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Some Unpublished Works of (Late) Dr. S.P. Khare

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ABSTRACT

The senior author had a long association with Dr. Satgur Prasad Khare, as his student both in graduate and postgraduate classes at the University of Allahabad, Prayagraj (India) for over *five* decades. Dr. Khare worked in the Theory of Numbers under the supervision of a celebrated Indian Number Theorist (Prof. H. Gupta). Even after his retirement, he continued his researches in Number Theory and, recently, he developed interest in cryptography. Due to lack of senior workers in his field, he trusted his own teacher and often used to send his researches to the author for review. He met a tragic and premature end of his life on 30 April, 2021 leaving a lot of his unpublished work that needs to be highlighted. He was completing a research monograph on Number Theory before his sudden cardiac arrest. The present article is a compilation of some of his bizarre researches, which he had been sending to the author. An attempt is made to edit and revise the works of Dr. Khare and is presented here. It consists of certain recurrence relations for square and cubes of large integers and other interesting problems detailed below. Further works of Dr. Khare dwelling upon the "*Primality and factorization*" of RSA numbers will be presented in a series of papers.

Keywords: Theory of Numbers, Matrices, Recursive relations, Mathematical computation

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1. Formula for Computation of Square and Cube of a Number

1.1. Computation of square

Let $T_n \equiv n^2$ be the n^{th} term so that

$$T_{n+1} = (n+1)^2 = n^2 + 2n + 1 = T_n + 2n + 1. \tag{1.1}$$

This is a *recursive formula* for computation of squares of the first n natural numbers. Particularly, for $n = 0, 1, 2, 3, \ldots$, we know

$$T_1 = 1, \qquad T_2 = 4, \qquad T_3 = 9, \ldots .$$

Knowing

$$T_{897} = (897)^2 = (900 - 3)^2 = 810000 - 5400 + 9 = 804609,$$

T_{898} can be computed by the relation (1.1):

$$T_{898} = T_{897} + 2\,(897) + 1 = 804609 + 1794 + 1 = 806404 = (898)^2.$$

1.2. Computation of cube [1]

Let $T_n \equiv n^3$ be the n^{th} term so that

$$T_{n+1} = (n+1)^3 = n^3 + 3n^2 + 3n + 1 = T_n + 3n\,(n+1) + 1. \tag{1.2}$$

Above formula is the recurrence relation. Particularly, for $n = 0, 1, 2, 3, 4, \ldots$, we know

$$T_0 = 0, \qquad T_1 = 1, \qquad T_2 = 8, \qquad T_3 = 27, \qquad T_4 = 64, \ldots .$$

Knowing T_n, T_{n+1} can be computed by above recurrence relation (1.2).

For example, $(97)^3 = 912673$. Therefore, $(98)^3$ can be computed by Eq. (1.2):

$$(98)^3 = 912673 + 3 \times 97 \times 98 + 1 = 912673 + 28518 + 1 = 941192. //$$

2. To Find the Second Order Non-Singular Matrices with Elements

$$0, 1, 2, \ldots, p-1,$$

for a given (positive integral) number p.

Proof. Let the desired matrix be of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For a nonsingular matrix both its columns have to be non-vanishing. To choose a non-null matrix $\begin{bmatrix} a \\ c \end{bmatrix}$, there are $p^2 - 1$ choices. Further, the second column $\begin{bmatrix} b \\ d \end{bmatrix}$ should not be any multiple of the first column. This allows $p^2 - p$ choices of such matrices. Hence, total number of choices is

$$(p^2 - 1).\,(p^2 - p) = p^4 - p^3 - p^2 + p. //$$

3. Theorem: Sum of the cubes of the first n natural numbers divides third multiple of the sum of fifth powers of the (first n natural) numbers:

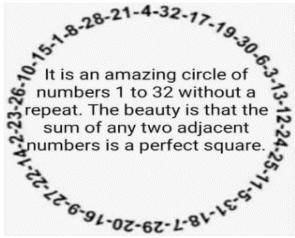
$$p \equiv \sum n^3 \quad \text{divides} \quad q \equiv 3 \sum n^5 . \tag{3.1}$$

Proof. We compute the cubic and 5^{th} powers of the numbers $n = 1, 2, 3, 4, 5$ and build the following table:

n	p	q	d (quotient)
1	1	3	3
2	9	99	11
3	36	828	23
4	100	3900	39
5	225	13275	59

The statement can be easily seen to be true up to $n = 5$. For further numbers the method of finite induction may be applied.

4. Arranging distinct (positive) integers 1 to 32 in a cycle so that every pair of adjacent two numbers is a perfect square.



The squares: 3^2, 4^2, 5^2, 6^2, 7^2 can be split into two arguments x and y each one not exceeding 32. In other words, partitioning the integer 32 into exactly two distinct parts whose sum is a perfect square:

$3^2 = 9$ partitioned as {(8, 1), (7, 2), (6, 3), (5, 4)};

$4^2 = 16$ partitioned as

{(15, 1), (14, 2), (13, 3), (12, 4) (11, 5), (10, 6), (9, 7)};

$5^2 = 25$ partitioned as

{(24, 1), (23, 2), (22, 3), (21, 4), (20, 5), (19, 6), (18, 7), (17, 8), (16, 9), (15, 10), (14, 11), (13, 12)};

$6^2 = 36$ partitioned as {(32, 4), (31, 5), (30, 6), (29, 7), (28, 8), (27, 9), (26, 10), (25, 11),

(24, 12), (23, 13), (22, 14), (21, 15), (20, 16), (19, 17);

$7^2 = 49$ partitioned as {(32, 17), (31, 18), (30, 19), (29, 20), (28, 21), (27, 22), (26, 23), (25, 24).

For $n^2 = x + y = y + x$, two distinct pairs (x, y) and (x', y') are arranged on the circumference of the circle such that $y + x'$ is a perfect square. As such, we pick up the following 32 groups amongst above pairs and start writing with the number 32 clockwise:

(32, 17), (19, 17), (19, 30), (6, 30), (6, 3), (13, 3), (13, 12), (24, 12), (24, 25), (11, 25),

(11, 5), (31, 5), (31, 18), (7, 18), (7, 29), (20, 29), (20, 16), (9, 16), (9, 27), (22, 27), (22, 14),

(2, 14), (2, 23), (26, 23), (26, 10), (15, 10), (15, 1), (8, 1), (8, 28), (21, 28), (21, 4) and (32, 4);

whose arguments can be arranged in a way satisfying above requirements.

Note 4.1. However, more arrangements are also possible.

5. Computing the value of π up to 100 decimal places: [2]
3·14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679.

6. Euler's Constant γ

$$\gamma = \lim_{n\to\infty} \{(1 + 1/2 + 1/3 + \ldots + 1/n) - \log_e n\}$$

$$= 0{\cdot}57721\ 56649\ 01532\ 86060\ 65120\ 90082.$$

7. A paradox. Computing the sum of the continued fraction $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1\ldots\infty}}}}$.

Solution. Setting value of the fraction as x, we have

$$x = \frac{1}{1-x} \quad \Rightarrow \quad x - x^2 = 1 \quad \Rightarrow \quad x^2 - x + 1 = 0,$$

having imaginary solutions

$$x = \{1 \pm \sqrt{(-1)^2 - 4}\}/2 = \{1 \pm \sqrt{-3}\}/2.\ //$$

Note 7.1. The fraction being real but it sums to imaginary numbers – hence a paradox?

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