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On The Diophantine Equation $8^\alpha + 67^\beta = \gamma^2$

[1]Sudhanshu Aggarwal*, [2]Lalit Mohan Upadhyaya

Author's Affiliation:
[1]Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur-273402, Uttar Pradesh, India
E-mail: sudhanshu30187@gmail.com
[2]Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun-248179, Uttarakhand, India
E-mail: lmupadhyaya@rediffmail.com, hetchres@gmail.com

***Corresponding Author: Sudhanshu Aggarwal,** Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur-273402, Uttar Pradesh, India
E-mail: sudhanshu30187@gmail.com

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ABSTRACT

In this paper, authors have examined the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$, where α, β, γ are non-negative integers, for non-negative integer solutions. Authors used Catalan's conjecture for this purpose. Results of the present paper show that the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$, where α, β, γ are non-negative integers, has a unique solution in non-negative integers and this solution is given by $(\alpha, \beta, \gamma) = (1, 0, 3)$.

KEYWORDS: Diophantine Equation; Catalan Conjecture; Solution; Integers.

AMS Subject Classification: 11D61

1. Introduction

Diophantine equations have numerous applications in Physics, Chemistry, Mathematics, Astronomy, Cryptography and Knot theory [1-3]. Many of the puzzles such as Mahavira puzzle, the monkey and coconuts puzzle and hundred fowls puzzle of ancient times depend for their solution on the consideration of Diophantine equations of the first and second degree [4]. Nowadays, scholars examined various Diophantine equations of different kind (Linear, Non-linear, Homogeneous, Non-homogeneous, Exponential). Sroysang [5-7] examined the Diophantine equations $8^x + 19^y = z^2, 31^x + 32^y = z^2$ and $8^x + 13^y = z^2$. The Diophantine equation $193^x + 211^y = z^2$ was studied by Aggarwal [8] and he told that this equation has no non-negative integer solution. Aggarwal and Sharma [9] studied the non-linear Diophantine equation $379^x + 397^y = z^2$. The exponential Diophantine equation $421^p + 439^q = r^2$ was solved by Bhatnagar and Aggarwal [10]. Recently, Gupta et al. [11-12] examined two Diophantine equations $(x^a + 1)^m + (y^b + 1)^n = z^2$ and $x^\alpha + (1 + my)^\beta = z^2$. They showed that these two equations have no solution in non-negative integers.

The main aim of this paper is to determine the non-negative integer solution of the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$, where α, β, γ are non-negative integers.

2. Preliminaries

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Proposition 2.1: Catalan's conjecture [13]: The Diophantine equation ${\omega_1}^{\alpha} - {\omega_2}^{\beta} = 1$, where $\omega_1, \omega_2, \alpha, \beta$ are integers such that $\min\{\omega_1, \omega_2, \alpha, \beta\} > 1$, has a unique solution and it is given by $(\omega_1, \omega_2, \alpha, \beta) = (3, 2, 2, 3)$.

Lemma 2.2: The Diophantine equation $8^{\alpha} + 1 = \gamma^2$, where α, γ are non-negative integers, has a unique solution and this solution is given by $(\alpha, \gamma) = (1, 3)$.

Proof: Suppose α and γ are non-negative integer such that $8^{\alpha} + 1 = \gamma^2$. If $\alpha = 0$, then $\gamma^2 = 8^0 + 1 = 2$ which is not possible due to the nature of γ. Then $\alpha \geq 1$. It follows that $\gamma^2 = 8^{\alpha} + 1 \geq 8^1 + 1 = 9$. Then $\gamma \geq 3$. Now, we consider on the equation $\gamma^2 - 8^{\alpha} = 1$. By Proposition 2.1, we have $\alpha = 1$. We obtain that $\gamma^2 = 9$ and then $\gamma = 3$. Hence the Diophantine equation $8^{\alpha} + 1 = \gamma^2$, where α, γ are non-negative integers, has a unique solution which is given by $(\alpha, \gamma) = (1, 3)$.

Lemma 2.3: The Diophantine equation $67^{\beta} + 1 = \gamma^2$, where β, γ are non-negative integers, has no non-negative integer solution.

Proof: Suppose β and γ are non-negative integer such that $67^{\beta} + 1 = \gamma^2$. If $\beta = 0$, then $\gamma^2 = 67^0 + 1 = 2$ which is not possible due to the nature of γ. Then $\beta \geq 1$. It follows that $\gamma^2 = 67^{\beta} + 1 \geq 67^1 + 1 = 68$. Then $\gamma \geq 9$. Now, we consider on the equation $\gamma^2 - 67^{\beta} = 1$. By Proposition 2.1, we have $\beta = 1$. We obtain that $\gamma^2 = 68$. This is a contradiction due to the nature of γ.

Hence the Diophantine equation $67^{\beta} + 1 = \gamma^2$, where β, γ are non-negative integers, has no non-negative integer solution.

3. Main Results

Theorem 3.1: The Diophantine equation $8^{\alpha} + 67^{\beta} = \gamma^2$, where α, β, γ are non-negative integers, has a unique solution in non-negative integers and this solution is given by $(\alpha, \beta, \gamma) = (1, 0, 3)$.

Proof: Let α, β, γ be non-negative integers such that $8^{\alpha} + 67^{\beta} = \gamma^2$. By Lemma 2.3, we obtain that $\alpha \geq 1$. Then γ is odd. It follows that $\gamma = 2r + 1$ for some non-negative integer r. Then $8^{\alpha} + 67^{\beta} = (2r + 1)^2 = 4(r^2 + r) + 1$. It follows that $67^{\beta} \equiv 1 (mod\ 4)$. Thus, β is even. Then $\beta = 2m$ for some non-negative integer m. Now, there are two possibilities for m as:

CASE: $m = 0$. Then $\beta = 0$. By Lemma 2.2, we obtain that $(\alpha, \gamma) = (1, 3)$.

CASE: $m \geq 1$. Then $\gamma^2 - 67^{2m} = 2^{3\alpha}$. It follows that $(\gamma - 67^m)(\gamma + 67^m) = 2^{3\alpha}$. Then $(\gamma - 67^m) = 2^{\theta}$ and $(\gamma + 67^m) = 2^{3\alpha - \theta}$, where θ is a non-negative integer.

Now we have $2(67)^m = (\gamma + 67^m) - (\gamma - 67^m) = 2^{3\alpha - \theta} - 2^{\theta} = 2^{\theta}\left(2^{3\alpha - 2\theta} - 1\right)$. We have two sub-cases as:

SUBCASE: $\theta = 0$. Then $(\gamma - 67^m) = 2^0 = 1$. It follows that γ is even. This is a contradiction.

SUBCASE: $\theta = 1$. Then we have $(67)^m = (2^{3\alpha - 2} - 1) \Rightarrow 2^{3\alpha - 2} - (67)^m = 1$. If $\alpha = 1$ then $m = 0$. This is a contradiction. This implies that $\alpha \geq 2$. By Proposition 2.1, we have $m = 1$. It follows that $2^{3\alpha - 2} - (67)^1 = 1 \Rightarrow 2^{3\alpha - 2} = 68$. This is impossible.

Hence, the Diophantine equation $8^{\alpha} + 67^{\beta} = \gamma^2$, where α, β, γ are non-negative integers, has a unique solution and this solution is given by $(\alpha, \beta, \gamma) = (1, 0, 3)$.

Corollary 3.2: The Diophantine equation $8^{\alpha} + 67^{\beta} = \delta^4$, where α, β, δ are non-negative integers, has no solution in non-negative integers.

Proof: Let α, β, δ be non-negative integers such that $8^{\alpha} + 67^{\beta} = \delta^4$. Let $\gamma = \delta^2$ then γ must be a non-negative integer and $8^{\alpha} + 67^{\beta} = \delta^4$ becomes $8^{\alpha} + 67^{\beta} = \gamma^2$. Now by Theorem 3.1, we have $(\alpha, \beta, \gamma) = (1, 0, 3)$. It follows that $\gamma = \delta^2 = 3$. This is a contradiction due to the nature of δ. Hence, the Diophantine equation $8^{\alpha} + 67^{\beta} = \delta^4$, where α, β, δ are non-negative integers, has no solution in non-negative integers.

Corollary 3.3: The Diophantine equation $2^{\mu} + 67^{\beta} = \gamma^2$, where μ, β, γ are non-negative integers, has a solution in non-negative integers and it is given by $(\mu, \beta, \gamma) = (3, 0, 3)$.

Proof: This follows from Theorem 3.1, where $\mu = 3\alpha$.

4. Conclusion

Authors successfully examined the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$, where α, β, γ are non-negative integers, for non-negative integer solutions by using Catalan's conjecture. Results indicate that$(\alpha, \beta, \gamma) = (1, 0, 3)$ is the only solution of the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$, where α, β, γ are non-negative integers. The present scheme can be apply to solve other Diophantine equations in future.

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