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An update on the Upadhyaya transform *

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Abstract It is almost two years now when the Upadhyaya transform was introduced by the first author (Upadhyaya, Lalit Mohan, Introducing the Upadhyaya integral transform, Bull. Pure Appl. Sci. Sect. E Math. Stat., 38(E)(1), 471–510, doi 10.5958/2320-3226.2019.00051.1 https://www.researchgate.net/publication/334033797)) as the most powerful, versatile and robust generalization and unification of a number of variants of the classical Laplace transform which have appeared in the mathematics research literature during the years 1993 to 2019. In this paper we provide an update on the Upadhyaya transform, where we explain the definition the one-dimensional Upadhyaya transform and its n-dimensional generalization in more detail and we show that how many other various variants of the classical Laplace transform, that have come to our notice since then and most of which are introduced into the mathematics research literature during the past two years by a number of authors after the advent of the Upadhyaya transform, follow as a special case of the Upadhyaya transform. We find the Upadhyaya transform of some trigonometric and hyperbolic functions, the sine integral, the generalized hypergeometric function and the Bessel function of the first kind in order to exemplify the vast power of the Upadhyaya transform and we also correct a minor typo in the aforementioned paper of the first author.

Key words Upadhyaya transform, n-dimensional Upadhyaya transform, degenerate Upadhyaya transform, Dinesh Verma transform, Raj transform, β-Laplace transform,

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ARA transform, generalized Laplace transform, Jafari transform, Abaoub-Shkheam transform, AF-transform (Pourreza transform), α-integral Laplace transform, triple Aboodh transform, modified Sumudu transform, degenerate Sumudu transform, degenerate Elzaki transform.

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1 Introduction

The utility of the methods of operational calculus in addressing the vast multitude of problems arising in mathematical physics, applied mathematics, engineering, social sciences, economics, life sciences and many other fields remains undisputable. This has led to a vast interest in the study of integral transforms as an indispensable tool for exploring the solutions of many problems that are met in the study of a number of theoretical and experimental phenomena of engineering, applied sciences, economics, social sciences, etc., just to mention a few. Many of the problems arising in these fields, when formulated mathematically, result in the formation of either ordinary or partial differential equations, or, sometimes, also in the form of fractional differential equations, integrodifferential equations, or, also as integral equations and the like. It is sometimes required to solve these equations analytically to arrive at their exact solutions, or, else it may be desirable to solve them by numerical methods. When the existing mathematical tools do not suffice for the purpose of achieving the solution of a problem at hand, the researchers from the other disciplines of study either themselves embark upon the strategy to develop new mathematical methods and tools which may solve the problems that they are stuck up with, or, the mathematicians also try to develop the new tools which may provide solutions to the existing problems of research in the applied sciences and other interdisciplinary disciplines of study, or, the new tools developed by the mathematicians may have a potential for future applications. With this aim many researchers have contributed significantly to the development of the various variants of the classical Laplace type integral transforms, mostly during the past twenty seven years, especially from the year 1993 onwards, when Watugala [2] introduced his integral transform by the name Sumudu transform. Many variants of the classical Laplace transform today exist in the mathematical research literature, almost all of them, which appeared in the mathematics research literature during the years 1993 till the month of April, 2019 are described exhaustively as the special cases of the Upadhyaya transform [1] and its various generalizations (possibly with the exception of a few of them by whose previous existence in the literature the first author may not have been aware of till that time). It may be pointed out here that many mathematicians have introduced these numerous variants of the classical Laplace transform in the literature and these variants of the classical Laplace transforms have been enormously employed either by these mathematicians themselves, or, by other researchers to an extremely wide diversity of phenomenon to solve a number of problems arising in the vast fields of study. Many variants of the classical Laplace transform introduced during the past twenty eight years (1993–2021) have some specialities in them on account of their being applicable to some peculiar types of phenomenon, where they prove to be more elegant in providing the solutions to the problems than their classical counterpart, the celebrated Laplace transform. Perhaps this may be the motivating factor for a number of researchers for introducing such a large variety of the variants of the classical Laplace transform into the mathematics research literature during the past almost three decades. The first author, after being introduced to the extant wide varieties of the classical Laplace transform decided to put forth the most powerful generalization and unification of all the extant variants of the classical Laplace transform, by the name Upadhyaya transform (UT, in brief) [1]. One very obvious and simple thing about all the extant variants of the classical Laplace transforms, including the Upadhyaya transform (UT) itself, is that all these variants are ultimately reducible into and also very intimately related to their great grandfather - the classical Laplace transform. This fact is evident from the so called "duality relations" which are abundant in the literature and provide a pathway of converting one variant of the classical Laplace transform into its another variant, or, into the classical Laplace transform itself and we mention here that the Upadhyaya transform itself is no exception to this universal rule (see, Upadhyaya [1, Theorem 5.2, p. 482]), although this transform can boast of itself

as the hitherto known most powerful generalization and unification of all the extant variants of the classical Laplace transforms existing till date in the mathematics literature!
In this paper we add more details and explanations to the definitions of the one-dimensional Upadhyaya transform and its n-dimensional generalization as given in Upadhyaya [1]. The scheme of the paper is as follows: in section 2 of this paper we recall the definitions of the one-dimensional Upadhyaya transform and the n-dimensional Upadhyaya transform from the introductory work [1] and provide more mathematical explanatory details to these definitions which the first author could not provide in his introductory work [1] owing to some unavoidable and highly unpleasant circumstances prevailing then in his professional and institutional life. We now propose to fill up the gaps in those definitions, which the first author inadvertently forgot to explicitly mention at that time owing to the reasons just mentioned. Without modifying the definitions of the one-dimensional Upadhyaya transform and its n-dimensional generalization as proposed in the introductory work [1], we only propose here to augment the definitions of these two concepts in Section 2 of this paper. In section 3 we mention a number of variants of the classical Laplace transform most of which have appeared in the mathematics literature during the past two years after the advent of the Upadhyaya transform besides some other variants of the classical Laplace transform which existed in the literature prior to the introduction of the Upadhyaya transform and show how all these integral transforms follow as a *special or particular case* of the Upadhyaya transform, or, how some of them can be extracted from the Upadhyaya transform. This attempt of ours, we hope, would further highlight the extremely broad nature of generalization and unification that underlies the Upadhyaya transform. In section 4 we find the Upadhyaya transform of some trigonometric and hyperbolic functions and among the special functions we find the Upadhyaya transform of the sine integral, the generalized hypergeometric function and the Bessel function of the first kind with a view only to illustrate the vast potential of this versatile transform much of which remains untapped till date. A minor typo in the introductory work of the first author [1] is mentioned and corrected in section 5 and the conclusions are summarized very briefly in the concluding section 6 of the paper.

2 Upadhyaya transform and its n-dimensional generalization

We recall below the definitions of the one dimensional Upadhyaya transform (UT) and its n-dimensional generalization from the previous work of Upadhyaya [1].

Definition 2.1. The one dimensional Upadhyaya transform: Following Upadhyaya [1, (2.1), p. 473] we define below three complex parameters λ_1, λ_2 and λ_3 as follows:

$$\lambda_1 = \frac{\left(\sum_{i=0}^{r_1} a_i z_1^i\right)^{m_1}}{\left(\sum_{i=0}^{r_2} b_i z_2^i\right)^{m_2}}, \lambda_2 = \frac{\left(\sum_{i=0}^{r_3} c_i z_3^i\right)^{m_3}}{\left(\sum_{i=0}^{r_4} d_i z_4^i\right)^{m_4}}, \lambda_3 = \frac{\left(\sum_{i=0}^{r_5} k_i z_5^i\right)^{m_5}}{\left(\sum_{i=0}^{r_6} l_i z_6^i\right)^{m_6}}, \tag{2.1}$$

where, m_i and r_i denote nonnegative integers $(i = 1, 2, \ldots, 6)$ and a_i(for $i = 0, 1, 2, \ldots, r_1$); b_i(for $i = 0, 1, 2, \ldots, r_2$); c_i(for $i = 0, 1, 2, \ldots, r_3$); d_i(for $i = 0, 1, 2, \ldots, r_4$); k_i(for $i = 0, 1, 2, \ldots, r_5$); l_i(for $i = 0, 1, 2, \ldots, r_6$) are complex constants, and z_1(for $i = 0, 1, 2, \ldots, r_1$); z_2(for $i = 0, 1, 2, \ldots, r_2$);z_3(for $i = 0, 1, 2, \ldots, r_3$);z_4(for $i = 0, 1, 2, \ldots, r_4$);z_5(for $i = 0, 1, 2, \ldots, r_5$); z_6(for $i = 0, 1, 2, \ldots, r_6$) are either independent complex variables or independent complex parameters. In case the situation at hand demands then we can also choose the parameters $\lambda_1, \lambda_2, \lambda_3$, the constants $a_i, b_i, c_i, d_i, k_i, l_i$ and the independent variables (or independent parmeters) $z_1, \ldots, z_6$ to be real numbers also. From (2.1) it is clear that the three complex parameters $\lambda_1, \lambda_2, \lambda_3$ are respectively rational functions of the six independent complex variables (or independent complex parameters) $z_1, \ldots, z_6$, i.e, the complex parameter λ_1 is a rational function of the two independent complex variables or complex parameters z_1 and z_2; the complex parameter λ_2 is a rational function of the two independent complex variables or complex parameters z_3 and z_4 and the complex parameter λ_3 is a rational function of the two independent complex variables or complex parameters z_5 and z_6 .
With a view of a more broader generalization of (2.1) we also propose here that the parameters $\lambda_1, \lambda_2, \lambda_3$ are some arbitrary functions of the pairs of independent complex variables or independent complex pa-

rameters z_1, z_2; z_3, z_4 and z_5, z_6 respectively, i.e.,

$$\lambda_1 = \lambda_1 (z_1, z_2), \lambda_2 = \lambda_2 (z_3, z_4), \lambda_3 = \lambda_3 (z_5, z_6). \tag{2.2}$$

Let a real valued function $F(t)$ belong to the set of functions defined by

$$A = \left\{ F(t) : \exists M, \eta_1 \text{ and/or } \eta_2 > 0, |F(t)| < M e^{|t|/\eta_j}, \text{ if } t \in (-1)^j \times [0, \infty), j = 1, 2 \right\}, \tag{2.3}$$

where, the constant M is finite while η_1, η_2 may not exist simultaneously. The function $F(t)$ is defined such that $F(t) = 0$ for $t = 0$, then the *Upadhyaya transform* (UT) of the function $F(t)$ is defined as

$$\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \int_0^\infty e^{-\lambda_2 t} F(\lambda_3 t)\, dt \tag{2.4}$$

provided this integral converges and the parameters $\lambda_1, \lambda_2, \lambda_3$ are given by (2.1) or (2.2) and $\mathcal{U}$ is called the *Upadhyaya Transform Operator.*

Definition 2.2. The n-dimensional Upadhyaya transform: Following Upadhyaya [1, (6.1), p. 495] we define the complex parameters $\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}$ by the relations

$$\lambda_1^{(j)} = \frac{\left(\sum_{i=0}^{r_1^{(j)}} a_i^{(j)} \left(z_1^{(j)}\right)^i\right)^{m_1^{(j)}}}{\left(\sum_{i=0}^{r_2^{(j)}} b_i^{(j)} \left(z_2^{(j)}\right)^i\right)^{m_2^{(j)}}}, \lambda_2^{(j)} = \frac{\left(\sum_{i=0}^{r_3^{(j)}} c_i^{(j)} \left(z_3^{(j)}\right)^i\right)^{m_3^{(j)}}}{\left(\sum_{i=0}^{r_4^{(j)}} d_i^{(j)} \left(z_4^{(j)}\right)^i\right)^{m_4^{(j)}}}, \lambda_3^{(j)} = \frac{\left(\sum_{i=0}^{r_5^{(j)}} k_i^{(j)} \left(z_5^{(j)}\right)^i\right)^{m_5^{(j)}}}{\left(\sum_{i=0}^{r_6^{(j)}} l_i^{(j)} \left(z_6^{(j)}\right)^i\right)^{m_6^{(j)}}} \tag{2.5}$$

where, $j = 1, \ldots, n$; $m_i^{(j)}$ and $r_i^{(j)}$ are nonnegative integers $(i = 1, 2, \ldots, 6)$ and $a_i^{(j)}, b_i^{(j)}, c_i^{(j)}, d_i^{(j)}, k_i^{(j)}, l_i^{(j)}$ are complex constants; $z_1^{(j)}$(for $i = 0, 1, 2, \ldots, r_1^{(j)}$); $z_2^{(j)}$(for $i = 0, 1, 2, \ldots, r_2^{(j)}$); $z_3^{(j)}$(for $i = 0, 1, 2, \ldots, r_3^{(j)}$); $z_4^{(j)}$(for $i = 0, 1, 2, \ldots, r_4^{(j)}$); $z_5^{(j)}$(for $i = 0, 1, 2, \ldots, r_5^{(j)}$); $z_6^{(j)}$(for $i = 0, 1, 2, \ldots, r_6^{(j)}$) are either independent complex variables or independent complex parameters. When the situation demands, we may also take the parameters $\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}$, the constants $a_i^{(j)}, b_i^{(j)}, c_i^{(j)}, d_i^{(j)}, k_i^{(j)}, l_i^{(j)}$ and the independent variables (or independent parameters) $z_1^{(j)}, \ldots, z_6^{(j)}$ to be real numbers also. We also remark that in (2.5) the $3n$ independent complex parameters $\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}$ are respectively rational functions of the $6n$ independent complex variables (or independent complex parameters) $z_1^{(j)}, \ldots, z_6^{(j)}$, i.e., the complex parameter $\lambda_1^{(1)}$ is a rational function of the two independent complex variables or independent complex parameters $z_1^{(1)}$ and $z_2^{(1)}$, etc. and so on.

With a view to provide a greater degree of generalization of (2.5) we mention here that the $3n$ independent complex parameters $\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}$ for $j = 1, \ldots, n$ are some arbitrary functions of the $6n$ pairs of independent complex variables or independent complex parameters $z_1^{(j)}, z_2^{(j)}$; $z_3^{(j)}, z_4^{(j)}$; and $z_5^{(j)}, z_6^{(j)}$ respectively, i.e.,

$$\lambda_1^{(j)} = \lambda_1^{(j)} \left(z_1^{(j)}, z_2^{(j)}\right), \lambda_2^{(j)} = \lambda_2^{(j)} \left(z_3^{(j)}, z_4^{(j)}\right), \lambda_3^{(j)} = \lambda_3^{(j)} \left(z_5^{(j)}, z_6^{(j)}\right), \text{ for } j = 1, \ldots, n. \tag{2.6}$$

We now consider a function $F(t_1, \ldots, t_n)$ of n-variables $t_1, \ldots, t_n$ such that the function $F(t_1, \ldots, t_n)$ is of exponential order $a\,(a > 0)$ for each of the n-variables $t_1, \ldots, t_n$.

The *n-dimensional Upadhyaya transform* of a function $F(t_1, \ldots, t_n)$ of n-variables $t_1, \ldots, t_n$ which is of exponential order $a\,(a > 0)$ for each of the n-variables $t_1, \ldots, t_n$ is denoted by

$$\mathcal{U}_n \left\{ F(t_1, \ldots, t_n); \lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}; \ldots; \lambda_1^{(n)}, \lambda_2^{(n)}, \lambda_3^{(n)} \right\}$$

and is defined by

$$\begin{aligned} \mathcal{U}_n \left\{ F(t_1, \ldots, t_n); \lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}; \ldots; \lambda_1^{(n)}, \lambda_2^{(n)}, \lambda_3^{(n)} \right\} &= \mathfrak{u}_n \left\{ \lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}; \ldots; \lambda_1^{(n)}, \lambda_2^{(n)}, \lambda_3^{(n)} \right\} \\ &= \prod_{i=1}^n \lambda_1^{(i)} \int_0^\infty \ldots (n) \ldots \int_0^\infty e^{-\left(\sum_{i=1}^n \lambda_2^{(i)} t_i\right)} F\left(\lambda_3^{(1)} t_1, \ldots, \lambda_3^{(n)} t_n\right) dt_1 \ldots dt_n, \end{aligned} \tag{2.7}$$

provided the integral exists, where, the parameters $\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}$ for $j = 1, \ldots, n$ are given by (2.5) or (2.6). The symbol $\int_0^\infty \ldots (n) \ldots \int_0^\infty$ in (2.7) means that the integral sign $\int_0^\infty$ appears n times in this equation corresponding to the n variables of integration $t_1, \ldots, t_n$ and the subscript n attached to the symbol $\mathcal{U}$ in the symbol $\mathcal{U}_n$ depicts the n-dimensional UT of the function $F(t_1, \ldots, t_n)$.

We mention here that the above interpretations provided by us in this paper also apply to the various other generalizations of the one-dimensional Upadhyaya transform like the one-dimensional and the n-dimensional degenerate Upadhyaya transforms, the one-dimensional and the n-dimensional modified degenerate Upadhyaya transforms, the basic one-dimensional and the basic n-dimensional Upadhyaya transforms and the fractional one-dimensional and the fractional n-dimensional Upadhyaya transforms as mentioned and discussed by the first author in [1].

3 Relation between the Upadhyaya transform and some other extant variants of the classical Laplace transform existing in the literature

We now enumerate some variants of the classical Laplace transform with which we are aware of at present and which have appeared in the mathematics research literature during the past two years after the introduction of the Upadhyaya transform by the first author in [1] and we also mention a few variants of the classical Laplace transform which existed in the literature prior to the introduction of the Upadhyaya transform and which were not covered by the first author in his study [1] and show how all these variants of the classical Laplace transform follow as a special case of the Upadhyaya transform, or, how some of them can be extracted from the Upadhyaya transform. This attempt on our part highlights the extremely general character and the very broad power of unification possessed by the Upadhyaya transform.

3.1 Gupta transform

The Gupta transform was introduced by Gupta et al. [3, p. 14] in March 2020, where, in the second section of their paper they define the Gupta transform for a continuous function $g(y)$ on any interval $y \geq 0$ by the relation

$$\dot{R}\{g(y)\} = \frac{1}{q^3} \int_0^\infty e^{-qy} g(y)\, dy = G(q) \tag{3.1}$$

where, q is a real or complex parameter. We can see at once from (2.4) that the Gupta transform is a special case of the Upadhyaya transform if we choose the parameters of the UT as $\lambda_1 = \frac{1}{q^3}, \lambda_2 = q, \lambda_3 = 1$ and the following relation between the UT of a function $F(t)$ and its Gupta transform $G(q)$ is at once self evident:

$$\mathcal{U}\left\{F(t); \frac{1}{q^3}, q, 1\right\} = \mathfrak{u}\left(\frac{1}{q^3}, q, 1\right) = \dot{R}\{F(t)\} = G(q).$$

3.2 Dinesh Verma transform

The Dinesh Verma transform (DVT) was introduced in April 2020 by Verma [4, p. 19], who defined the Dinesh Verma transform (DVT) of a 'well-defined function' $f(t)$ of real numbers $t \geq 0$, denoted by $D\{f(t)\}$ for a real or complex paramete p by the relation

$$D\{f(t)\} = p^5 \int_0^\infty e^{-pt} f(t)\, dt = \bar{f}(p). \tag{3.2}$$

For the choice of the parameters $\lambda_1 = p^5, \lambda_2 = p, \lambda_3 = 1$ in (2.4), the Dinesh Verma transform becomes a special case of the Upadhyaya transform and the following relation between the UT of a function $F(t)$ and its Dinesh Verma transform $D\{F(t)\}$ is plainly obvious:

$$\mathcal{U}\{F(t); p^5, p, 1\} = \mathfrak{u}(p^5, p, 1) = D\{F(t)\} = \bar{F}(p).$$

3.3 AF transform or Pourreza transform

This transform was introduced by Ahmadi et al. [6] in 2019 and is referred to by the name "AF transform" by Ahmadi and Ahmadi [5] and as "Pourreza transform" by Jafari [7]. It is defined for a function $f(t)$ which is piecewise continuous and of exponential order on every finite interval of $[0,\infty)$ by the relation (see, [5, (1.1)] by the relation

$$p(v)=AF\{f(t)\}=v\int_0^\infty e^{-v^2t}f(t)\,dt. \tag{3.3}$$

We can at once observe that the choice of the parameters $\lambda_1=v,\lambda_2=v^2,\lambda_3=1$ in (2.4), the AF transform (or, the Pourreza transform) becomes a special case of the Upadhyaya transform and the relation between the two is:

$$\mathcal{U}\left\{F(t);v,v^2,1\right\}=\mathfrak{u}\left(v,v^2,1\right)=AF\{F(t)\}=p(v).$$

We further remark here that the AF transform (or, the Pourreza transform) of (3.3) is not a new transform. In fact for the transformation $u\to\frac{1}{v}$ in the Tarig transform introduced by T.M. Elzaki and S.M. Elzaki in 2011 [19–21] (see, Upadhyaya [1, (4.14), p. 479] and for the transformation $v\to\frac{1}{v}$ in the Kashuri and Fundo transform (see, Upadhyaya [1, (4.8), p. 477]) introduced by Kashuri and Fundo in 2013 [22] (which itself, in fact, is also not a new transform and is itself identical to the earlier introduced Tarig transform in 2011 of T.M. Elzaki and S.M. Elzaki [19–21]), we obtain the AF transform (or, the Pourreza transform) of (3.3).

3.4 Raj transform

The Raj transform was introduced in 2020 by Jesuraj and Rajkumar [8], who defined it for a function $f(\varsigma)$ for $\varsigma\geq 0$ by the relation

$$Z(f(\varsigma))=\int_0^\infty e^{-\varsigma}f\left(\frac{\varsigma}{s}\right)d\varsigma \tag{3.4}$$

It can be very easily observed that for the choice of the parameters $\lambda_1=1,\lambda_2=1,\lambda_3=\frac{1}{s}$ in (2.4), the Raj transform becomes a special case of the Upadhyaya transform and the simple relation between the two is:

$$\mathcal{U}\left\{F(t);1,1,\frac{1}{s}\right\}=\mathfrak{u}\left(1,1,\frac{1}{s}\right)=Z(F(t)).$$

We also remark here that the result given by Jesuraj and Rajkumar [8] for the Raj transform of 1 *(see, [8, case (A), section III, p. 959]) is in error. It should read correctly as $Z(1)=1$, i.e., the Raj transform of unity w.r.t. the parameter s should be unity and not $\frac{1}{s}$ as incorrectly mentioned by Jesuraj and Rajkumar [8]!*

3.5 β-Laplace transform

The β-Laplace transform was introduced by Gaur and Agarwal [9] in 2020 (the another introduced version of which in a slightly different notation is the modified Laplace transform by Saif et al. in 2020 [11, (2.1), p. 129]). For a function $f(t),t\geq 0$, which is of exponential order and sectionally continuous on $[0,\infty)$, the β-Laplace transform for the parameter $\beta>1$ is defined by (see, Agarwal et al. [10, p. 316]) the relation

$$\mathcal{L}_\beta\{f(t)\}_{(s)}=\int_0^\infty \beta^{-st}f(t)\,dt,\qquad \beta>1. \tag{3.5}$$

We mention that the β-Laplace transform of (3.5) is also a special case of the Upadhyaya transform for the choice of the parameters $\lambda_1=1,\lambda_2=s\ln\beta\,(\text{for }\operatorname{Re}(\beta)>1),\lambda_3=1$ in (2.4) and the following relation between the two transforms easily follows:

$$\mathcal{U}\{F(t);1,s\ln\beta,1\}=\mathfrak{u}(1,s\ln\beta,1)=\mathcal{L}_\beta\{F(t)\}_{(s)}.$$

Here it is worth mentioning that if in the above discussion of this subsection we replace the symbol β with a then we at once see that the β-Laplace transform introduced by Gaur and Agarwal [9] in 2020 becomes the same as the modified Laplace transform introduced by Saif et al. in 2020 [11], thus with the replacement $\beta\to a$ in the above discussion, it follows immediately that the modified Laplace transform of Saif et al. in 2020 [11] also becomes a *special case* of the Upadhyaya transform.

3.6 Generalized Laplace transform G-transform

The generalized Laplace transform G-transform is due to Kim et al. [12], which is defined by them (obviously for a function $f(t)$ of exponential order and which is sectionally continuous in any finite subinterval of $[0,\infty)$) for an integer value of α (as remarked by Kim et al. on p. 258 of [12]) by the relation

$$G\left(f\right)=u^{\alpha}\int_{0}^{\infty}e^{-\frac{t}{u}}f\left(t\right)dt=F\left(u\right). \tag{3.6}$$

It is obvious that the G-transform of (3.6) is also a special case of the Upadhyaya transform for the choice of the parameters $\lambda_1=u^{\alpha},\lambda_2=\frac{1}{u},\lambda_3=1$ in (2.4) and the plain relation between these two transforms is

$$\mathcal{U}\left\{F\left(t\right);u^{\alpha},\frac{1}{u},1\right\}=\mathfrak{u}\left(u^{\alpha},\frac{1}{u},1\right)=G\left(F\left(t\right)\right)=f\left(u\right).$$

It is pertinent to mention here that the above mentioned G-transform of Kim et al. [12] is a particular case of the Sadik transform given in 2018 by Shaikh [13] for the functions $F(t)$ (belonging to the set A defined by (2.3)) according to the relation (see, [13, (1), p. 101])

$$S\left[F\left(t\right);v^{\alpha},\beta\right]=\mathcal{F}\left(v^{\alpha},\beta\right)=\frac{1}{v^{\beta}}\int_{0}^{\infty}e^{-tv^{\alpha}}F\left(t\right)dt \tag{3.7}$$

where, v is a complex variable, α is any nonzero real number and β is any real number. We can at once see that for the transformations $v\to u,\beta\to-\alpha$ and $\alpha\to-1$ in (3.7) (Sadik transform) reduce it to (3.6) (G-transform). *Both the Sadik transform and the G- transform are particular cases of the Upadhyaya transform (see, [1, subsection 4.14, p. 479]).*

3.7 New Laplace type integral transform of Albayrak et al. [14]

A new Laplace type integral transform was announced by Albayrak et al. [14] in June 2020. For a function $f(t)$ (where, $t\geq 0$) which is of exponential order and is sectionally continuous in any finite subinterval of $[0,\infty)$ this new Laplace type integral transform of Albayrak et al. (see, [14, (3)]) is defined by

$$\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}=\int_{0}^{\infty}t^{\alpha-1}e^{-y^{\mu}t^{\mu}}f\left(t\right)dt=F\left(y\right),\mathrm{Re}\left(\alpha\right)>0,\mathrm{Re}\left(\mu\right)>0,\mathrm{Re}\left(y\right)>0 \tag{3.8}$$

For the particular case of (3.8) when $\alpha=\mu$, Albayrak et al. [14] also mention the following integral transform (see, [14, (4)])

$$\mathcal{L}_{\mu}\left\{f\left(t\right);y\right\}=\int_{0}^{\infty}t^{\mu-1}e^{-y^{\mu}t^{\mu}}f\left(t\right)dt=\tilde{F}\left(y\right),\mathrm{Re}\left(\mu\right)>0,\mathrm{Re}\left(y\right)>0. \tag{3.9}$$

We mention that for the choice of the parameters $\lambda_1=\frac{1}{\mu},\lambda_2=y^{\mu},\lambda_3=1$ in (2.4) and by defining the function

$$F_{\alpha,\mu}\left(t\right)=\begin{cases}t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right),t\geqslant 0,\\ 0,\quad\text{otherwise},\end{cases}$$

the new Laplace type integral transform given by Albayrak et al. [14] in (3.8) becomes a particular case of the Upadhyaya transform and the relation between the two can be written as

$$\mathcal{U}\left\{F_{\alpha,\mu}\left(t\right);\frac{1}{\mu},y^{\mu},1\right\}=\mathfrak{u}\left(\frac{1}{\mu},y^{\mu},1\right)=\mathcal{L}_{\alpha,\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}. \tag{3.10}$$

Similarly, for the choice of the parameters $\lambda_1=\frac{1}{\mu},\lambda_2=y^{\mu},\lambda_3=1$ in (2.4) and by defining the function

$$F_{\mu}\left(t\right)=\begin{cases}f\left(t^{\frac{1}{\mu}}\right),t\geqslant 0,\\ 0,\quad\text{otherwise},\end{cases}$$

it follows that the new Laplace type integral transform defined by Albayrak et al. [14] in (3.9) also becomes a special case of the Upadhyaya transform and the following relation between the two is evident

$$\mathcal{U}\left\{F_{\mu}\left(t\right);\frac{1}{\mu},y^{\mu},1\right\}=\mathfrak{u}\left(\frac{1}{\mu},y^{\mu},1\right)=\mathcal{L}_{\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}. \tag{3.11}$$

We also record below the corrected forms of equation nos. (5) and (6) given by Albayrak et al. [14] (see, [14, (5),(6)]) wherein they point out the relations between their $\mathcal{L}_{\alpha,\mu}$-transform and $\mathcal{L}_{\mu}$-transform and the classical Laplace transform denoted by the symbol $\mathcal{L}$:

$$\mathcal{L}_{\alpha,\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}=\frac{1}{\mu}\mathcal{L}\left\{t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\}, \tag{3.12}$$

and

$$\mathcal{L}_{\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}=\frac{1}{\mu}\mathcal{L}\left\{f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\}. \tag{3.13}$$

One can at once deduce by combining (3.10) and (3.12) that

$$\mathcal{U}\left\{t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right);\frac{1}{\mu},y^{\mu},1\right\}=\mathcal{L}_{\alpha,\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}=\frac{1}{\mu}\mathcal{L}\left\{t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\},$$

and the combination of (3.11) and (3.13) yields the relation

$$\mathcal{U}\left\{f\left(t^{\frac{1}{\mu}}\right);\frac{1}{\mu},y^{\mu},1\right\}=\mathcal{L}_{\mu}\left\{f\left(t^{\frac{1}{\mu}}\right);y\right\}=\frac{1}{\mu}\mathcal{L}\left\{f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\}.$$

3.8 ARA transform

The ARA transform was introduced in the year 2020 by Saadeh et al. [15]. For a continuous function $g(t)$ on the interval $(0,\infty)$ the ARA integral transform of order n is defined by the relation (see, Saadeh et al. [15, (3)])

$$\mathcal{G}_n\left[g\left(t\right)\right]\left(s\right)=G\left(n,s\right)=s\int_0^{\infty}t^{n-1}e^{-st}g\left(t\right)dt,\quad s>0. \tag{3.14}$$

First we remark that the ARA transform is *a particular case of the $\mathcal{L}_{\alpha,\mu}$ transform of* (3.8) *for the case $\alpha=n, y=s, \mu=1$ and is an s-multiple of it.* Further, for the choice of the parameters $\lambda_1=s,\lambda_2=s,\lambda_3=1$ in (2.4) and by defining the function

$$F_n\left(t\right)=\begin{cases}t^{n-1}g\left(t\right), t\geqslant 0,\\ 0,\quad \text{otherwise},\end{cases}$$

we observe that the ARA transform of (3.14) becomes a special case of the Upadhyaya transform and the relation between the two is

$$\mathcal{U}\left\{F_n\left(t\right);s,s,1\right\}=\mathfrak{u}\left(s,s,1\right)=\mathcal{G}_n\left[g\left(t\right)\right]\left(s\right)=G\left(n,s.\right)$$

3.9 Jafari transform

The Jafari transform was introduced by Jafari [7] in the month of August 2020. If $f(t)$ be an integrable function defined for $t\geq 0, p(s)\neq 0$ and $q(s)$ are positive real functions, then, the Jafari transform (general integral transform) $\mathcal{T}(s)$ of $f(t)$ is defined by the relation (see, [7, Definition 1])

$$T\left\{f\left(t\right);s\right\}=\mathcal{T}\left(s\right)=p\left(s\right)\int_0^{\infty}e^{-q(s)t}f\left(t\right)dt, \tag{3.15}$$

provided the integral exists for some $q(s)$.
First of all we remark here that *the Jafari transform of* (3.15) *is so far the most general transform introduced after the introduction of the Upadhyaya transform of* (2.4) *in as much as the Jafari transform of* (3.15) *incorporates a number of variants of the classical Laplace transform that exist in the mathematics research literature for special values of the functions $p(s)$ and $q(s)$ as has been depicted by Jafari himself in Table 1 of [7].*
Now we mention that from (2.1) it is clear that the three complex parameters $\lambda_1,\lambda_2,\lambda_3$ are rational functions of the six independent complex variables (or complex parameters) $z_1,\ldots,z_6$ as already mentioned in the Definition 2.1. Alternatively, we can also see from the more general (2.2) that the three complex parameters $\lambda_1,\lambda_2,\lambda_3$ are functions of the six independent complex variables (or complex parameters) $z_1,\ldots,z_6$. If we now take in the present case that all the six independent complex variables $z_1,\ldots,z_6$ are all real variables and choose, as a very special case, that each of the four independent

variables $z_1, \ldots, z_4$ is equal to s, where $s > 0$ is a real variable (i.e., if we take in (2.1) (resp. (2.2)) that $z_1 = z_2 = z_3 = z_4 = s, s > 0$) and then observe that from (2.1) (resp. (2.2)) both the parameters λ_1 and λ_2 are rational (resp. general) functions of the positive real variable s, which we denote respectively by $\lambda_1 = p(s)$ and $\lambda_2 = q(s)$, (where of course $p(s) \neq 0$) and further choose in (2.1) (resp. (2.2)) that $\lambda_3 = 1$ (note that this choice of λ_3 can be made in a number of possible ways, see, for instance, the discussion in Upadhyaya [1, subsections 4.12, 4.13 and 4.14]) then *the Jafari transform of* (3.15) *also becomes a special case of the Upadhyaya transform defined by* (2.4) *and the relation between these two transforms can be easily written as:*

$$\mathcal{U}\{F(t); p(s), q(s), 1\} = \mathfrak{u}(p(s), q(s), 1) = T\{F(t); s\} = \mathcal{T}(s).$$

3.10 α-integral Laplace transform

The α-integral Laplace transform was introduced by Medina et al. [16]. For a function $f(t)$ of exponential order, which is piecewise continuous in any finite subinterval of $(0, \infty)$ the α-integral Laplace transform is defined by (see also, Jafari [7, Table 1])

$$L_\alpha\{f(t)\} = \int_0^\infty e^{-s^{\frac{1}{\alpha}}t} f(t)\, dt, \quad \alpha \in \mathbb{R}_0^+ \tag{3.16}$$

We can at once infer from (3.16) that for the choice of the parameters $\lambda_1 = 1, \lambda_2 = s^{\frac{1}{\alpha}}, \lambda_3 = 1$ in (2.4), the α-integral Laplace transform becomes a special case of the Upadhyaya transform and the relation between the two is

$$\mathcal{U}\left\{F(t); 1, s^{\frac{1}{\alpha}}, 1\right\} = \mathfrak{u}\left(1, s^{\frac{1}{\alpha}}, 1\right) = L_\alpha(F(t)).$$

3.11 Abaoub-Shkheam transform or Q-transform

The Q-transform or Abaoub-Shkheam transform was given in 2020 by Abaoub and Shkheam [17]. For a function $f(t)$ defined for all $t \geq 0$, the Abaoub-Shkheam transform or Q-transform is defined by the relation (see, [17, (1)])

$$T(u, s) = Q[f] = \int_0^\infty e^{-\frac{t}{s}} f(ut)\, dt \tag{3.17}$$

provided the integral exists for some $s \in (-t_1, t_2)$.
It is obvious that the choice of the parameters $\lambda_1 = 1, \lambda_2 = \frac{1}{s}, \lambda_3 = u$ in (2.4), the Abaoub-Shkheam transform (or the Q-transform) of (3.17) becomes a special case of the Upadhyaya transform and the following relation between these two transforms follows immediately

$$\mathcal{U}\left\{F(t); 1, \frac{1}{s}, u\right\} = \mathfrak{u}\left(1, \frac{1}{s}, u\right) = Q(F(t)) = T(u, s).$$

3.12 New integral transform of Jabber and Tawfiq

A new varaint of the classical Laplace transform was introduced in 2018 by Jabber and Tawfiq [18]. For a function $f(t)$ which is of exponential order and is sectionally continuous in any finite subinterval of $(0, \infty)$, the new transform of Jabber and Tawfiq is defined, for a real number u for which the integral is convergent, by the relation (see, [18, 1, p. 152])

$$\bar{f}(u) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt. \tag{3.18}$$

It is obvious that the choice of the parameters $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = \frac{1}{u}$ in (2.4) shows that the the new transform of Jabber and Tawfiq defined by (3.18) becomes a special case of the Upadhyaya transform and the relation between them is

$$\mathcal{U}\left\{F(t); 1, 1, \frac{1}{u}\right\} = \mathfrak{u}\left(1, 1, \frac{1}{u}\right) = \mathbb{T}\{F(t)\} = \bar{F}(u).$$

3.13 New integral transform of Yang [23, 24]

Yang [23] gave a new variant of the classical Laplace transform in 2016, which is defined by the relation (see, Liang et al. [24, Definition 2.1, p. 529])

$$\Omega(\gamma) = N_I[\omega(\lambda)] = \frac{1}{\gamma} \int_0^\infty e^{-\gamma\lambda} \omega(\lambda)\, d\lambda, \quad \gamma > 0. \tag{3.19}$$

It is not difficult to see that for the choice of parameters $\lambda_1 = \frac{1}{\gamma}, \lambda_2 = \gamma, \lambda_3 = 1$ in (2.4) the new transform of Yang given by (3.19) becomes a special case of the Upadhyaya transform and the relation between these two transforms can be expressed as

$$\mathcal{U}\left\{F(t);\frac{1}{\gamma},\gamma,1\right\} = \mathfrak{u}\left(\frac{1}{\gamma},\gamma,1\right) = N_I\left[F(t)\right] = \Omega(\gamma).$$

We also mention here that the new integral transform of Yang [23,24] given by (3.19) *is not a new integral transform, but it is the same as the Aboodh transform introduced by K.S. Aboodh [26] in 2013, which itself (the Aboodh transform) is also a special case of the Upadhyaya transform (see, Upadhyaya [1, (4.7), p. 477]).*

3.14 Triple Aboodh transform

The triple Aboodh transform was introduced by Alfaqeih and Özis [27] in March 2019. For a continuous function$f(x,y,t)$ of three variables x, y, t the triple Aboodh transform is defined by (see, Alfaqeih and Özis [27, Definition 2.1, p. 42])

$$K(p,q,r) = A_x A_y A_t \left(f(x,y,t)\right) = \frac{1}{pqr}\int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} f(x,y,t)\, dxdydt. \tag{3.20}$$

Now we recall below the particular case corresponding to $n = 3$ of (2.7) (keeping in mind (2.5) (resp. (2.6))) which is called the *triple Upadhyaya transform (TUT)* (see, Upadhyaya [1, (6.19), p. 500])

Definition 3.1. The triple Upadhyaya transform (TUT) of a function $F(t_1,t_2,t_3)$ of three variables t_1, t_2, t_3 which is of exponential order $a\,(a > 0)$ for each of the variables t_1, t_2, t_3 is denoted by

$$\mathcal{U}_3\left\{F(t_1,t_2,t_3);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}$$

and is defined by

$$\begin{aligned}
&\mathcal{U}_3\left\{F(t_1,t_2,t_3);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}\\
&\quad = \mathfrak{u}_3\left\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}\\
&\quad = \lambda_1^{(1)}\lambda_1^{(2)}\lambda_1^{(3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\lambda_2^{(1)}t_1+\lambda_2^{(2)}t_2+\lambda_2^{(3)}t_3\right)} F\left(\lambda_3^{(1)}t_1,\lambda_3^{(2)}t_2,\lambda_3^{(3)}t_3\right) dt_1 dt_2 dt_3.
\end{aligned} \tag{3.21}$$

We observe that for the choice of the parameters $\lambda_1^{(1)} = \frac{1}{p}, \lambda_1^{(2)} = \frac{1}{q}, \lambda_1^{(3)} = \frac{1}{r}, \lambda_3^{(1)} = 1 = \lambda_3^{(2)} = \lambda_3^{(3)}$ and $\lambda_2^{(1)} = p, \lambda_2^{(2)} = q, \lambda_2^{(3)} = r$ in (3.21) it reduces to (3.20), thereby showing that the triple Aboodh transform of (3.20) is a special case of the triple Upadhyaya transform of (3.21) and the relation between these two transforms can be written as

$$\begin{aligned}
\mathcal{U}_3\left\{F(t_1,t_2,t_3);\frac{1}{p},p,1;\frac{1}{q},q,1;\frac{1}{r},r,1\right\} &= \mathfrak{u}_3\left\{\frac{1}{p},p,1;\frac{1}{q},q,1;\frac{1}{r},r,1\right\}\\
&= A_{t_1}A_{t_2}A_{t_3}\left\{F(t_1,t_2,t_3);p,q,r\right\} = K(p,q,r).
\end{aligned}$$

Remark 3.2. We remark here that besides the above discussed newly introduced variants of the classical Laplace transforms the modified Sumudu transform introduced by Duran in 2021 [25] and the new Laplace-type integral transform introduced in 2021 by Maitama and Zhao [28] - are also the *particular or special cases* of the one-dimensional Upadhyaya transform defined by (2.4) in conjunction with (2.1), (2.2) and (2.3). The degenerate Sumudu transform introduced by Duran in the month of December 2020 [34] and the recently introduced degenerate Elzaki transform of Kalavathi et al. [35] are both *particular or special cases* of the one-dimensional degenerate Upadhyaya transform [1, (7.4), p. 502]. In a separate communication of ours in future we will show how these newly introduced variants of the classical Laplace transforms are also the *particular or special cases* of the above mentioned Upadhyaya transforms and their relations with these cases of the Upadhyaya transforms will also be discussed by us there.

4 Upadhyaya transform of some trigonometric, hyperbolic and special functions

In this section we evaluate the Upadhyaya transforms of some trigonometric and hyperbolic functions and among the special functions we evaluate the Upadhyaya transform of the sine integral, the generalized hypergeometric function and the Bessel function of the first kind.
We consider the function $F(t) = \frac{\sin at}{t}$. To find the Upadhyaya transform of this function we set that $G(t) = \sin at$, and recall the following theorem from Upadhyaya [1]:

Theorem 4.1. *(Upadhyaya [1, Theorem 5.17, p. 490]). Let* $\mathcal{U}\{G(t); \lambda_1, \lambda_2, \lambda_3\} = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3)$, *if* $\lim_{t\to 0} \frac{G(t)}{t}$ *exists then*

$$\mathcal{U}\left\{\frac{G(t)}{t}; \lambda_1, \lambda_2, \lambda_3\right\} = \frac{1}{\lambda_3}\int_{\lambda_2}^{\infty} \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3)\, d\lambda_2. \tag{4.1}$$

Observing that $\lim_{t\to 0} \frac{G(t)}{t} = \lim_{t\to 0} \frac{\sin at}{t} = a$, which exists for finite values of a and we also have from Upadhyaya [1, (5.17), p. 485] that

$$\mathcal{U}\{\sin at; \lambda_1, \lambda_2, \lambda_3\} = \frac{a\lambda_1\lambda_3}{\lambda_2^2 + a^2\lambda_3^2},\ \operatorname{Re}(\lambda_2 - ia\lambda_3) > 0, \tag{4.2}$$

we can now write with the help of (4.1) and (4.2) that

$$\begin{aligned}\mathcal{U}\left[\frac{\sin at}{t}; \lambda_1, \lambda_2, \lambda_3\right] &= \frac{1}{\lambda_3}\int_{\lambda_2}^{\infty} \mathcal{U}(\sin at; \lambda_1, \lambda_2, \lambda_3)\, d\lambda_2 = \frac{1}{\lambda_3}\int_{\lambda_2}^{\infty} \frac{a\lambda_1\lambda_3}{\lambda_2^2 + a^2\lambda_3^2} d\lambda_2 \\ &= \frac{\lambda_1}{\lambda_3}\left[\tan^{-1}\left(\frac{\lambda_2}{a\lambda_3}\right)\right]_{\lambda_2=\lambda_2}^{\infty} = \frac{\lambda_1}{\lambda_3}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\lambda_2}{a\lambda_3}\right)\right] = \frac{\lambda_1}{\lambda_3}\cot^{-1}\left(\frac{\lambda_2}{a\lambda_3}\right).\end{aligned} \tag{4.3}$$

Noting that in the left hand side of (4.3) $\mathcal{U}\left[\frac{\sin at}{t}; \lambda_1, \lambda_2, \lambda_3\right] = \lambda_1 \int_0^\infty e^{-\lambda_2 t} \frac{\sin a\lambda_3 t}{\lambda_3 t} dt$, (from (2.4)) we can easily deduce that

$$\int_0^\infty e^{-\lambda_2 t} \frac{\sin a\lambda_3 t}{t} dt = \cot^{-1}\left(\frac{\lambda_2}{a\lambda_3}\right),$$

which is equivalent to saying that

$$\mathcal{L}\left\{\frac{\sin a\lambda_3 t}{t}; \lambda_2\right\} = \cot^{-1}\left(\frac{\lambda_2}{a\lambda_3}\right),$$

where, $\mathcal{L}$ is the classical Laplace transform operator. As a special case, if we put $\lambda_2 = 0$ in (4.3) we easily deduce that

$$\mathcal{U}\left[\frac{\sin at}{t}; \lambda_1, 0, \lambda_3\right] = \frac{\pi\lambda_1}{2\lambda_3}.$$

We can also suitably evaluate some definite integrals by using the UT. As an illustration, from (4.2) we get for the case $a = 1$,

$$\mathcal{U}\{\sin t; \lambda_1, \lambda_2, \lambda_3\} = \frac{\lambda_1\lambda_3}{\lambda_2^2 + \lambda_3^2},\ \operatorname{Re}(\lambda_2 - i\lambda_3) > 0,$$

to which a simple application of Theorem 4.2 gives

$$\begin{aligned}&\mathcal{U}[t\sin t; \lambda_1, \lambda_2, \lambda_3] = -\lambda_3 \frac{d}{d\lambda_2}[\mathcal{U}[\sin t; \lambda_1, \lambda_2, \lambda_3]] \\ &= -\lambda_3 \frac{d}{d\lambda_2}\left[\frac{\lambda_1\lambda_3}{(\lambda_2^2 + \lambda_3^2)}\right] = \frac{2\lambda_1\lambda_2\lambda_3^2}{(\lambda_2^2 + \lambda_3^2)^2}.\end{aligned} \tag{4.4}$$

If we write the definition of $\mathcal{U}[t\sin t; \lambda_1, \lambda_2, \lambda_3]$ in the left hand side of (4.4) with the help of (2.4) followed by the cancellation of λ_1 and λ_3 from the numerators on both the sides (as $\lambda_1 \neq 0$ and $\lambda_3 \neq 0$), we find that

$$\int_0^\infty e^{-\lambda_2 t} t \sin \lambda_3 t dt = \frac{2\lambda_2\lambda_3}{(\lambda_2^2 + \lambda_3^2)^2}, \tag{4.5}$$

which can also be rewritten in terms of the classical Laplace transform operator $\mathcal{L}$ as

$$\mathcal{L}\left\{t\sin\lambda_3 t;\lambda_2\right\}=\frac{2\lambda_2\lambda_3}{\left(\lambda_2^2+\lambda_3^2\right)^2}.$$

For finding now the Upadhyaya transform of the function $t^2\cos at$ we recall below the following theorem from Upadhyaya [1]:

Theorem 4.2. *(Upadhyaya [1, Theorem 5.15, p. 490]) For any natural number n, if the UT of any function $F(t)$, i.e., $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$ is differentiable n-times with respect to the parameter λ_2, then we have*

$$\mathcal{U}\left\{t^nF(t);\lambda_1,\lambda_2,\lambda_3\right\}=(-\lambda_3)^n\frac{d^n}{d\lambda_2^n}\left\{\mathcal{U}\left\{F(t);\lambda_1,\lambda_2,\lambda_3\right\}\right\}=(-\lambda_3)^n\frac{d^n}{d\lambda_2^n}\left\{\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)\right\}, \quad (4.6)$$

and the following result from Upadhyaya [1, (5.16), p. 485] giving the UT of $\cos at$

$$\mathcal{U}\left\{\cos at;\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2},\quad \operatorname{Re}(\lambda_2-ia\lambda_3)>0. \quad (4.7)$$

With the help of (4.6) and (4.7) we find that

$$\begin{aligned}\mathcal{U}\left[t^2\cos at;\lambda_1,\lambda_2,\lambda_3\right]&=(-\lambda_3)^2\frac{d^2}{d\lambda_2^2}\left[\mathcal{U}\left(\cos at;\lambda_1,\lambda_2,\lambda_3\right)\right]\\&=\lambda_3^2\frac{d^2}{d\lambda_2^2}\left[\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2}\right]=\frac{2\lambda_1\lambda_2\lambda_3^2\left(\lambda_2^2-3a^2\lambda_3^2\right)}{\left(\lambda_2^2+a^2\lambda_3^2\right)^3}.\end{aligned}$$

Next we proceed to find the UT of the function $\sin\sqrt{t}$ w.r.t. the parameters $\lambda_1,\lambda_2,\lambda_3$, i.e., we aim to evaluate $\mathcal{U}\left[\sin\sqrt{t};\lambda_1,\lambda_2,\lambda_3\right]$. For this we expand the function $\sin\sqrt{t}$ around the origin by its Taylor expansion

$$\sin\sqrt{t}=t^{1/2}-\frac{t^{3/2}}{3!}+\frac{t^{5/2}}{5!}-\frac{t^{7/2}}{7!}+\dots \quad (4.8)$$

Now taking the UT of both sides of (4.8) w.r.t. the parameters $\lambda_1,\lambda_2,\lambda_3$ noting that (see Upadhyaya [1, (5.4), p. 483])

$$\mathcal{U}\left\{t^a;\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\lambda_1\lambda_3^a\Gamma(a+1)}{\lambda_2^{a+1}},\qquad \operatorname{Re}(\lambda_2)>0,\operatorname{Re}(a)>-1, \quad (4.9)$$

we obtain

$$\mathcal{U}\left[\sin\sqrt{t};\lambda_1,\lambda_2,\lambda_3\right]=\frac{\lambda_1\lambda_3^{1/2}\Gamma\left(\frac{3}{2}\right)}{\lambda_2^{3/2}}-\frac{\lambda_1\lambda_3^{3/2}\Gamma\left(\frac{5}{2}\right)}{3!\lambda_2^{5/2}}+\frac{\lambda_1\lambda_3^{5/2}\Gamma\left(\frac{7}{2}\right)}{5!\lambda_2^{7/2}}-\frac{\lambda_1\lambda_3^{7/2}\Gamma\left(\frac{9}{2}\right)}{7!\lambda_2^{9/2}}+\dots,$$

which, after a little simplification yields that

$$\mathcal{U}\left[\sin\sqrt{t};\lambda_1,\lambda_2,\lambda_3\right]=\frac{\sqrt{\pi}\lambda_1\lambda_3^{1/2}}{2\lambda_2^{3/2}}e^{-\frac{\lambda_3}{4\lambda_2}}=\frac{\lambda_1}{2\lambda_2}\sqrt{\frac{\pi\lambda_3}{\lambda_2}}e^{-\frac{\lambda_3}{4\lambda_2}}. \quad (4.10)$$

Proceeding further, we set $F(t)=\sin\sqrt{t}$, then $F'(t)=\frac{\cos\sqrt{t}}{2\sqrt{t}}$. Now recalling the following result from Upadhyaya [1]

Theorem 4.3. *(Upadhyaya [1, Theorem 5.11, p. 488]) If $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$ and if $F(t)$ is continuous for $0\leqslant t\leqslant N$ and is of exponential order for $t>N$, while $F'(t)$ is sectionally continuous for $0\leqslant t\leqslant N$ then*

$$\mathcal{U}\left\{F'(t);\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\lambda_2}{\lambda_3}\mathcal{U}\left\{F(t);\lambda_1,\lambda_2,\lambda_3\right\}-\frac{\lambda_1}{\lambda_3}F(0), \quad (4.11)$$

we can write with the help of (4.11) for the function $F(t) = \sin\sqrt{t}$ (noting that here $F(0) = 0$),

$$\mathcal{U}\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}; \lambda_1, \lambda_2, \lambda_3\right] = \frac{\lambda_2}{\lambda_3}\mathcal{U}\left[\sin\sqrt{t}; \lambda_1, \lambda_2, \lambda_3\right],$$

which on using the linearity property of the UT and (4.10) at once yields that

$$\mathcal{U}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}; \lambda_1, \lambda_2, \lambda_3\right] = \lambda_1\sqrt{\frac{\pi}{\lambda_2\lambda_3}}e^{-\frac{\lambda_3}{4\lambda_2}}.$$

Next we aim to evaluate the UT of the function $e^{iat}\sinh at$ w.r.t. the parameters $\lambda_1, \lambda_2, \lambda_3$. For this purpose the we recall the following result from Upadhyaya [1, (5.20), p. 485]

$$\mathcal{U}\{\sinh at; \lambda_1, \lambda_2, \lambda_3\} = \frac{a\lambda_1\lambda_3}{\lambda_2^2 - a^2\lambda_3^2}, \mathrm{Re}(\lambda_2 \pm a\lambda_3) > 0, \tag{4.12}$$

along with the following First Translation Property of UT (see Upadhyaya [1])

Theorem 4.4. *(Upadhyaya [1, Theorem 5.4, p. 484]) If $\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3)$ then*

$$\mathcal{U}\left\{e^{at}F(t); \lambda_1, \lambda_2, \lambda_3\right\} = \mathcal{U}\{F(t); \lambda_1, \lambda_2 - a\lambda_3, \lambda_3\} = \mathfrak{u}(\lambda_1, \lambda_2 - a\lambda_3, \lambda_3). \tag{4.13}$$

With the help of (4.12) and (4.13) we can immediately write that

$$\mathcal{U}\left[e^{iat}\sinh at; \lambda_1, \lambda_2, \lambda_3\right] = \mathcal{U}\left[\sinh at; \lambda_1, \lambda_2 - ia\lambda_3, \lambda_3\right],$$

which on using the elementary relation $e^{iat} = \cos at + i\sin at$ on the left hand side with the linearity property of UT and (4.12) gives

$$\mathcal{U}[\cos at\sinh at; \lambda_1, \lambda_2, \lambda_3] + i\mathcal{U}[\sin at\sinh at; \lambda_1, \lambda_2, \lambda_3] = \frac{a\lambda_1\lambda_3}{(\lambda_2 - ia\lambda_3)^2 - a^2\lambda_3^2}.$$

An elementary simplification of the above equation and equating the real and imaginary parts on both sides of it yields the following relations

$$\mathcal{U}[\cos at\sinh at; \lambda_1, \lambda_2, \lambda_3] = \frac{a\lambda_1\lambda_3\left(\lambda_2^2 - 2a^2\lambda_3^2\right)}{\lambda_2^4 + 4a^4\lambda_3^4},$$

and

$$\mathcal{U}[\sin at\sinh at; \lambda_1, \lambda_2, \lambda_3] = \frac{2a^2\lambda_1\lambda_2\lambda_3^2}{\lambda_2^4 + 4a^4\lambda_3^4}.$$

In the next step we proceed below to find the Upadhyaya transform of the sine integral, the generalized hypergeometric series and the Bessel function of the first kind.

4.1 Upadhyaya transform of the Sine integral

The Sine integral is defined by (see Spiegel [29, Entry No. 8, Appendix C, p. 255])

$$\mathrm{Si}(t) = \int_0^t \frac{\sin x}{x}dx. \tag{4.14}$$

To find the UT of the Sine integral we will use the result of (4.3) with the help of which we can write here, as a special case of that result when $a = 1$ in it, that

$$\mathcal{U}\left[\frac{\sin t}{t}; \lambda_1, \lambda_2, \lambda_3\right] = \frac{\lambda_1}{\lambda_3}\cot^{-1}\left(\frac{\lambda_2}{\lambda_3}\right).$$

Now we require the following result from Upadhyaya [1]:

Theorem 4.5. *(Upadhyaya [1, Theorem 5.13, p. 489]) Let $\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3)$ then*

$$\mathcal{U}\left\{\int_0^t F(u)\,du; \lambda_1, \lambda_2, \lambda_3\right\} = \frac{\lambda_3}{\lambda_2}\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} = \frac{\lambda_3}{\lambda_2}\mathfrak{u}(\lambda_1, \lambda_2, \lambda_3), \tag{4.15}$$

which for the case $F(u) = \frac{\sin u}{u}$ gives

$$\mathcal{U}\left\{\int_0^t \frac{\sin u}{u} du; \lambda_1, \lambda_2, \lambda_3\right\} = \frac{\lambda_3}{\lambda_2}\mathcal{U}\left\{\frac{\sin t}{t}; \lambda_1, \lambda_2, \lambda_3\right\}$$

which at once yields that

$$\mathcal{U}\{\mathrm{Si}(t); \lambda_1, \lambda_2, \lambda_3\} = \frac{\lambda_1}{\lambda_2}\cot^{-1}\left(\frac{\lambda_2}{\lambda_3}\right) = \frac{\lambda_1}{\lambda_2}\tan^{-1}\left(\frac{\lambda_3}{\lambda_2}\right). \tag{4.16}$$

4.2 Upadhyaya transform of the generalized hypergeometric function

The well known generalized hypergeometric function is defined by the infinite series (see, Slater [30, (2.1.1.1), p. 40 and (2.1.1.2), p. 41])

$${}_AF_B[a_1, \ldots, a_A; b_1, \ldots, b_B; z] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_A)_n}{(b_1)_n \ldots (b_B)_n} \frac{z^n}{n!} \tag{4.17}$$

where,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, (a)_0 = 1 \tag{4.18}$$

is the usual Pochhammer symbol (see, Slater [30, (I.1), (I.2), Appendix I, p. 239]). The series (4.17) is convergent when $A \leq B$. When $A = B + 1$, the series (4.17) is convergent when $|z| < 1$, for $z = 1$ this series converges when $\mathrm{Re}\left(\sum_{\nu=1}^{B} b_\nu - \sum_{\nu=1}^{A} a_\nu\right) > 0$ and when $z = -1$ it converges when $\mathrm{Re}\left(\sum_{\nu=1}^{B} b_\nu - \sum_{\nu=1}^{A} a_\nu\right) > -1$ (see, Slater [30, Section 2.2, p. 45]).
The following result giving the Laplace transform of a generalized hypergeometric function is well known in the theory of hypergeometric functions (see, Erdélyi et al. [31, Entry No. (17), p. 219]

$$\begin{aligned}&\int_0^\infty e^{-st} t^{\nu-1} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; kt]\, dt \\ &= \frac{\Gamma(\nu)}{s^\nu} {}_{r+1}F_q\left[a_1, \ldots, a_r, \nu; b_1, \ldots, b_q; \frac{k}{s}\right], \mathrm{Re}(\nu) > 0.\end{aligned} \tag{4.19}$$

A generalized form of (4.19) is proven by the first author for the case of real symmetric positive definite matrix arguments in his doctoral dissertation (see Upadhyaya [32, Theorem 2.3.2, pp. 27–28]).
We now consider the UT of the function $t^{\nu-1} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; kt]$ w.r.t. the variables $\lambda_1, \lambda_2, \lambda_3$ as below using (2.4)

$$\begin{aligned}&\mathcal{U}\left[t^{\nu-1} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; kt]; \lambda_1, \lambda_2, \lambda_3\right] \\ &= \lambda_1 \int_0^\infty e^{-\lambda_2 t} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; k\lambda_3 t] dt,\end{aligned}$$

which, on using the definition of a generalized hypergeometric function from (4.17) yields

$$\begin{aligned}&\mathcal{U}\left[t^{\nu-1} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; kt]; \lambda_1, \lambda_2, \lambda_3\right] \\ &= \lambda_1 \int_0^\infty e^{-\lambda_2 t}\left[\lambda_3^{\nu-1} t^{\nu-1} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_q)_n} \frac{(k\lambda_3 t)^n}{n!}\right] dt.\end{aligned}$$

The orders of summation and integration in the above expression can be interchanged under the assumption that the concerned series is convergent in the range of integration, which gives us

$$\begin{aligned}&\mathcal{U}\left[t^{\nu-1} {}_rF_q[a_1, \ldots, a_r; b_1, \ldots, b_q; kt]; \lambda_1, \lambda_2, \lambda_3\right] \\ &= \lambda_1 \lambda_3^{\nu-1} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_q)_n} \frac{(k\lambda_3)^n}{n!}\left[\int_0^\infty e^{-\lambda_2 t} t^{\nu+n-1} dt\right].\end{aligned}$$

The integral within the square brackets on the right hand side of the last expression can be evaluated as a gamma function and then using (4.18) and interpreting the resulting series as a generalized hypergeometric function by the aid of (4.17) we obtain the desired result as under

$$\begin{aligned} &\mathcal{U}\left[t^{\nu-1}{}_rF_q\left[a_1,\dots,a_r;b_1,\dots,b_q;kt\right];\lambda_1,\lambda_2,\lambda_3\right] \\ &= \frac{\lambda_1\lambda_3^{\nu-1}\Gamma(\nu)}{\lambda_2^{\nu}}{}_{r+1}F_q\left[a_1,\dots,a_r,\nu;b_1,\dots,b_q;\frac{k\lambda_3}{\lambda_2}\right]. \end{aligned} \tag{4.20}$$

4.3 Upadhyaya transform of the Bessel function

We proceed now to evaluate the Upadhyaya transform of the Bessel function of the first kind of index n which is defined by the series (see Rainville [33, (3), p. 109])

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}. \tag{4.21}$$

We take the UT of both sides of (4.21) w.r.t. the parameters $\lambda_1,\lambda_2,\lambda_3$ and using (4.9) under the condition of interchangeability of the orders of summation and integration yields (as the series defining the Bessel function $J_n(t)$ is absolutely convergent) immediately the following result for the UT of a Bessel function of the first kind of index n

$$\mathcal{U}\left[J_n(t);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1\lambda_3^n}{2^n\lambda_2^{n+1}}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(n+2r+1)}{r!\Gamma(n+r+1)}\left(\frac{\lambda_3}{2\lambda_2}\right)^{2r}. \tag{4.22}$$

Some special cases of (4.22) deserve a special mention. When we set $n=0$ in (4.22), we get the UT of a Bessel function of order zero as

$$\mathcal{U}\left[J_0(t);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1}{\lambda_2}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(2r+1)}{r!\Gamma(r+1)}\left(\frac{\lambda_3}{2\lambda_2}\right)^{2r}. \tag{4.23}$$

If we explicitly write a few starting terms of the series appearing on the right hand side of (4.23) followed by the elementary simplification of these terms then we can rewrite (4.23) as

$$\begin{aligned} \mathcal{U}\left[J_0(t);\lambda_1,\lambda_2,\lambda_3\right] &= \frac{\lambda_1}{\lambda_2}\left[1-\frac{1}{2}\left(\frac{\lambda_3}{\lambda_2}\right)^2+\frac{3}{2\cdot 4}\left(\frac{\lambda_3}{\lambda_2}\right)^4-\frac{5\cdot 3}{2\cdot 4\cdot 6}\left(\frac{\lambda_3}{\lambda_2}\right)^6+\dots\right] \\ &= \frac{\lambda_1}{\lambda_2}\left[1+\left(\frac{\lambda_3}{\lambda_2}\right)^2\right]^{-\frac{1}{2}} = \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}. \end{aligned} \tag{4.24}$$

A simple application of the following Change of Scale Property of the UT (see Upadhyaya [1])

Theorem 4.6. *(Upadhyaya [1, Theorem 5.9, p. 487]) Let* $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$ *then*

$$\mathcal{U}\{F(at);\lambda_1,\lambda_2,\lambda_3\} = \mathcal{U}\left\{F(t);\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right\} = \mathfrak{u}\left(\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right), \tag{4.25}$$

to (4.24) yields that

$$\mathcal{U}\left[J_0(at);\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[J_0(t);\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right] = \frac{\lambda_1}{\sqrt{\lambda_2^2+a^2\lambda_3^2}}. \tag{4.26}$$

Similarly an application of Theorem 4.2 to (4.26) yields that

$$\begin{aligned} \mathcal{U}\left[tJ_0(at);\lambda_1,\lambda_2,\lambda_3\right] &= -\lambda_3\frac{d}{d\lambda_2}\left\{\mathcal{U}\left[J_0(at);\lambda_1,\lambda_2,\lambda_3\right]\right\} \\ &= -\lambda_3\frac{d}{d\lambda_2}\left[\frac{\lambda_1}{\sqrt{\lambda_2^2+a^2\lambda_3^2}}\right] = \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2^2+a^2\lambda_3^2)^{3/2}}. \end{aligned} \tag{4.27}$$

Now we discuss the special case of (4.22) corresponding to $n = 1$. Noting that for a nonnegative integer r, as $\Gamma(2r+2) = (2r+1)!$ and $\Gamma(r+2) = (r+1)!$, we have

$$\begin{aligned}\mathcal{U}[J_1(t);\lambda_1,\lambda_2,\lambda_3] &= \frac{\lambda_1\lambda_3}{2\lambda_2^2}\sum_{r=0}^{\infty}\frac{(-1)^r(2r+1)!}{r!(r+1)!}\left(\frac{\lambda_3}{2\lambda_2}\right)^{2r}\\ &= \frac{\lambda_1\lambda_3}{2\lambda_2^2}\left[1-3\left(\frac{\lambda_3}{2\lambda_2}\right)^2+10\left(\frac{\lambda_3}{2\lambda_2}\right)^4-35\left(\frac{\lambda_3}{2\lambda_2}\right)^6+\ldots\right]\\ &= \frac{\lambda_1}{\lambda_3}\left[1-\left\{1-\frac{1}{2}\left(\frac{\lambda_3}{\lambda_2}\right)^2+\frac{3}{8}\left(\frac{\lambda_3}{\lambda_2}\right)^4-\frac{5}{16}\left(\frac{\lambda_3}{\lambda_2}\right)^6+\frac{35}{128}\left(\frac{\lambda_3}{\lambda_2}\right)^8-\ldots\right\}\right]\\ &= \frac{\lambda_1}{\lambda_3}\left[1-\left\{1+\left(\frac{\lambda_3}{\lambda_2}\right)^2\right\}^{-1/2}\right],\end{aligned}$$

or,

$$\mathcal{U}[J_1(t);\lambda_1,\lambda_2,\lambda_3] = \frac{\lambda_1}{\lambda_3}\left[1-\frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]. \tag{4.28}$$

5 Correction of a minor typo in the introductory paper of the first author (Upadhyaya [1])

In this section we correct a minor typo, that has so far caught our attention, which inadvertently crept in the earlier paper on this topic of the first author (see, Upadhyaya [1]). We refer the reader to the Example 5.16 of Upadhyaya [1, p. 490] in which one of the equations reads as below:

$$c(-\lambda_3)\frac{d}{d\lambda_2}\left\{\frac{\lambda_1}{\lambda_2-2\lambda_3}\right\} = \frac{\lambda_1\lambda_3}{(\lambda_2-2\lambda_3)^2},$$

which should be *correctly read as below:*

$$\mathcal{U}\left\{te^{2t};\lambda_1,\lambda_2,\lambda_3\right\} = (-\lambda_3)\frac{d}{d\lambda_2}\left[\mathcal{U}\left\{e^{2t};\lambda_1,\lambda_2,\lambda_3\right\}\right] = (-\lambda_3)\frac{d}{d\lambda_2}\left\{\frac{\lambda_1}{\lambda_2-2\lambda_3}\right\} = \frac{\lambda_1\lambda_3}{(\lambda_2-2\lambda_3)^2}.$$

6 Conclusion

In this paper we exhibited how a number of variants of the classical Laplace transform that have appeared in the mathematics research literature during the past two years after the introduction of the Upadhyaya transform by the first author in 2019 (Upadhyaya [1]) can be expressed as special cases of the Upadhyaya transform. This attempt of ours underlines the great power of unification and generalization that is possessed by this versatile transform. We also evaluated the Upadhyaya transform of some trigonometric and hyperbolic functions and also found the Upadhyaya transform of the Sine integral, the generalized hypergeometric function and the Bessel's function of the first kind. These examples only highlight the vast potential of the applications of the Upadhyaya transform, which still remains to be fully exploited by the researchers worldwide. We conclude by invoking the world's research academia to further unravel the mysteries of this stupendous transform as outlined by the first author himself in [1].

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Note added in proof We mention below that the the new integral transforms – the "SEE transform" (Sadiq, Emad and Eman transform) introduced in June 2021 by Mansour et al. [36] and the "complex SEE transform" (complex Sadaq-Emad-Emann transform) introduced by Mansour et al. [37], which have come to our (the first author's) notice only during the past few days are also both *particular cases* of the Upadhyaya transform, which we shall show in one of our future communications as mentioned by us in Remark 3.2. Further, we remark here that the results published in the month of June 2021 by the highly learned and our very respected authors Mansour et al. [38] regarding the SEE transforms of the Bessel functions can be seen to be deducible as the *special cases* from our more general results of the Upadhyaya transform of the Bessel functions discussed in subsection 4.3 of this paper. As these three papers [36–38] of these very learned authors were uploaded on the ResearchGate `https://www.researchgate.net` only on 17 June 2021 [36], 07 June 2021 [37] and 25 June 2021 [38] and the first author noticed their presence only just a few days back and a very close resemblance of the topics discussed in these papers and in the current paper of ours necessitated the addition of this note in the final proof stage of this paper of ours. We wish to point out here that the learned readers may notice the vast and robust power of generalization and unification possessed by the Upadhyaya transform for the numerous extant variants of the classical Laplace transform that are present in the mathematics research literature at present. Before closing this note we also observe here that the "Al-Zughair transform" as mentioned by Kuffi et al. in [39] and Mohammed et al. in [40] *can also be extracted from the one dimensional Upadhyaya transform* (2.4) *for particular values of the parameters* λ_1, λ_2 *and* λ_3, which we shall show in a future communication of ours as mentioned by us in Remark 3.2.

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