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Fixed point results in complete S_b -metric spaces using contractive mappings *

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Abstract In this paper we prove the existence and uniqueness of fixed points for mappings satisfying contractive conditions on the complete S_b-metric spaces and show that these mappings are S_b-continuous at such fixed points.

Key words Fixed point theorems, complete S_b-metric spaces, contractive mappings.

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1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem and is generalized in various directions, (see [1–9]). Bakhtin [1] and Czerwik [3,4] introduced b-metric spaces and proved the contraction principle in this framework. Many authors have earlier obtained fixed point results for single-valued functions, in the setting of b-metric spaces, e.g. see [3,4]. Mustafa and Sims [9] introduced the notion of G-metric spaces.

Definition 1.1. Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following conditions:

1. $G(x, y, z) = 0$ if $x = y = z$,
2. $0 < G(x, x, y)$, for all x, y in X and $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ and $z \neq y$,
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all the three variables),
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$. (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Sedghi et al. [5] introduced the concept of S-metric space by modifying G-metric space. The definition of S-metric space is as follows:

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Definition 1.2. Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

1. $S(x, y, z) \geq 0$,
2. $S(x, y, z) = 0$ if and only if $x = y = z$,
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the pair (X, S) is called an S-metric space.

Lemma 1.3. *([5]) In an S-metric space , we have*

$$S(x, x, y) = S(y, y, x) \ \ for\ all\ x, y \in X\ .$$

Sedghi and Dung [6] remarked that every S-metric space is topologically equivalent to a metric space. Souayah and Mlaiki [2] introduced the concept of S_b-metric space as follows:

Definition 1.4. ([2]) Let X be a nonempty set. A function $S_b : X^3 \to [0, \infty)$ is said to be an S_b–metric if and only if for all $x, y, z, t \in X$, the following conditions hold:

S1 $S_b(x, y, z) = 0$ if and only if $x = y = z$,

S2 $S_b(x, x, y) = S_b(y, y, x)$ for all $x, y \in X$,

S3 $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$, where, $s \geq 1$ be a given number .

The pair (X, S_b) is then called an S_b-metric space. See also ([7, Definition 1.7]). For $s = 1$, the space S_b becomes an S-metric space.

Proposition 1.5. *([7]) Let (X, S_b) be an S_b–metric space.*

1. *A sequence $\{x_n\}$ in X converges to x if and only if $S_b(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S_b(x_n, x_n, x) < \varepsilon$. It is denoted by* $\lim_{n\to\infty} x_n = x$.
2. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that* $S_b(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
3. *The S_b-metric space (X, S_b) is said to be complete if every Cauchy sequence is convergent.*

Proposition 1.6. *([7]). Let $(X, S_b), (X', S_b')$ be S_b-metric spaces, and let $f : X \to X'$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X, $S_b(x_n, x_n, a) \to 0$ implies that $S_b(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous in X if and only if it is continuous at all $a \in X$.*

Mustafa [8] proved the following propositions for the existence of fixed points in G-metric space:

Proposition 1.7. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping which satisfies the following condition for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq k \max \left\{ \begin{array}{c} G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), \\ G(z, Tz, Tz), G(x, Ty, Ty), G(y, Tz, Tz), \\ G(z, Tx, Tx) \end{array} \right\},$$

where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point (say, u) and T is G- continuous at u.

Proposition 1.8. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping which satisfies the following condition for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq k\ max \left\{ \begin{array}{c} [G(x, Ty, Ty) + G(y, Tx, Tx)], \\ [G(y, Tz, Tz) + G(z, Ty, Ty)], \\ [G(x, Tz, Tz) + G(z, Tx, Tx)] \end{array} \right\},$$

where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point (say u) and T is G-continuous at u.

2 Main results

In this section, we prove Proposition 1.7 and Proposition 1.8 for S_b–metric space.

Theorem 2.1. *Let* (X, S_b) *be a complete* S_b*-metric space and let* $T : X \to X$ *be a mapping which satisfies the following condition for all* $x, y, z \in X$,

$$S_b(Tx, Ty, Tz) \le k \ max \left\{ \begin{array}{c} S_b(x, y, z), S_b(Tx, Tx, x), S_b(Ty, Ty, y), \\ S_b(Tz, Tz, z), S_b(Ty, Ty, x), S_b(Tz, Tz, y), \\ S_b(Tx, Tx, z) \end{array} \right\}, \tag{2.1}$$

where $k \in [0, \frac{1}{2})$*. Then* T *has a unique fixed point (say,* u*) and* T *is* S_b*- continuous at* u*.*

Proof. Suppose that T satisfies condition (2.1). Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n(x_0)$,

$$x_1 = T^1(x_0) = T(x_0),$$
$$x_2 = T^2(x_0) = T\{T(x_0)\} = T(x_1),$$
$$\vdots$$
$$\vdots$$
$$x_n = T^n(x_0) = T(x_{n-1}),$$

then by (2.1), we have

$$S_b(x_n, x_n, x_{n+1}) \le k \ \max \left\{ \begin{array}{c} S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, x_{n-1}), \\ S_b(x_{n+1}, x_{n+1}, x_n), S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_{n-1}), \\ S_b(x_n, x_n, x_n) \end{array} \right\}$$

By using (**S2**), we have

$$S_b(x_n, x_n, x_{n+1}) \le k \ \max \left\{ \begin{array}{c} S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1}), \\ S_b(x_{n-1}, x_{n-1}, x_{n+1}) \end{array} \right\} \tag{2.2}$$

Now, if $S_b(x_n, x_n, x_{n+1}) \le kS_b(x_n, x_n, x_{n+1})$, then $k \ge 1$, which is contradiction as $k < \frac{1}{2}$. So, (2.2) becomes

$$S_b(x_n, x_n, x_{n+1}) \le k \ \max\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_{n-1}, x_{n-1}, x_{n+1}) \tag{2.3}$$

But by (**S3**), we have

$$S_b(x_{n-1}, x_{n-1}, x_{n+1}) \le s\{2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_{n+1}, x_{n+1}, x_n)\}$$

and by (**S2**), we have

$$S_b(x_{n-1}, x_{n-1}, x_{n+1}) \le s\{2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})\} \tag{2.4}$$

So, (2.3) becomes

$$S_b(x_n, x_n, x_{n+1}) \le k \ \max \left\{ \begin{array}{c} S_b(x_{n-1}, x_{n-1}, x_n), \\ s\{S_b(x_n, x_n, x_{n+1}) + 2S_b(x_{n-1}, x_{n-1}, x_n)\} \end{array} \right\} \tag{2.5}$$

Hence, it must be the case that

$$S_b(x_n, x_n, x_{n+1}) \le ks[S_b(x_n, x_n, x_{n+1}) + 2S_b(x_{n-1}, x_{n-1}, x_n)] \tag{2.6}$$

which implies

$$S_b(x_n, x_n, x_{n+1}) \le \frac{2ks}{1 - ks}\{S_b(x_{n-1}, x_{n-1}, x_n)\}$$

Let $q = \frac{2ks}{1-ks}$, then $q < 1$

$$S_b(x_n, x_n, x_{n+1}) \le qS_b(x_{n-1}, x_{n-1}, x_n) \tag{2.7}$$

and by repeated application of (2.7), we have

$$S_b\left(x_n, x_n, x_{n+1}\right) \leq q^n S_b\left(x_0, x_0, x_1\right). \tag{2.8}$$

Then for all $n, m \in N, n < m$, we have by repeated use of (**S3**) and (2.8) that

$$\begin{aligned}
S_b\left(x_n, x_n, x_m\right) &\leq s[2S_b\left(x_n, x_n, x_{n+1}\right) + S_b\left(x_m, x_m, x_{n+1}\right)] \\
&= s[2S_b\left(x_n, x_n, x_{n+1}\right) + S_b\left(x_{n+1}, x_{n+1}, x_m\right)] \\
&\leq s[2S_b\left(x_n, x_n, x_{n+1}\right) + s\{2S_b\left(x_{n+1}, x_{n+1}, x_{n+2}\right) + S_b\left(x_m, x_m, x_{n+2}\right)\}] \\
&= s\left[2S_b\left(x_n, x_n, x_{n+1}\right) + 2sS_b\left(x_{n+1}, x_{n+1}, x_{n+2}\right) + sS_b\left(x_{n+2}, x_{n+2}, x_m\right)\right] \\
&\leq s\left[2S_b\left(x_n, x_n, x_{n+1}\right) + 2sS_b\left(x_{n+1}, x_{n+1}, x_{n+2}\right) + s^2\left\{2S_b\left(x_{n+2}, x_{n+2}, x_{n+3}\right) + S_b\left(x_m, x_m, x_{n+3}\right)\right\}\right] \\
&\leqslant s\left[2S_b\left(x_n, x_n, x_{n+1}\right) + 2sS_b\left(x_{n+1}, x_{n+1}, x_{n+2}\right) + 2s^2 S_b\left(x_{n+2}, x_{n+2}, x_{n+3}\right)\right. \\
&\quad \left. + \ldots + 2s^{m-n-1} S_b\left(x_{m-1}, x_{m-1}, x_m\right)\right] \\
&\leq 2s[q^n + sq^{n+1} + s^2 q^{n+2} + \cdots + s^{m-n-1} q^{m-1}] S_b\left(x_0, x_0, x_1\right) \\
&= 2sq^n[1 + sq + (sq)^2 + \cdots + (sq)^{m-1-n}] S_b\left(x_0, x_0, x_1\right) \\
&\leq 2sq^n[1 + sq + (sq)^2 + \ldots \cdots \cdots] S_b\left(x_0, x_0, x_1\right) \\
&= 2sq^n\left(\frac{1}{1-sq}\right) S_b\left(x_0, x_0, x_1\right)
\end{aligned} \tag{2.9}$$

Then $\lim_{n,m\to\infty} S_b\left(x_n, x_n, x_m\right) = 0$, since $\lim_{n,m\to\infty} 2sq^n\left(\frac{1}{1-sq}\right) S_b\left(x_0, x_0, x_1\right) = 0$.
For $n, m, l \in N$ (**S3**) implies that $S_b\left(x_n, x_m, x_l\right) \leq s[S_b\left(x_n, x_n, x_m\right) + S_b\left(x_m, x_m, x_m\right) + S_b\left(x_l, x_l, x_m\right)]$, taking limit as $n, m, l \to \infty$, we get $S_b\left(x_n, x_m, x_l\right) \to 0$. So $\{x_n\}$ is S_b-Cauchy sequence. By completeness of (X, S_b), there exists $u \in X$ such that $\{x_n\}$ is S_b-convergent to u. Suppose that $T(u) \neq u$, then

$$S_b\left(T(u), T(u), x_n\right) \leq k \max\left\{\begin{array}{c} S_b\left(u, u, x_{n-1}\right), S_b\left(Tu, Tu, u\right), S_b\left(Tu, Tu, u\right), \\ S_b\left(x_n, x_n, x_{n-1}\right), S_b\left(Tu, Tu, u\right), S_b\left(x_n, x_n, u\right), \\ S_b\left(Tu, Tu, x_{n-1}\right) \end{array}\right\},$$

$$S_b\left(T(u), T(u), x_n\right) \leq k \max\left\{\begin{array}{c} S_b\left(u, u, x_{n-1}\right), S_b\left(Tu, Tu, u\right), \\ S_b\left(x_n, x_n, x_{n-1}\right), S_b\left(x_n, x_n, u\right), \\ S_b\left(Tu, Tu, x_{n-1}\right) \end{array}\right\}, \tag{2.10}$$

Taking the limit as $n \to \infty$ and using the fact that the function S_b is continuous in its variables, we have $S_b\left(Tu, Tu, u\right) \leq kS_b\left(Tu, Tu, u\right)$, which is a contradiction since $0 \leq k < \frac{1}{2}$. So, $u = Tu$.
To prove the uniqueness, suppose that $v \neq u$ is such that $Tv = v$, then (2.1) implies that

$$S_b\left(v, v, u\right) \leq k \max\left\{S_b\left(v, v, u\right), S_b\left(u, u, v\right)\right\}, \tag{2.11}$$

thus $S_b\left(v, v, u\right) \leq kS_b\left(v, v, u\right)$, which is a contradiction, since, $0 \leq k < \frac{1}{2}$. So, $u = v$.
To see that T is S_b-continuous at u, let $\{y_n\} \subseteq X$ be a sequence such that $\lim\left(y_n\right) = u$, then

$$S_b\left(T(y_n), Tu, T(y_n)\right) \leq k \max\left\{\begin{array}{c} S_b(y_n, u, y_n), S_b(T\left(y_n\right), T\left(y_n\right), y_n), \\ S_b(Tu, Tu, u), S_b(Tu, Tu, y_n), \\ S_b\left(T(y_n), T\left(y_n\right), u\right) \end{array}\right\}. \tag{2.12}$$

And we deduce that

$$S_b\left(T(y_n), u, T(y_n)\right) \leq k \max\left\{\begin{array}{c} S_b(y_n, u, y_n), S_b(T\left(y_n\right), T\left(y_n\right), y_n), \\ S_b\left(u, u, y_n\right), S_b\left(T(y_n), T\left(y_n\right), u\right) \end{array}\right\}. \tag{2.13}$$

But (**S3**) implies that

$$S_b(T\left(y_n\right), T\left(y_n\right), y_n) \leq s[S_b\left(y_n, y_n, u\right) + 2S_b(Ty_n, Ty_n, u)]. \tag{2.14}$$

And (2.13) leads to the following cases,

1. $S_b\left(T(y_n), u, T(y_n)\right) \leq qS_b(y_n, y_n, u)$,
2. $S_b\left(T(y_n), u, T(y_n)\right) \leq kS_b(y_n, u, y_n)$,
3. $S_b\left(T(y_n), u, T(y_n)\right) \leq kS_b(u, u, y_n)$.

In each case we take the limit as $n \to \infty$ to see that $S_b(T(y_n), T(y_n), u) \to 0$ and so, by Proposition 1.5, we have that the sequence$\{T(y_n)\}$ is S_b-convergent to $u = Tu$, therefore, Proposition 1.6 implies that T *is* S_b-continuous at u.

□

Remark 2.2. If the S_b- metric space is bounded (that is, for some $M > 0$ we have $S_b(x, y, z) \leq M$ for all $x, y, z \in X$) then an argument similar to that used above establishes the results for $0 \leq k < 1$.

Corollary 2.3. *Let (X, S_b) be a complete S_b-metric space and let $T : X \to X$ be a mapping which satisfies the following condition for some $m \in N$* and for all $x, y, z \in X$*:*

$$S_b\left(T^m(x), T^m(y), T^m(z)\right) \leq k \max\left\{\begin{array}{c} S_b(x, y, z), S_b\left(T^m(x), T^m(x), x\right), \\ S_b\left(T^m(y), T^m(y), y\right), S_b\left(T^m(z), T^m(z), z\right), \\ S_b\left(T^m(y), T^m(y), x\right), S_b\left(T^m(z), T^m(z), y\right), \\ S_b\left(T^m(x), T^m(x), z\right) \end{array}\right\}, \quad (2.15)$$

where $k \in [0, \frac{1}{2})$, then T has a unique fixed point (say, u), and T^m is S_b-continuous at u.

Proof. From the previous theorem, we have that T^m has a unique fixed point (say, u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 2.4. *Let (X, S_b) be a complete S_b-metric spaces and let $T : X \to X$ be a mapping which satisfies the following condition for all $x, y, z \in X$:*

$$S_b(T(x), T(y), T(z)) \leq k \max\left\{\begin{array}{c} \left[S_b(Ty, Ty, x) + S_b(Tx, Tx, y)\right], \\ \left[S_b(Tz, Tz, y) + S_b(Ty, Ty, z)\right], \\ \left[S_b(Tz, Tz, x) + S_b(Tx, Tx, z)\right], \end{array}\right\}, \quad (2.16)$$

where $k \in [0, \frac{1}{2})$, then T has a unique fixed point (say, u), and T is S_b-continuous at u.

Proof. Suppose that T satisfies the condition (2.16), let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n(x_0)$, then by (2.16) we get

$$S_b(x_n, x_n, x_{n+1}) \leq k \max\left\{\begin{array}{c} \left[S_b(x_n, x_n, x_{n-1}) + S_b(x_n, x_n, x_{n-1})\right], \\ \left[S_b(x_{n+1}, x_{n+1}, x_{n-1}) + S_b(x_n, x_n, x_n)\right], \\ \left[S_b(x_{n+1}, x_{n+1}, x_{n-1}) + S_b(x_n, x_n, x_n)\right] \end{array}\right\}, \quad (2.17)$$

$$= k \max\left\{2S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_{n-1})\right\}.$$

By using (**S2**), we have

$$S_b(x_n, x_n, x_{n+1}) \leq k \max\left\{2S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_{n-1}, x_{n-1}, x_{n+1})\right\}. \quad (2.18)$$

By (**S3**), we have

$$S_b(x_{n-1}, x_{n-1}, x_{n+1}) \leq s\left\{2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_{n+1}, x_{n+1}, x_n)\right\},$$

and from (**S2**), we have

$$S_b(x_{n-1}, x_{n-1}, x_{n+1}) \leq s\left\{2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})\right\}. \quad (2.19)$$

Then (2.18) becomes

$$S_b(x_n, x_n, x_{n+1}) \leq k \max\left\{\begin{array}{c} 2S_b(x_{n-1}, x_{n-1}, x_n), \\ s[2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})] \end{array}\right\}.$$

So, it must be the case that

$$S_b(x_n, x_n, x_{n+1}) \leq ks\{2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})\},$$

which implies

$$S_b(x_n, x_n, x_{n+1}) \leq \frac{2ks}{1-ks}\{S_b(x_{n-1}, x_{n-1}, x_n)\}.$$

Let $q = \frac{2ks}{1-ks}$, then $q < 1$ and

$$S_b(x_n, x_n, x_{n+1}) \leq q\{S_b(x_{n-1}, x_{n-1}, x_n)\}. \tag{2.20}$$

By the repeated application of (2.20), we have

$$S_b(x_n, x_n, x_{n+1}) \leq q^n S_b(x_0, x_0, x_1). \tag{2.21}$$

Then for all $n, m \in \mathbb{N}, n < m$, we have by repeated use of (**S2**),(**S3**) and (2.21) that

$$\begin{aligned}
S_b(x_n, x_n, x_m) &\leq s[2S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})] \\
&= s[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m)] \\
&\leq s[2S_b(x_n, x_n, x_{n+1}) + s\{2S_b(x_{n+1}, x_{n+1}, x_{n+2}) + S_b(x_m, x_m, x_{n+2})\}] \\
&= s[2S_b(x_n, x_n, x_{n+1}) + 2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + sS_b(x_{n+2}, x_{n+2}, x_m)] \\
&\leqslant s[2S_b(x_n, x_n, x_{n+1}) + 2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad + s^2\{2S_b(x_{n+2}, x_{n+2}, x_{n+3}) + S_b(x_m, x_m, x_{n+3})\}] \\
&\leqslant s\left[2S_b(x_n, x_n, x_{n+1}) + 2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + 2s^2S_b(x_{n+2}, x_{n+2}, x_{n+3})\right. \\
&\quad \left. + \ldots + 2s^{m-n-1}S_b(x_{m-1}, x_{m-1}, x_m)\right] \\
&\leq 2s[q^n + sq^{n+1} + s^2q^{n+2} + \cdots + s^{m-n-1}q^{m-1}]S_b(x_0, x_0, x_1) \\
&= 2sq^n[1 + sq + (sq)^2 + \cdots + (sq)^{m-1-n}]S_b(x_0, x_0, x_1) \\
&\leq 2sq^n[1 + sq + (sq)^2 + \ldots \cdots]S_b(x_0, x_0, x_1) \\
&= 2sq^n\left(\frac{1}{1-sq}\right)S_b(x_0, x_0, x_1)
\end{aligned}$$

Then $\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = 0$, since, $\lim_{n,m\to\infty} 2sq^n\left(\frac{1}{1-sq}\right)S_b(x_0, x_0, x_1) = 0$ and $\{x_n\}$ is an S_b-Cauchy sequence. By the completeness of (X, S_b), there exists $u \in X$ such that $\{x_n\}$ is S_b-convergent to u. Suppose that $T(u) \neq u$, then

$$S_b(Tu, Tu, x_n) \leq k \max\left\{\begin{array}{c} [S_b(Tu, Tu, u) + S_b(Tu, Tu, u)], \\ [S_b(x_n, x_n, u) + S_b(Tu, Tu, x_{n-1})], \\ [S_b(x_n, x_n, u) + S_b(Tu, Tu, x_{n-1})] \end{array}\right\}. \tag{2.22}$$

Taking the limit as $n \to \infty$, and using the fact that the function S_b is continuous in its variables, we get

$$S_b(Tu, Tu, u) \leq k \max\{2S_b(Tu, Tu, u), S_b(Tu, Tu, u)\}. \tag{2.23}$$

Since, $0 \leq k < \frac{1}{2}$, this contradiction implies that $u = Tu$.
To prove the uniqueness, suppose that $v \neq u$ such that $Tv = v$, then

$$S_b(v, v, u) \leq k \max\left\{\begin{array}{c} [S_b(v, v, v) + S_b(v, v, v)], \\ [S_b(u, u, v) + S_b(v, v, u)], \\ [S_b(u, u, v) + S_b(v, v, u)] \end{array}\right\}. \tag{2.24}$$

So we deduce that $S_b(v, v, u) \leq k[2S_b(v, v, u)]$, which is a contradiction since $0 \leq k < \frac{1}{2}$. So, $u = v$.
To show that T is S_b-continuous at u, let $\{y_n\} \subseteq X$ be a sequence such that $\lim\{y_n\} = u$ in (X, S_b), then

$$S_b(Tu, Tu, Ty_n) \leq k \max\left\{\begin{array}{c} [S_b(Tu, Tu, u) + S_b(Tu, Tu, u)], \\ [S_b(Ty_n, Ty_n, u) + S_b(Tu, Tu, y_n)], \\ [S_b(Ty_n, Ty_n, u) + S_b(Tu, Tu, y_n)] \end{array}\right\}. \tag{2.25}$$

Thus, (2.25) becomes

$$S_b(u,u,Ty_n) \leq k\left[S_b(Ty_n,Ty_n,u)+S_b(u,u,y_n)\right] \tag{2.26}$$

But by (**S3**) we have , $S_b(Ty_n,Ty_n,u) \leq 2sS_b(Ty_n,Ty_n,u)$, therefore, (2.26) implies that

$$S_b(Ty_n,Ty_n,u) \leq kS_b(u,u,y_n)+2ksS_b(Ty_n,Ty_n,u)$$

and we deduce that

$$S_b(Ty_n,Ty_n,u) \leq \frac{k}{1-2ks}S_b(u,u,y_n). \tag{2.27}$$

Taking the limit of (2.27) as $n \to \infty$, we see that $S_b(Ty_n,Ty_n,u) \to 0$ and so, by Proposition 1.6, we have $Ty_n \to u = Tu$ which implies that T is S_b-continuous at u.

□

Corollary 2.5. *Let (X,S_b) be a complete S_b-metric space and let $T: X \to X$ be mapping which satisfies the following condition for some $m \in \mathbb{N}$ and for all $x,y,z \in X$:*

$$S_b(T^m(x),T^m(y),T^m(z)) \leq k \max\left\{\begin{array}{l}\left[S_b(T^my,T^my,x)+S_b(T^mx,T^mx,y)\right],\\ \left[S_b(T^mz,T^mz,y)+S_b(T^my,T^my,z)\right],\\ \left[S_b(T^mz,T^mz,x)+S_b(T^mx,T^mx,z)\right],\end{array}\right\}, \tag{2.28}$$

where, $k \in [0, \frac{1}{2})$, then T has a unique fixed point (say, u), and T^m is S_b-continuous at u.

Proof. The proof follows from the previous theorem and the same argument as is used in the proof of the Corollary 2.3. □

References

[1] Bakhtin, I.A. (1989). The contraction mapping principle in quasi metrics spaces, *Functional Analysis*, 30, 26–37.

[2] Souayah, N. and Mlaiki, N. (2016). A fixed point theorem in S_b-metric spaces, *J. Math. Computer Sci.*, 16, 131–139.

[3] Czerwik, S. (1993). Contraction mapping in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, 1, 5–11.

[4] Czerwik, S. (1998). Nonlinear set-valued contraction mappings in b-metric spaces., *Atti Semin. Mat. Fis. Univ. Modena*, 46, 263–276.

[5] Sedghi, S., Shobe, N. and Aliouche, A. (2012). A generalization of fixed point theorems in S-metric spaces, *Matematički Vesnik*, 64(30, 258–266.

[6] Sedghi, S. and Dung, Nguyen Van (2014). Fixed point theorems on S-metric spaces, *Matematički Vesnik*, 66(1), 113–124.

[7] Sedghi, S., Gholidahneh, A., Došenović, T., Esfahani, J. and Radenović, S. (2016). Common fixed point of four maps in S_b- metric spaces, *J. Linear Topol. Algebra*, 5(2), 93–104.

[8] Mustafa, Z. and Sims, B. (2009). Fixed point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory and Applications*, Volume 2009, Article ID 917175.

[9] Mustafa, Z. and Sims, B. (2006). A new approach to generalized metric space, *J. Nonlinear Convex Anal.*, 7(2), 289–297.