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Non-homogeneous binary cubic equation $a(x-y)^3 = 8\,x\,y\,, a > 0$ *

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Abstract A search is made for determining the different solutions in integers to the binary cubic equation $a(x-y)^3 = 8\,x\,y\,, a > 0$ by employing linear transformations. Some special relations for the solutions are obtained. The process for getting second order Ramanujan numbers, sequence of Diophantine triples with suitable property and Dio-3 tuples is also illustrated.

Key words Not-homogeneous third degree, Binary third degree, Solutions in integers, Ramanujan numbers, 3-tuples, Dio-3 tuples.

2020 Mathematics Subject Classification 11D25, 11D99.

Nomenclature of symbols

$P_\alpha^5 = \frac{\alpha^2(\alpha+1)}{2}$, $t_{m,s} = \frac{s\,[2+(s-1)\,(m-2)]}{2}$.

1 Introduction

The third degree Diophantine equations are numerous [1,2] and offer expansion in this subject. In [3–28] a broad collection of different forms of equations is given. An interesting non-homogeneous third degree Diophantine equation with two variables $a(x-y)^3 = 8\,x\,y\,, a > 0$ is studied in this paper for finding its solutions in integers. Some fascinating relations among the solutions are obtained. The process of obtaining second order Ramanujan numbers, sequence of Diophantine triples with suitable property and Dio-3 tuples is illustrated.

1.1 Method of analysis

Consider

$$a\;(x-y)^3 = 8\,x\,y\,, a > 0 \tag{1.1}$$

Taking

$$x = c + k\,d\,, y = c - k\,td, c \neq k\,d \neq 0 \tag{1.2}$$

in (1.1) , leads to

$$c^2 = k^2\,d^2\,(1 + a\,k\;d) \tag{1.3}$$

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which is satisfied by

$$d = a\,k\,s^2 - 2\,s\ ,\ c = k\,(a\,k\,s^2 - 2\,s)\,(a\,k\,s - 1) \tag{1.4}$$

Substituting (1.4) in (1.2), one has

$$\begin{aligned} x &= x(a\,,k\,,s) = a\,k^2\,s\,(a\,k\,s^2 - 2\,s)\,, \\ y &= y(a\,,k\,,s) = k\,(a\,k\,s^2 - 2\,s)\,(a\,k\,s - 2) \end{aligned} \tag{1.5}$$

Observe that (1.5) satisfies (1.1).

Note 1.1. It is to mention that (1.3) is also satisfied by

$$d = a\,k\,s^2 + 2\,s\ ,\ c = k\,(a\,k\,s^2 + 2\,s)\,(a\,k\,s + 1)$$

For this choice , (1.1) is satisfied by

$$\begin{aligned} x &= x(a\,,k\,,s) = k\,(a\,k\,s + 2)\,\,(a\,k\,s^2 + 2\,s)\,, \\ y &= y(a\,,k\,,s) = a\,k^2\,s\,(a\,k\,s^2 + 2) \end{aligned}$$

To obtain various relations among the solutions, one has to choose special values to $k\,,s$. For illustration, a few observations from the solutions (1.5) when $k = 1\,,s = n + 1$ are presented below:
To start with, the solutions to (1.1) when $k = 1\,,s = n + 1$ in (1.5) are found to be

$$\begin{aligned} x &= x(a,n) = a\,(n + 1)^2\,(a\,n + a - 2)\,, \\ y &= y(a,n) = \,(n + 1)\,(a\,n + a - 2)^2. \end{aligned} \tag{1.6}$$

Observations

1 $a\,\,(n + 1)\,[x(a,n) - y(a,n)] = 2\,x(a,n)$,

2 $3\,[a\,(x(a,n) - y(a,n)) + 2]$ is a nasty number,

3 $a^3\,(n + 1)^3[(x(a,n))^3 - (y(a,n))^3] = \,2\,(x(a,n))^2\,[4\,x(a,n) - 3\,a^2\,(n + 1)^2\,y(a,n)]$,

4 $(n + 1)\,[x(a,n) + n + 1]$ is a perfect square,

5 $\begin{aligned} &[x(b,u + v + 1) - y(b,u + v + 1)] - 2\,[x(b,u + v) - y(b,u + v)] \\ &+[x(b,u + v - 1) - y(b,u + v - 1)] = 4\,b, \end{aligned}$

6 $\sum_{n=1}^{N} y(a,n) = a^2\,(t_{3,N})^2 + (3\,a^2 - 4\,a)\,P_N^4 + (3\,a^2 - 8\,a + 4)\,(t_{3,N}) + (a - 2)^2\,N$,

7 $\sum_{n=1}^{N}(x(a,n) - y(a,n)) = 2\,a\,P_N^4 + 4\,(a - 1)\,t_{3,N} + 2\,(a - 2)\,N$,

8 $\sum_{n=1}^{N}(x(a,n) - y(a,n)) = 4\,a\,P_N^5 + (14\,a - 12)\,t_{3,N} + 6\,(a - 2)\,N$.

2 Formulation of the second order Ramanujan numbers

From each of the solutions of (1.1) given by (1.6), one can find Ramanujan numbers of order two having real integers as base numbers.

Illustration 2.1. Consider

$$\begin{aligned} x(a,n) &= a(n + 1)^2\,(\,a\,n + a - 2) \\ &= a\,(n + 1)^2\,\,*\,(\,a\,n + a - 2) = \,(n + 1\,)^2\,\,*\,a\,(a\,n + a - 2) \\ &= \,P * Q = R * S \text{ say} \end{aligned}$$

From the above relation, one may observe that

$$\begin{aligned} &(P + Q)^2 + (R - S)^2 = (P - Q)^2 + (R + S)^2 \,= P^2 + Q^2 + R^2 + S^2 \\ &(a\,n^2 + 3\,a\,n + 2\,a - 2)^2 + (n^2 + (2 - a^2)\,n + 1 - a^2 + 2\,a)^2 \\ &\qquad = (a\,n^2 + \,a\,n + 2\,)^2 + (n^2 + (2 + a^2)\,n + 1 + a^2 - 2\,a)^2 \\ &= (a^2 + 1)\,n^4 + 4\,(a^2 + 1)\,n^3 + (a^4 + 7\,a^2 + 6)\,n^2 + (2\,a^4 - 4\,a^3 + 6\,a^2 - 4\,a + 4)\,n \\ &\qquad +(\,a^4 - 4\,a^3 + 6\,a^2 - 4\,a + 5) \end{aligned}$$

Thus,

$$(a^2+1)\,n^4+4\,(a^2+1)\,n^3+(a^4+7\,a^2+6)\,n^2+(2\,a^4-4\,a^3+6\,a^2-4\,a+4)\,n+(\,a^4-4\,a^3+6\,a^2-4\,a+5)$$

represents a second order Ramanujan number.

3 Formation of sequence of Diophantine triples with the property $D(a^2\,n + 2\,a\,(a-1))$

Consider

$$A = \frac{x(a,n)}{a(n+1)^2} = a\,n + a - 2\,, B = \frac{x(a,n+1)}{a(n+2)^2} = a\,n + 2\,a - 2$$

It is observed that
$A\,B + (a^2\,n + 2\,a\,(a-1)) = [a\,n + 2\,(a-1)]^2$, a perfect square.
Thus, (A, B) is a Diophantine 2-tuple with property$D(a^2 n + 2\,a\,(a-1))$.
If C is the third tuple, then it is satisfied by the simultaneous equations

$$A\,C + a^2 n + 2\,a\,(a-1) = p^2 \tag{3.1}$$

$$B\,C + a^2 n + 2\,a\,(a-1) = q^2 \tag{3.2}$$

Eliminating C between (3.1) and (3.2),, we have

$$(a\,n + 2\,a - 2)\,p^2 - (a\,n + a - 2)\,q^2 = a\,(a^2 n + 2\,a\,(a-1)) \tag{3.3}$$

Taking

$$p = X + (a\,n + a - 2)\,T\,, \tag{3.4}$$

$$q = X + (a\,n + 2\,a - 2)\,T, \tag{3.5}$$

in (3.3) and simplifying , we get

$$X^2 = [a^2 n^2 + (3\,a - 4)\,a\,n\, + 2\,a^2 - 6\,a + 4]\,T^2 + a^2 n + 2a(a-1)$$

which is satisfied by

$$T = 1\,, X = a\,n + 2\,(a-1) \tag{3.6}$$

From (3.4) and (3.1), we get

$$C = 4\,a\,n + 7\,a - 8$$

Note that $(A\,, B\,, C)$ is Diophantine triple with property$D\,(a^2 n + 2\,a\,(a-1))$. The method for forming sequences of Diophantine Triples with property $D\,(a^2 n + 2\,a\,(a-1))$ is given below:
Let M denote the 3×3 matrix

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Then

$$(A\,, B\,, C)\, *\, M = (a\,n + a - 2\,, 4\,a\,n + 7\,a - 8\,, 9\,a\,n + 14\,a - 18)$$

Observe that

$$(a\,n + a - 2)\,(4\,a\,n + 7\,a - 8) + a^2 n + 2\,a\,(a-1) = (2\,a\,n + 3\,a - 4)^2$$
$$(a\,n + a - 2)\,(9\,a\,n + 14\,a - 18) + a^2 n + 2\,a\,(a-1) = (3\,a\,n + 4\,a - 6)^2$$
$$(4\,a\,n + 7a - 8)\,(9\,a\,n + 14\,a - 18) + a^2 n + 2\,a\,(a-1) = (6\,a\,n + 10\,a - 12)^2$$

Thus, the triple $(a\,n + a - 2\,, 4\,a\,n + 7\,a - 8\,, 9\,a\,n + 14\,a - 18)$ represents Diophantine 3-tuple with the property$D(a^2 n + 2\,a\,(a-1))$. Repeating the above process, sequence of Diophantine triples with the property $D(a^2\,n + 2\,a\,(a-1))$ is obtained.

3.1 Dio 3- tuple

Let (A, B) be a Dio 2-tuple with $D(-(n+1)a^2 + a + 1)$.
Let $A = a\,n + a - 2\,, B =\ a\,n + 2\,a - 2$ be two integers such that $AB + A + B + D(-a^2 + a + 1 - a^2 n)$ is a perfect square
Consider an integer C not equal to zero satisfying the relations

$$(an + a - 1)C + an + a - 2 - (n+1)a^2 + a + 1 = p^2 \tag{3.7}$$

$$(an + 2a - 1)C + an + 2a - 2 - (n+1)a^2 + a + 1 = q^2 \tag{3.8}$$

Eliminating C from (3.7) and (3.8), we obtain

$$\begin{aligned}(an+2a-1)p^2-(an+a-1)q^2 &= (an+2a-1)(an+2a-1-(n+1)a^2)\\ &\quad -(an+a-1)(an+3a-1-(n+1)a^2)\end{aligned} \tag{3.9}$$

Using the linear transformations

$$p = X+(an+a-1)T \tag{3.10}$$

$$q = X+(an+2a-1)T \tag{3.11}$$

in (3.9), it leads to the Pell equation

$$X^2=(A+1)(B+1)T^2+a-(n+1)a^2 \tag{3.12}$$

Let $T_0=1$ and $X_0=an+a-1$ be the initial solution of (3.12). Thus (3.10) yields $\alpha_0=2(an+a-1)$. And using (3.7), we get $C=4an+5a-5$

Hence $(A,B,C)=(a\,n+a-2\,,\,a\,n+2\,a-2, 4an+5a-5)$ is the Dio-3 tuple with property $D(-a^2+a+1-a^2n)$.

Let E be an integer not equal to zero satisfying the relations

$$(an+2a-1)E+an+3a-1-(n+1)a^2=p^2, \tag{3.13}$$

$$(4an+5a-4)E+4an+6a-4-(n+1)a^2=q^2. \tag{3.14}$$

Eliminating E from (3.13) and (3.14), we obtain

$$(4an+5a-4)p^2-(an+2a-1)q^2=-3a(an+a-1)^2. \tag{3.15}$$

Using the linear transformations

$$p=X+(an+2a-1)T, \quad q=X+(4an+5a-4)T \tag{3.16}$$

in (3.15), it leads to the Pell equation

$$X^2=(B+1)(C+1)T^2-(a^2n+a^2-a). \tag{3.17}$$

Let $T_0=1$ and $X_0=2an+3a-2$ be the initial solution of (3.17). Thus (3.16) yields $p=3an+5a-3$ and using (3.13), we get $E=9an+13a-10$.

Repeating the above process, sequences of Dio-3 tuple with the property $D(-a^2+a+1-a^2n)$ are obtained.

4 Conclusion

Different methods are illustrated to find solutions in integers to the third degree non-homogeneous Diophantine equation having two variables of the form $a(x-y)^3=8\,x\,y\,,a>0$. The researchers may attempt to find various other methods to find solutions in integers to this problem or attempt other such problems.

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