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Recent Advances on Fixed Point Theorems

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ABSTRACT

This paper mainly focuses on the recent advances in the fixed point theory. Some discussions are presented on the relation of fixed point theorems to applications, and areas are delineated in the future research directions as well.

KEYWORDS: Complete metric space, fixed point, contraction mappings, non-expansive mappings.
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1. THE ORIGIN OF FIXED POINT THEOREM

It's simple to show whether or not an equation has a solution by phrasing it as a fixed point problem. Therefore, we may deduce that finding the fixed points of a function $\xi\mu$ is equivalent to solving the equation $\zeta\mu = 0$, where $\mu = \zeta\mu + \mu$. To put it another way, the goal is to find a function ξ that has the property $\xi\mu = \mu$. To illustrate this fact, take the quadratic equation $\zeta^2 - 9\zeta + 8 = 0$. The roots of this simple quadratic equation are $\zeta = 1$ and $\zeta = 8$. Let us rewrite this easy equation as the following style:

$$\zeta^2 = 9\zeta - 8$$

Assumed that, $\zeta\mu = \frac{\zeta^2+8}{9}$ we get that $\mu = 1$ and $\mu = 8$ are fixed points of ζ. Thus, we cansay that the issue of locating the fixed points of $\zeta\mu$ is similar to the problem of finding thesolution of the equation $\zeta\mu = 0$, where $\xi\mu = \zeta\mu + \mu$. The findings which are concerned withthe existence of fixed points are called fixed point theory.

The challenge of identifying fixed points of a self-map ξ defined on a suitable space $\mathcal{H}$ can be connected to the difficulty of solving a system of equations in general.

2. INTRODUCTION TO FIXED POINT THEOREM

The theory of fixed points is concerned with the criteria that ensure the presence of points μ in a set $\mathcal{H}$ that satisfy an operator equation $\mu = \xi\mu$, where ξ is a transformation defined on a set $\mathcal{H}$. Such a problem's

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solution set might be empty, finite, infinite, countable, or uncountable. Fixed point theory is strongly connected to establishing integral and differential equation solutions [36-40]. Solving a fixed point problem encompasses a wide range of challenges. In general, fixed point theory research includes:
(i): Less severe mapping criteria that ensure the presence of a fixed point are being investigated.
(ii): Investigation of the conditions that assure the uniqueness of a fixed point.
(iii): Increasing the generic space through modifying, enriching, and expanding the structures of defining domains.
(iv): Approximations of fixed points and a study of the structure of the fixed point collection in the mapping under consideration.

The metric fixed point theorem is based on the Banach contraction principle, which was introduced in 1922 and has affected many aspects of nonlinear functional analysis. The method is based on Picard's idea of proving the existence and uniqueness of a solution to the equation $\xi\mu = \mu$ by employing successive approximations. The existence of solutions to differential or integral equations can be established using this approach. Due of its uses in mathematics and other related subjects, it has been generalized in many ways. Extensions of the Banach contraction principle have been obtained by either generalizing the domain of the mappings or extending the contractive condition on the mappings. In the literature, there have been numerous expansions and alterations that have resulted in far-reaching applications.

A brief history of fixed point theory is given below:

2.1. Historical Outline. The metric contraction principles and metric fixed point theory have their origins in the successive approximation approach for demonstrating the existence and uniqueness of differential equation solutions. This technique is related to the names of famous nineteenth-century mathematicians such as Cauchy, Liouville, Lipschitz, Peano, and most notably, Picard. Actually, the iterative procedure used in the proof of the contraction theorem is due to Picard iterates. Brouwer in [16], proved the following theorem in 1912:

Theorem 2.1. (Brouwer). Let $\mathcal{H}$ be a closed ball in $\mathcal{R}^n$. Then, any continuous mapping $\xi: \mathcal{H} \to \mathcal{H}$ has at least one fixed point.
This theory has a long history. The concepts utilized in its proof were known to Poincare as early as 1886. Brouwer in 1909 proved the theory when $n = 3$. And in 1910 Hadamard introduced the first proof for an arbitrary element, also Brouwer introduced another proof in 1912.

This paper has some material overlap with the popular introductions to fixed point theory by Aksoy [2], Goebel [33], Dugundji [27]. This paper will provide you an overview of the metric fixed point theorem. Single-valued or multi-valued mappings can be used to study a fixed point problem; we will focus on the first type in this paper.
The fixed point theorem has three main topics:
(i): Metric fixed point theorem.
(ii): Topological fixed point theorem.
(iii): Discrete fixed point theorem.
Historically, the boundary lines between the three areas were defined by the find of three main theorems:
(i): Banach [13] fixed point theorem.
(ii): Brouwer [16] fixed point theorem.
(iii): Tarski [79] fixed point theorem.
In this paper, we will focus primarily on the first area. The following section discusses the most important fixed point theorems as well as their developments.

2.2. Metric Fixed Point Theorem. Banach's fixed point theorem [13], commonly known as the Banach contraction principle, was published in 1922.
Definition 2.1. [46] Let $(\mathcal{H}, \mathfrak{Q})$ be a metric space and let $\xi: \mathcal{H} \to \mathcal{H}$ be a mapping.
(i): A point $\mu \in \mathcal{H}$ is called a fixed point of ξ if $\mu = \xi\mu$.
(ii): ξ is called a contraction if there exists a fixed constant $\sigma < 1$ such that for all $\mu, \nu \in \mathcal{H}$,

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq \sigma(\mu, \nu). \quad (2.1)$$

Note that [34]:
(i): If (2.1) holds for $\sigma < 1$, then ξ is called contraction mapping.
(ii): If (2.1) holds for $\sigma \leq 1$ and $\mu \neq \nu$, then ξ is called contractive mapping.
(iii): If (2.1) holds for $\sigma = 1$, then ξ is called nonexpansive mapping.

(iv): If (2.1) holds for a fixed $\sigma < +\infty$, then ξ is called Lipschitz continuous mapping.

Remark 2.1. [34, 42] The following obvious consequences apply to the mapping ξ:
Contraction ⇒ contractive ⇒ nonexpansive ⇒ Lipschitz continuous.
Banach's fixed point theorem is the following theorem, it is also known as the Banach contraction principle.

Theorem 2.3.(Banach) [13] Let $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space and $\xi: \mathcal{H} \to \mathcal{H}$ fulfills the condition (2.1). Then there exists a unique fixed point of ξ in $\mathcal{H}$.

Theorem 2.4. (Banach contraction principle) [13] Let $(\mathcal{H}, \mathfrak{Q})$ be a complete metric spaceand let $\xi: \mathcal{H} \to \mathcal{H}$ be a contraction mapping with $\sigma < 1$. Then, ξ has a unique fixed point $\mu_* \in \mathcal{H}$ and for all $\mu \in \mathcal{H}$, we have:

$$\lim_{n\to\infty} \xi^n \mu = \mu. \tag{2.2}$$

Moreover, for all $\mu \in \mathcal{H}$, we have:

$$\mathfrak{Q}(\xi^n \mu, \mu_*) \le \frac{\sigma^n}{1-\sigma}(\xi\mu, \mu_*). \tag{2.3}$$

Remark 2.2. It is significant that should be $\sigma < 1$. It allows us to manage the rate ofconvergence of $\{\mu_n\} = \{\xi^n \mu_0\}$ to the fixed point since $\sigma^n \to 0$ as $n \to \infty$. If we consider ξ tobe a contractive mapping rather than a contraction, we lose that control over the convergence and the fixed point of ξ need not exist.
Edelstein's theorem [28] indicates that if the space is severely constrained, a unique fixed point of a contractive mapping is guaranteed.

Theorem 2.5. (Edelstein) [28]. Suppose $(\mathcal{H}, \mathfrak{Q})$ be a metric space and $\xi: \mathcal{H} \to \mathcal{H}$ be such that for all $\mu, \nu \in \mathcal{H}$ and $\mu \neq \nu$,

$$\mathfrak{Q}(\xi\mu, \xi\nu) < Q(\mu, \nu). \tag{2.4}$$

Then, a subsequence $\{\mu_{n_k}\} \subset \{\mu_n\} \subset \mathcal{H}$ of iteration $\xi^n \mu$ is the unique fixed point of ξ for some $\mu_0 \in \mathcal{H}$.

The fixed point theorem of non-expansive mappings differs significantly from that of contraction mappings, and we will not talk about it in this paper. Next the Banach and Edelstein theorems, authors are left to wonder if there is any another non-contractive type condition that does not need the map to be continuous but still has a unique fixed point. Using an iterative approach for a self-mapping that is neither contraction nor contractive in nature in complete metric space, Kannan in [48] discovered the answer and established a fixed point theorem.

Theorem 2.6. (Kannan) [48] Suppose $\xi: \mathcal{H} \to \mathcal{H}$, where $(\mathcal{H}, \mathfrak{Q})$ a complete metric space is and ξ for all $\mu, \nu \in \mathcal{H}$ satisfies the condition:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \le \gamma[\mathfrak{Q}(\mu, \xi\mu) + \mathfrak{Q}(\nu, \xi\nu)], \tag{2.5}$$

where, $0 < \gamma < \frac{1}{2}$. Then ξ has a unique fixed point in $\mathcal{H}$.

Note that when we compare equations (2.1) and (2.5), we can see that equation (2.1) entails mapping continuity, but the equation (2.5) does not. Sarkhel in [71] used Banach's fixed point theorem to show Kannan's fixed point theorem in a different way. Furthermore, the existence of a fixed point for a rigorous type Kannan contraction requires the assumption of mapping continuity and the compactness requirement on metric space, as proved in the next theorem:

Theorem 2.7. (Delefina) [25] Suppose $(\mathcal{H}, \mathfrak{Q})$ be a compact metric space and suppose $\xi: \mathcal{H} \to \mathcal{H}$ be a continuous mapping that fulfills for all $\mu, \nu \in \mathcal{H}$, the condition:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \le \frac{1}{2}[\mathfrak{Q}(\mu, \xi\mu) + \mathfrak{Q}(\nu, \xi\nu)], \tag{2.6}$$

with $\mu \neq \nu$. Then ξ has a unique fixed point in $\mathcal{H}$.

Chatterjee in [19] introduced the following:

Theorem 2.8. (Chatterjee) [19] Suppose $(\mathcal{H}, \mathfrak{Q})$ be complete metric space and suppose $\xi: \mathcal{H} \to \mathcal{H}$ be a continuous mapping that fulfills for all $\mu, \nu \in \mathcal{H}$, the condition:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \le \tau[\mathfrak{Q}(\nu, \xi\mu) + \mathfrak{Q}(\mu, \xi\nu)], \tag{2.7}$$

$0 < \tau < \frac{1}{2}$, with $\mu \neq \nu$. Then ξ has a unique fixed point in $\mathcal{H}$.

Then many attempts were made for expanded and developed equation (2.1), for e.g. Reich in [58] obtained the next result:

Theorem 2.9. [63]. Suppose $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space and suppose $\xi: \mathcal{H} \to \mathcal{H}$ be a continuous mapping that fulfills for all $\mu, \nu \in \mathcal{H}$, the condition:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq [c_1\mathfrak{Q}(\mu, \xi\mu) + c_2\mathfrak{Q}(\nu, \xi\nu) + c_3\mathfrak{Q}(\mu, \nu)], \quad (2.8)$$

for all $\mu, \nu \in \mathcal{H}$ such that $c_1 + c_2 + c_3 < 1$. Then ξ has a unique fixed point in $\mathcal{H}$.

In the same way, Shukla et al [72] developed the equation (2.8) as follows:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq [c_1\mathfrak{Q}(\mu, \xi\mu) + c_2\mathfrak{Q}(\nu, \xi\nu) + c_3\mathfrak{Q}(\xi\mu, \nu)], \quad (2.9)$$

for all $\mu, \nu \in \mathcal{H}$ such that $c_1 + c_2 + c_3 < 1$.
In [35] Hardy and Rogers introduced a generalization of Reich's fixed point theorem, as in the following theorem:

Theorem 2.10. (Hardy-Rogers) [35] Let $(\mathcal{H}, \mathfrak{Q})$be a metric space and ξ a self-mapping of $\mathcal{H}$. Suppose, $c_1, c_2, c_3, c_4, c_5 \in \mathfrak{R}^+$ and we set, $c_1 + c_2 + c_3 + c_4 + c_5 = e < 1$ such that

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq [c_1\mathfrak{Q}(\mu, \xi\mu) + c_2\mathfrak{Q}(\nu, \xi\nu) + c_3\mathfrak{Q}(\xi\mu, \nu) + c_4\mathfrak{Q}(\xi\nu, \mu) + c_5\mathfrak{Q}(\mu, \nu)],$$

for all $\mu, \nu \in \mathcal{H}$, under the conditions; $\mathcal{H}$ is complete, then ξ has a unique fixed point.
Or,

$$\mathfrak{Q}(\xi\mu, \xi\nu) < [c_1\mathfrak{Q}(\mu, \xi\mu) + c_2\mathfrak{Q}(\nu, \xi\nu) + c_3\mathfrak{Q}(\xi\mu, \nu) + c_4\mathfrak{Q}(\xi\nu, \mu) + c_5\mathfrak{Q}(\mu, \nu)], \quad (2.10)$$

for all$\mu, \nu \in \mathcal{H}$, under the conditions; $\mathcal{H}$ is compact,$\xi$is continuous and $\mu \neq \nu$then ξ has a unique fixedpoint.

There have been many expansions on a milder form of Banach contraction principle. In next section we introduce some of these expansions.

3. SOME OTHER GENERALIZATIONS OF BANACH CONTRACTION PRINCIPLE

The following conclusion by Rakotch [62] is the first extensions in this bearing.

Theorem 3.1. (Rakotch) [62] Let $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space, and let$\xi: \mathcal{H} \to \mathcal{H}$, for all $\mu, \nu \in \mathcal{H}$, satisfies:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq \delta[\mathfrak{Q}(\mu, \nu)]\mathfrak{Q}(\mu, \nu), \quad (3.1)$$

where$\delta: \mathfrak{R}^+ \to [0,1)$ is a decreasing function. Then ξ has a unique fixed point. After that, a different type of Rakotch's theorem in which the function fulfills a simpler requirement that $\delta(q_n) \to 1 \Longrightarrow q_n \to 0,\ q > 0$, has been presented.
Boyd and Wong in [14], established a more generalized finding as follows:

Theorem 3.2. (Boyd and Wong) [14] Suppose $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space and suppose $\xi: \mathcal{H} \to \mathcal{H}$, for all $\mu, \nu \in \mathcal{H}$, satisfies:

$$\mathfrak{Q}(\xi\mu, \xi\nu) \leq \beta[\mathfrak{Q}(\mu, \nu)], \quad (3.2)$$

where$\beta: \boldsymbol{\mathcal{R}}^+ \to [0, \infty)$ is upper semi-continuous function from the right, that is, for any sequence $q_n \to q \geq 0 \Longrightarrow \lim_{n\to\infty} \sup \beta(q_n) \leq \beta(q)$ and satisfies $0 \leq \beta(q) < q$. Then ξ has a unique fixed point.

Matkowski in [53], provides a version of Theorem 3.2 in which the continuity of is substituted with another appropriate assumption.

Theorem 3.3. (Matkowski) [53] Suppose $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space, let $\xi: \mathcal{H} \to \mathcal{H}$ such that for all$\mu, \nu \in \mathcal{H}$ the equation (3.2) is satisfied, where$\beta: \mathcal{R}^+ \to [0, \infty)$ is nondecreasing in a monotonous style, and fulfills $\lim_{n\to\infty} \mu_n(q) = 0$for each $q > 0$. Then ξ has a unique fixedpoint.
Meir and Keeler in [54], expanded Theorem 3.2 in a different way.

Theorem 3.4. (Meir and Keeler) [54] Suppose that $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space and let $\xi: \mathcal{H} \to \mathcal{H}$ for all $\mu, \nu \in \mathcal{H}$ and for all $\epsilon > 0$there exists $\varepsilon > 0$, fulfills the condition:

$$\mathfrak{Q}(\mu, \nu) \leq \epsilon \Rightarrow \mathfrak{Q}(\xi\mu, \xi\nu) \leq \varepsilon. \quad (3.3)$$

Then ξ has a unique fixed point.

An equation (3.3) shows clearly that the mapping ξ is contractive. therefore, ξ has anunique fixed point, if it exists. In addition, the condition (3.3) implies that $\mathfrak{Q}(\mu_n, \mu_{n+1})$ is decreasing. Finally, in a complete metric space$\mathcal{H}$, it is simple to prove that $\{\mu_n\}$is a Cauchy sequence.

One of the most interesting expansions of the Banach contraction principle is the Caristi-Ekeland extension. Caristi [18] and Ekeland [29] both investigated at a new type of mapping to see whether there was a fixed point. They used a lower semi continuous function σ, such that if $\lim_{n\to\infty} \sigma(\mu_n) = \mu_0$ and $\lim_{n\to\infty} \mu_n = \mu_*$then $\sigma(\mu_0) \leq \mu_*$for all $\mu_n \subset \mathcal{H}$.

Theorem 3.5. (Variational Principle of Ekeland) [29]. Suppose that$(\mathcal{H}, \mathfrak{Q})$ be complete metric space and $\sigma: \mathcal{H} \to \mathcal{R}^+$, for all for all $\mu, \nu \in \mathcal{H}$ define:

$$\sigma(\mu) - \sigma(v) \Rightarrow \mu \preccurlyeq v. \tag{3.4}$$

Then,$(\mathcal{H}, \preccurlyeq)$ has a maximal element.

Theorem 3.6. (Caristi Fixed Point Theorem) [18] Suppose that $(\mathcal{H}, \mathfrak{Q})$ be complete metric space and $\sigma: \mathcal{H} \to \mathcal{R}^+$,let$\xi: \mathcal{H} \to \mathcal{H}$ for all $\mu, \nu \in \mathcal{H}$, satisfies:

$$\mathfrak{Q}(\mu, \xi\mu) \leq \sigma(\mu) - \sigma(\xi\mu). \tag{3.5}$$

Then, ξ has a fixed point.

Ciric in [22], obtained the next generalization of the Banach contraction principle.

Theorem 3.7. (Ciric) [22] Suppose $(\mathcal{H}, \mathfrak{Q})$ be a complete metric space and suppose $\xi: \mathcal{H} \to \mathcal{H}$ be a quasi-contraction, such that:

$$\mathfrak{Q}(\xi\mu, \xi v) \leq \sigma \max\{\mathfrak{Q}(\mu, v), \mathfrak{Q}(\xi\mu, \xi v), \mathfrak{Q}(\xi\mu, \mu), \mathfrak{Q}(v, \xi v), \mathfrak{Q}(\mu, \xi v), \mathfrak{Q}(\xi\mu, v)\}, \tag{3.6}$$

for a fixed constant $\sigma < 1$. Then ξ has a unique fixed point.

Rhoades in [59], showed that Theorem 3.7 is also true if we replace equation (3.6) by:

$$\mathfrak{Q}(\xi\mu, \xi v) \leq \sigma \max\left\{\mathfrak{Q}(\mu, v), \mathfrak{Q}(\xi\mu, \xi v), \mathfrak{Q}(\xi\mu, \mu), \mathfrak{Q}(v, \xi v), \frac{1}{2}[\mathfrak{Q}(\mu, \xi v) + \mathfrak{Q}(\xi\mu, v)]\right\}. \tag{3.7}$$

Rhoades [64], Jachymaski [41] andCiric [23] have studied a sundry of such contractive conditions. Presic [60, 61] in 1965 generalized the Banach contraction principle to product spaces and utilized his generalized version to establish convergence of some specific sequences. This interesting result was later generalized and applied by a number of authors. Alber et al, in [3], proposed an extension of the Banach contraction principle in Hilbert spaces, Rhoades in [65] expanded and refined their conclusions to metric spaces. Furthermore, Suzuki in [76], defined the metric completeness of the underlying space and established a new type of extension of the Banach contraction principle. Suzuki's and Kannan's contractions are independent, according to [76], although both kinds define the metric completeness of the underlying spaces. Also, Suzuki, in 2009 [78] proved generalized versions of the result 2.5 in compact metric space.

In the literature of metric fixed point theory, az-contraction is a natural generalization of the Banach contraction principle. The$z$-contraction proposed by Wardowski in [83] is one of the more interesting generalized Banach's contractions:

Definition 3.1. Let $z: \mathcal{R}^+ \to \mathcal{R}$ be a mapping which is satisfying the following conditions:

(i): z is strictly increasing, i.e. for all a1; $a_1, a_2 \in \mathcal{R}^+$such that $z(a_1) < z(a_1)$ whenever $a_1 < a_1$.

(ii): For all sequence $\{a_n\}_{n\in N}$ of positive real numbers $\lim_{n\to\infty} a_n = 0$if and only if $\lim_{n\to\infty} z(a_n) = -\infty$.

(iii): There exists $\sigma < 1$, such that $\lim_{a_1\to+0} {a_1}^{\sigma} z(a_1) = 0$.

Spaces and their properties, functions used to get a fixed point and their clarification, contractive mappings and their generalization, and a lot of examples, we put them in the next section.

4. VARIOUS KINDS OF ABSTRACT SPACES

This review paper explains the existence and uniqueness of the fixed point in various spaces subject within certain conditions and various mappings. In this section, we will present definitions, concepts and examples for these spaces and mappings.

4.1. $\mathscr{b}$-Metric Space. Bakhtin [12] in 1989, defined $\mathscr{b}$-metric spaces as an expansion of metric spaces by establishing a $\mathscr{b}$-metric constant in triangle inequality of metric condition.

Definition 4.1. [12] Let $\mathcal{H}$ be a non-empty set and $\ell > 1$. A function $\mathfrak{Q}: \mathcal{H} \times \mathcal{H} \to [0, \infty)$is called ℓ-metric space if it satisfies the following for all$\mu_1, \mu_2, \mu_3 \in \mathcal{H}$
(i): $\mathfrak{Q}(\mu_1, \mu_2) = 0, \quad \mu_1 = \mu_2$;
(ii): $\mathfrak{Q}(\mu_1, \mu_2) = \mathfrak{Q}(\mu_2, \mu_1)$;
(iii): $\mathfrak{Q}(\mu_1, \mu_3) = \ell[\mathfrak{Q}(\mu_2, \mu_1) + \mathfrak{Q}(\mu_1, \mu_3)]$;
The pair $(\mathcal{H}, \mathfrak{Q})$ is called a ℓ-metric space.

Remark 4.1. [20] The type of ℓ-metric space is greater than the type of metric space, where a ℓ-metric space is a metric space when$\ell = 1$.
Additional research occurred in [46], Kamran in 2017 expanded the ℓ-metric and presented new fundamental ideas including convergence, Cauchy sequence and completeness in the extended ℓ-metric space.

Definition 4.2. [46] Let $\mathcal{H}$ be a non-empty set and $\mathfrak{L}_\ell: \mathcal{H} \times \mathcal{H} \to [1, \infty)$. A function $\mathfrak{L}_\ell: \mathcal{H} \times \mathcal{H} \to [0, \infty)$ is called an extended ℓ-metric space if for all$\mu_1, \mu_2, \mu_3 \in \mathcal{H}$satisfies:
(i): $\mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_1, \mu_2) = 0 \Leftrightarrow \mu_1 = \mu_2$;
(ii): $\mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_1, \mu_2) = \mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_2, \mu_1)$;
(iii): $\mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_1, \mu_3) = \mathfrak{L}_\ell(\mu_1, \mu_3)[\mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_2, \mu_1) + \mathfrak{Q}_{\mathfrak{L}_\ell}(\mu_1, \mu_3)]$;
The pair $(\mathcal{H}, \mathfrak{Q}_{\mathfrak{L}_\ell})$ is called a $\mathfrak{L}_\ell$ -metric space. If $\mathfrak{L}_\ell(\mu_1, \mu_3) = \ell$, for ℓ>1 then we reduce to Definition 4.1.

Definition 4.3. [41] An extended ℓ-metric space $(\mathcal{H}, \mathfrak{L}_\ell)$ is complete if every Cauchy sequence in $\mathcal{H}$ is convergent.

4.2. A Partial Metric Space. In 1994, Matthews in [52] proposed the concept of partial metric spaces by expanding the concept of metric spaces. Matthews presented an unexpected characteristic in this space: a point's self-distance can be non-zero (i.e., the self-distance$\mathcal{P}(\mu_1, \mu_2) = 0$, does not have to hold). Bukatin et al [17] discussed the idea of non-zero self-distance. In reality, Matthews' findings are crucial to the theory of computation. Matthews gave the following definitions and conclusions in partial metric spaces.

Definition 4.4. [47] Let $\mathcal{H}$ be non-empty set and function $\mathcal{P}: \mathcal{H} \times \mathcal{H} \to [1, \infty)$ be a function satisfying the following:
(i): $\mathcal{P}(\mu_1, \mu_1) = \mathcal{P}(\mu_2, \mu_2) = \mathcal{P}(\mu_1, \mu_2)$ if and only if $\mu_1 \neq \mu_2$,
(ii): $(\mu_1, \mu_2) = \mathcal{P}(\mu_2, \mu_1)$,
(iii): $(\mu_1, \mu_2) = \mathcal{P}(\mu_1, \mu_3) + \mathcal{P}(\mu_3, \mu_2) - \mathcal{P}(\mu_3, \mu_3)$.
Then $\mathcal{P}$ is said to be a partial metric on H and a pair $(\mathcal{H}, \mathcal{P})$ is called a partial metric space.
A basic example of a partial metric space is the pair $(\mathcal{H}, \mathcal{P})$ where $\mathcal{H} = [0,1)$ and $\mathcal{P}(\mu_1, \mu_2) = \max\{\mu_1, \mu_2\}$ for all $\mu_1, \mu_2 \in \mathcal{H}$.

4.3. A metric Like-Space. Amini-Harandi [6] in 2012, introduced the generalization notion of partial metric space which is metric-like space. Amini-Harandi showed some fixed point results and presented $\mathfrak{O}$-completeness in such space. Later, Shukla et al. [73], in 2013 have shown common fixed point results and gave 0-$\mathfrak{O}$-completeness in metric-like space which generalized Amini-Harandis theorems.

Definition 4.5. [6] A metric-like space on non-empty set $\mathcal{H}$ is a function $\mathfrak{O}: \mathcal{H} \times \mathcal{H} \to [1, \infty)$. If the following conditions hold:
(i): $\mathfrak{O}(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$,
(ii): $\mathfrak{O}(\mu_1, \mu_2) = \mathfrak{O}(\mu_2, \mu_1)$
(iii): $\mathfrak{O}(\mu_1, \mu_3) = \mathfrak{O}(\mu_1, \mu_2) + \mathfrak{O}(\mu_2, \mu_3)$
Then a pair $(\mathcal{H}, \mathfrak{O})$ is said to be a metric-like space.

4.4. Rectangular Metric Space. In 2000, Branciari [15] introduced the notion of a rectangular metric space where the triangle inequality of a metric space was replaced by another inequality, the so-called rectangular inequality. The author also extended the celebrated Banach contraction mapping principle in the context of rectangular metric spaces. Later, Azam et al .[10] obtained sufficient conditions for the existence of a unique fixed point of Kannan type mapping in the framework of rectangular metric spaces. Subsequently, Azam et al. [11] showed an analog of the Banach contraction principle in the setting of rectangular cone metric spaces. Following this trend, a number of authors focused on rectangular metric spaces and proved the existence and uniqueness of a fixed point for a specific type of mappings.

Definition 4.6. [17] Let $\mathcal{H}$ be a non-empty set. A function $\mathfrak{D}: \mathcal{H} \times \mathcal{H} \to [1, \infty)$ is arectangular metric on $\mathcal{H}$ if, for all $\mu_1, \mu_2 \in \mathcal{H}$ and all distinct points $v_1, v_2 \in \mathcal{H}$ each distinct from μ_1, μ_2, the following terms hold
(i): $\mathfrak{D}(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$
(ii): $\mathfrak{D}(\mu_1, \mu_2) = \mathfrak{D}(\mu_2, \mu_1)$
(iii): $\mathfrak{D}(\mu_1, \mu_2) \leq \mathfrak{D}(\mu_2, \mu_1) + \mathfrak{D}(\mu_2, \mu_1) + \mathfrak{D}(\mu_2, \mu_1)$.
Then $(\mathcal{H}, \mathfrak{D})$ is called a rectangular metric space.

4.5. Rectangular $\mathscr{b}$-Metric Space. In 2015 George et al [32] presented the notion of rectangular $\mathscr{b}$-metric space and which generalizes the concept of metric space, rectangular metricspace, and $\mathscr{b}$metric space. The authors gave an analog of the Banach contraction principleas well as the Kannan type fixed point theorem in rectangular $\mathscr{b}$-metric space.

Definition 4.7. [32, 44] Let $\mathcal{H}$ be a non-empty set and the mapping $\mathfrak{D}: \mathcal{H} \times \mathcal{H} \to [1, \infty)$ satisfies:
(i): $\mathfrak{D}(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$
(ii): $\mathfrak{D}(\mu_1, \mu_2) = \mathfrak{D}(\mu_2, \mu_1)$
(iii) there exists a real number $\mathscr{b} > 1$ such that:
$\mathfrak{D}(\mu_1, \mu_2) \leq \mathscr{b}[\mathfrak{D}(\mu_2, \mu_1) + \mathfrak{D}(\mu_2, \mu_1) + \mathfrak{D}(\mu_2, \mu_1)]$,
for all $\mu_1, \mu_2 \in \mathcal{H}$ and $v_1 \neq v_2 \notin \{\mu_1, \mu_2\}$.
Then $\mathfrak{D}$is called a rectangular $\mathscr{b}$-metric on $\mathcal{H}$ and$(\mathcal{H}, \mathfrak{D})$ is called a rectangular $\mathscr{b}$-metric spacewith coefficient$\mathscr{b}$.

Remark 4.2. [32] Every metric space is a rectangular metric space and every rectangular metric spaces is rectangular $\mathscr{b}$-metric space, with $\mathscr{b} = 1$. However the opposite of implyingabove is not valid.

4.6. $\mathcal{S}$-Metric Space. In 2012, Sedghi et al [69] introduced the idea of $\mathcal{S}$-metric spaces andgave some of their properties. Then, the Authors showed a common fixed point theorem for self-mapping on complete S-metric spaces is given.

Definition 4.8. [57, 70] Let $\mathcal{H}$ be a non-empty set and $\mathcal{S}: \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to [1, \infty)$be a function fulfilling the next conditions for all $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{H}$
(i): $\mathcal{S}(\mu_1, \mu_2, \mu_3) \geq 0$,
(ii): $\mathcal{S}(\mu_1, \mu_2, \mu_3) = 0$ if and only if$\mu_1 = \mu_2 = \mu_3$,
(iii): $\mathcal{S}(\mu_1, \mu_2, \mu_3) \leq \mathcal{S}(\mu_1, \mu_1, \mu_4) + \mathcal{S}(\mu_2, \mu_2, \mu_4) + \mathcal{S}(\mu_3, \mu_3, \mu_4)$.
Then the function $\mathcal{S}$ is called an $\mathcal{S}$-metric on $\mathcal{H}$ and the pair $(\mathcal{H}, \mathcal{S})$ is called an $\mathcal{S}$-metric spaces.

4.7. Real Banach Space. Banach spaces are named after Stefan Banach, a Polish mathematician who, with Hans Hahn and Eduard Helly, invented and researched this notion in1920-1922. The word "Banach space" was invented by Frechet, who in turn coined the term Frechet space. The study of function spaces by Hilbert, Frchet, and Riesz earlier in the century gave rise to Banach spaces. In functional analysis, Banach spaces are crucial. The spaces under examination in various fields of analysis are frequently Banach spaces. The idea of normed linear spaces was around since the 1906, the modern definition was given first by Banach and Norbert independently in 1922.

Definition 4.9. [55, 66] A normed space is a pair $(\|\cdot\|, \mathcal{H})$ on a vector space $\mathcal{H}$ is a map $\|\cdot\|: \mathcal{H} \times \mathcal{H} \to [1, \infty)$ such that for all $\mu, v \in \mathcal{H}$and $c \in \mathfrak{R}$satisfies:
(i): $\|\mu\| > 0, \ \|\mu\| = 0 \to \mu = 0$.
(ii):): $\|c\mu\| = |c|\|\mu\|$
(iii): $\|\mu + v\| \leq \|\mu\| + \|v\|$.

Definition 4.10. [82] A Banach space$\mathcal{B}$ is a complete normed space$(\|\cdot\|, \mathcal{H})$.

Huang et al [51] in 2007, generalized the metric spaces by introducing the notion of cone metric spaces or simply substituting the set of real numbers with a generic Banach space $\mathcal{B}$ which is partially ordered with regard to a cone $\mathscr{p} \to \mathcal{B}$ and showed some fixed point results for contraction maps in normal cone metric space. Also, Liu et al [50] worked on cone metric spaces over Banach Algebras. The concept of a cone metric space includes the metric space, because each metric space is a cone metric space where $\mathcal{B} = \mathfrak{R}\ and\ \mathscr{p} = [0, +1)$. The principle of 2-normed metric spaces or 2-metric space was established by Gahler [30, 31] by generalizing the metric space and proved several fixed point theorems in such space. Singh et al. [74] in 2012, presented cone 2-metric spaces generalizing both 2-metric and cone metric spaces, and proved some fixed point results for self-mappings satisfying certain contractive conditions. For more information on this side see [47, 80, 81]. Dhamodharan et al. [26] in 2017, presented the concept of cone $\mathcal{S}$-metric space and prove some fixed point theorems by several contractive conditions.

5. VARIOUS KINDS OF HELPING FUNCTIONS

On the other hand, Popa [58, 59] examined an implicit function type condition instead of the normal explicit contraction condition. This research direction produced consistent literature on fixed point and common fixed point theorems in different ambient spaces. Saluja [67] in 2020, established some fixed point theorems in the framework of cone $\mathcal{S}$-metric spaces using implicit relation. The auxiliary functions used to find the fixed point are many and varied. We have included some of them in this paper:

5.1. $\mathcal{C}$-function. Ansari in [7] presented the definition of the type $\mathcal{C}$ -function as following:
Definition 5.1. A function$\boldsymbol{\varphi}: \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}$is called $\mathcal{C}$ -function, if
(i): $\boldsymbol{\varphi}(q_1, q_2) \leq q_1$,
(ii): $\boldsymbol{\varphi}(q_1, q_2) = q_1$implicit, that either $q_1 = 0$or $q_2 = 0$,
for all $q_1, q_2 \in \mathcal{R}^+$,

5.2. Altering distance function. Delbosco [24] and Skof [75] have established fixed point theorems for self -mappings of complete metric spaces by altering the distances between the points with the use of a function $\hbar$ where $\hbar$defined as:

Definition 5.2. A non-decreasing continuous function $\hbar \in \vartheta: \mathcal{R}^+ \to \mathcal{R}$is called an alteringdistance mapping if it satisfying the following properties:
(i): $\hbar$ is continuous and strictly increasing in $\mathcal{R}^+$,
(ii): $\hbar(q) = 0 \Leftrightarrow q = 0$,
(iii): $\hbar(q) \leq \mathcal{A}q^{\mathfrak{d}}$for every $q > 0$ where $\mathcal{A}, \mathfrak{d} \in \mathcal{R}^+$,
whereϑ is the class of altering distance functions.
Khan et al [47], established some fixed point theorems in a complete and compact metric space using an altering distance function. After that, many authors used the altering distance function to prove many fixed point results in different spaces. Fixed point problems in metric and probabilistic metric spaces have employed this function and its extension.

5.3. $\boldsymbol{w}$-Distance. In metric spaces, Kada et al [45], presented and studied the idea of $\boldsymbol{w}$-distance. With the use of this notion, they were able to extend a number of significant theorems, including Caristi's fixed point theorem and Ekeland's variational principle. Suzukiand Takahashi [76] almost simultaneously discovered a metric completeness characterization using $\boldsymbol{w}$-distances as an extension of the Banach contraction principle. Many writers have utilized and expanded w-distances since then, primarily in the context of fixed point theory and from various perspectives (see, for example, [4, 49, 56, 83] and their bibliographies).

Definition 5.3. [43] Let $(\mathcal{H}, \mathfrak{D})$be a metric space,a function $\boldsymbol{w}: \mathcal{H} \times \mathcal{H} \to [0,1)$ is called $\boldsymbol{w}$ -distance on $\mathcal{H}$ if it satisfies the following properties:
(i): $\boldsymbol{w}(\mu_1, \mu_3) \leq \boldsymbol{w}(\mu_1, \mu_2) + \boldsymbol{w}(\mu_2, \mu_3)$ for all $\mu_1, \mu_2, \mu_3 \in \mathcal{H}$;
(ii): $\boldsymbol{w}$ is lower semi continuous with respect to the second variable, i.e., if $\mu \in \mathcal{H}$ and $v_n \to v$ then $\boldsymbol{w}(\mu, v) \leq \lim_{n\to\infty} \boldsymbol{inf}\, \boldsymbol{w}(\mu, v_n)$;
(iii): for all$\varepsilon > 0$, there is $\epsilon > 0$such that $\boldsymbol{w}(\mu_1, \mu_2) \leq \epsilon$ and $\boldsymbol{w}(\mu_1, \mu_3) \leq \epsilon \Rightarrow \mathfrak{D}(\mu_2, \mu_3) \leq \varepsilon$.
Some concepts connected with generalized contractive, contraction, non-expansive mappings and generalized admissible mappings will be discussed in next:

6. VARIOUS KINDS OF CONTRACTIVE-COTRACTION-NONEXPANSION MAPPINGS

6.1. $(\boldsymbol{\varphi}, \boldsymbol{\gamma})$-Contractive Mappings. By constructing a novel family of non-decreasing functions and getting certain fixed points results, in 2012 Samet et al. [68] introduced the concept of contractive mappings and admissible mappings.

6.2. $(\boldsymbol{\varphi}, \boldsymbol{\mathfrak{y}})_{(\xi,\zeta)}$-Contraction Mappings. Alqahtani et al. [5], in 2018 introduced the notion of $(\boldsymbol{\varphi}, \boldsymbol{\mathfrak{y}})_{(\zeta)}$-contraction in the setting of extended $\mathscr{b}$-metric spaces and investigate the existence of a fixed point, also the authors in [5] instituted the concept of $(\boldsymbol{\varphi}, \boldsymbol{\eta})_{(\xi)}$-contraction to mitigate the continuity case on the specified self-mappings.

6.3. $(\boldsymbol{\varphi}, \boldsymbol{\ell})$**Non-expansive Mappings**. A new class of non-expansive type mappings introduced by Suzuki [77] in 2008, known as mappings fulfilling the following condition:
$\|\mu_1 - \boldsymbol{\xi}\mu_2\| \leq 2\|\mu_1 - \mu_2\| \Leftrightarrow \|\boldsymbol{\xi}\mu_1 - \boldsymbol{\xi}\mu_2\| \leq \|\mu_1 - \mu_2\|$,(6.1)

for all $\mu_1, \mu_2 \in \mathcal{K}$, where $\mathcal{K}$ is a nonempty subset of a Banach space and $\boldsymbol{\xi}: \mathcal{K} \to \mathcal{K}$, Suzuki showed some significant fixed point theorems for these mappings and established that these mappings don't have to be continuous in contrast to non-expansive mappings .

7. COMMON/COINCIDE FIXED POINT.

One of the most interesting research topics in fixed point theorems is the study of the existence and uniqueness of coinciding points of various operators in the setting of various metric spaces. (see [1, 9, 21]).

Definition 7.1. [8] Let $\boldsymbol{\xi}$and $\boldsymbol{\zeta}$ are self-mappings in $\mathcal{H}$, then:
(i): A point$\mu_0 \in \mathcal{H}$ is said to be a common fixed point of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ if$\boldsymbol{\xi}\mu_0 = \boldsymbol{\zeta}\mu_0 = \mu_0$.
(ii): A point $\mu_0 \in \mathcal{H}$ is called a coincidence point of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ if $\boldsymbol{\xi}\mu = \boldsymbol{\zeta}\mu = \mu_0$
(iii): The mappings $\boldsymbol{\xi}, \boldsymbol{\zeta}: \mathcal{H} \to \mathcal{H}$ are said to be weakly compatible if they commute at their coincidence point that is, $\boldsymbol{\xi}\,\boldsymbol{\zeta}\mu = \boldsymbol{\zeta}\boldsymbol{\xi}\,\mu$ whenever $\boldsymbol{\zeta}\mu = \boldsymbol{\xi}\,\mu$.

8. OPEN PROBLEMS

Researchers face many problems that make the fixed point and its study in constant development, whether in terms of restricting contraction contractive and non-expansive conditions or using auxiliary functions in different spaces. Among these open problems we mention:
(i): When does any non-expansive mapping $\boldsymbol{\xi}$ on $\mathcal{H}$, have at least one fixed point?
(ii): Structure of the fixed points sets.
(iii): Approximation of fixed points.
(iv): Abstract metric theory.

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