = \int_E \psi(x)\, g(x)dx. \end{aligned} \tag{3.12}$$

Thus we find that $\psi \in \alpha$ and ψ is an $\alpha\beta$-limit of $f_t\,(x)$, that is under a given definition of $\alpha\beta$-convergence of $f_t\,(x)$ is in α. Thus α is convergence closed. □

Theorem 3.3. *Let $\alpha \supset \phi$ then α is convergence closed under $\alpha\beta$-convergence provided $\beta \subseteq L_1$.*

Proof. Let $\psi(x) \in \zeta$. Also let $f_t\,(x) = \begin{cases} \psi\,(x)\,,\ \text{for } 0 \le x \le t \\ 0\ ,\ \text{for } x{>}t \end{cases}$, where t is in E.
Then $f_t\,(x)$ is a section of $\psi(x)$. Also $f_t\,(x) = \psi(x) \in \zeta$. Thus $\psi(x) \in L_\infty$ (as $L_\infty \supset \zeta$). Thus $f_t\,(x) = \psi(x) \in L_\infty$.
Therefore, $f_t(x)$ is bounded in $[0, t]$ and $f_t(x) = 0\ \forall x > t$ thus $f_t(x) \in \phi \subseteq \alpha$, thus $f_t(x)$ is in α, but $\psi(x) = f_t(x) \in \alpha$.
Hence $\psi(x) \in \alpha$. Now if $\psi(x)$ be the parametric limit of $f_t(x)$ and $f_t(x) \in \alpha$, then for each given $\varepsilon > 0$, $\exists$ a positive number $T(\varepsilon)$, such that $\forall x \ge 0$, and for all $t \ge T(\varepsilon)$

$$|f_t(x) - \psi(x)| \leqslant \varepsilon. \tag{3.13}$$

Let $[0, \infty) = E$. Also let $g(x) \in \beta$. Thus due to the definition of β, $g(x)$ is integrable. Thus,

$$\int_E |g(x)|dx < \infty. \tag{3.14}$$

Now in order to prove the theorem it is sufficient to show that $\psi(x)$ is $\alpha\beta$-limit of $f_t(x)$ and $\psi(x)$ is in α. But to prove that $\psi(x)$ is $\alpha\beta$-limit of $f_t(x)$. We have to show that

(i) $\int_0^\infty |g(x)\psi(x)|dx < \infty$, for $g(x)$ is any function of x and is in β and $\psi(x)$ is in α.
(ii) $\lim_{t\to\infty} \int_E f_t\,(x)\,g(x)dx \ = \int_E \psi\,(x)\,g(x)dx$, for $f_t(x)$ is a section of $\psi\,(x)$ in α and $g(x) \in \beta$.

To show (i) we proceed as follows:
since, $\int_0^\infty |g(x)\psi(x)|dx$ for $g(x)$ is in β, $\varepsilon > 0, t \ge T(\varepsilon)$

$$\begin{aligned} &= \int_E |g\,(x)|\,|\psi\,(x) - f_t\,(x) + f_t\,(x)\,|dx \\ &\le \int_E |g\,(x)|\,|\psi\,(x) - f_t\,(x)\,|dx + \int_E |g(x) f_t\,(x)\,|dx \\ &\le \varepsilon k(g) + \int_E |g(x) f_t\,(x)\,|dx, \end{aligned}$$

from (3.12) and (3.13), where $k(g)$ is independent of t running through 0 to ∞.
But $\int_E |g(x)f_t\,(x)\,|dx < \infty$. Hence $\int_E |g(x)\psi(x)|dx < \infty$. Also,

$$|\int_E f_t\,(x)\,g(x)dx - \int_E \psi\,(x)\,g(x)dx| \le \int_0^\infty |g\,(x)|\,|f_t\,(x) - \psi\,(x)\,|dx \le\ \varepsilon k(g),$$

where $k(g)$ depends on g but not on t. Hence $\lim_{t\to\infty} \int_E f_t\,(x)\,g(x)dx \ = \int_E \psi\,(x)\,g(x)dx$.
Thus the conditions (i) and (ii) hold good. Hence the function space α is convergence closed under $\alpha\beta$-convergence (or projective convergence)Thus the theorem is established . □

Theorem 3.4. *Let $\alpha \supset \phi$ then L_1^* is convergence closed under $L_1^* L_1^{**}$convergence, provided L_1 is perfect.*

Proof. Let $\psi(x) \in L_1^*$. Also let $f_t\,(x) = \begin{cases} \psi\,(x)\,,\ \text{for } 0 \le x\ \ \le t \\ 0\ ,\ \text{for } x{>}t \end{cases}$ where t is in $E = [0, \infty)$.
Then $f_t\,(x)$ is a section of $\psi(x)$. Thus $f_t\,(x) = \psi(x) \in L_1^*$, that is $f_t\,(x) = \psi(x)$ is in the dual space of the space of the integrable functions. Hence, $f_t\,(x) = \psi(x)$ will be bounded function of x. Therefore, $f_t(x)$ is bounded in $[0, t]$, as well as $f_t(x)$ is a section of $\psi(x)$. Hence, $f_t(x) = 0$ for every $x > t$, thus $f_t(x) \in \phi \subseteq \alpha$ also $f_t(x) = \psi(x) \in \alpha$ so it is shown that

$$\psi(x) \in \alpha. \tag{3.15}$$

Let the parametric limit of $f_t(x)$ be $\psi(x)$. Then for every given $\varepsilon > 0, \exists$ a positive number $T(\varepsilon)$ such that $\forall x \ge 0$,

$$f_t(x) - \psi(x) \le \varepsilon,\ \ \forall t \le T(\varepsilon). \tag{3.16}$$

Now for proving the theorem it remains to show that

(i) the projective limit of $f_t(x)$ is $\psi(x)$,

(ii) $\psi(x) \in \alpha$.

In order to prove (i) we observe that

$$\int_E |g(x)\psi(x)|dx < \infty \tag{3.17}$$

for $g(x)$ be any function in L_1^{**} and $\psi(x)$ is in α since

$$\begin{aligned}
&\int_E |g(x)\psi(x)|dx \\
&= \int_E |g(x)|\,|\psi(x) - f_t(x) + f_t(x)|dx \\
&\leq \int_E |g(x)|\,|\psi(x) - f_t(x)|dx + \int_E |g(x) f_t(x)|dx \\
&\leq \varepsilon k(g) + \int_E |g(x) f_t(x)|dx,
\end{aligned}$$

from (3.15) and the assumption that $g(x)$ is in L_1^{**} and L_1 is perfect.
Also $\int_E |g(x) f_t(x)|dx < \infty$ for $g(x) \in L_1$ and $f_t(x) \in L_1^*$ thus $\int_E |g(x)\psi(x)|dx < \infty$.
Also $|\int_E f_t(x)\, g(x)dx - \int_E \psi(x)\, g(x)dx| \leq \int_0^\infty |g(x)|\,|f_t(x) - \psi(x)|dx \leq \varepsilon k(g(x))$.
Thus $\lim_{t\to\infty} \int_E f_t(x)\, g(x)dx = \int_E \psi(x)\, g(x)dx$.
Thus $\psi(x)$ is the projective limit of $f_t(x)$. Also for (ii) we refer to (3.14). Hence the theorem is established. □

4 Conclusion

Thus the results through which the t-limits are the $\alpha\beta$-limits, which is not necessary for β to be normal, as the same is essential for the case of the sequence spaces for which fact we refer to previous research works (?) hence we find that in function spaces. By proving certain results it is shown that some of the existing function spaces are convergence closed. The important role played by the dual space of a function space in deducing these three results of section 3 is also visible.

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