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On Intuitionistic Nil Radicals of an Intuitionistic Fuzzy Ideals of Hemirings

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ABSTRACT

After the introduction of intuitionistic fuzzy sets by Atanassov K.T, many mathematical researchers have been devoted in the study of different aspects of this concept and they have been successful. The purpose of this paper is to introduce the notion of intuitionistic nil radicals of intuitionistic fuzzy ideals of hemirings. The author also investigated some of their properties.

KEYWORDS: Hemiring; intuitionistic fuzzy ideal; Intuitionistic nil radical; Intuitionistic intrinsic product; Intuitionistic sum

1. Introduction

On a non-empty set X, Zadeh in his classic paper [1] produced a function $\mu : X \rightarrow [0,1]$, called fuzzy set , which give the degree of membership of an element in a given set. It's generalization called intuitionistic fuzzy set was introduced by Atanassov K.T in [2] , which give both the degree of membership and the degree of non –membership of an element to the given set. Several researches have been made in the nil radicals of fuzzy ideals and intuitionistic fuzzy ideals of rings.[4-6]. A lot of researchers extend many concepts in ring theory to hemiring theory with some modfications [7-15]. In particular the author of [13] introduced the concept of nil radicals of fuzzy ideals of hemirings and discussed their properties. In this paper, we introduced the notion of intuitionistic nil radicals of an intuitionistic fuzzy ideal of hemirings and studied some of their related properties.

2. Preliminaries

Definition 2.1: A system $(\mathrm{R},+\cdot)$ where R is non empty set and $+$ and $\cdot$ are binary operations on R is called a hemiring if:
(i).$(\mathrm{R},+)$ is commutative semi group with zero element 0.
(ii). $(\mathrm{R},\cdot)$is a semigoup.
(iii). $\mathrm{a}(\mathrm{b}+\mathrm{c}) = \mathrm{ab}+\mathrm{ac}$ and $(\mathrm{a}+\mathrm{b})\mathrm{c} = \mathrm{ac}+\mathrm{bc}$and $0\cdot x = x.0$ for all $\mathrm{a},\mathrm{b},\mathrm{c},$ and, $\mathrm{x} \in \mathrm{R}$.

A hemiring $(\mathrm{R},+\cdot)$ is said to be commutative if$a \cdot b = b \cdot a$ $\forall a,b \in R$.

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Throughout this paper we assume Rto be a commutative hemiringwith unity and L stands for a complete heyting algebra.

Definition 2.2: A non-empty subsetIofR, closed under addition of R is such that ; for all $x \in R$ and $a \in I$ we have $xa \in I$ is called left ideal of R .

Definition 2.3: A left ideal I of R satisfying the property : if $y, z \in I$ *and* $x \in R$,$x + y = z$ implies $x \in I$ is called left k- ideal.

Analogous definitions can be given for right cases.

Definition 2.4: If I is an ideal of R , then its radical (also called nil radical) is defined as $\sqrt{I} = \{ x \in R : x^n \in I$, for some integer $n > 0\}$

Definition 2.5: An L-fuzzy set μ of non – empty arbitrary set $X$is a mapping $\mu : X \rightarrow L$.
The intersection and union of two fuzzy sets can be defined as follow :
(i). $\mu \cap v(x) = \min\{\mu(x), v(x)\}$.
(ii). $\mu \cup v(x) = \max\{\mu(x), v(x)\}$

Definition 2.6: A fuzzy set $\mu : X \rightarrow L$ satisfying the conditions:
(i).$\mu(x + y) \geq \mu(x) \wedge \mu(y)$
(ii).$\mu(xy) > \mu(y)$for x *and* y in R is called fuzzy left ideal.

Similarly right case can be defined.

Definition 2.7: Let μ be a fuzzy ideal of. The fuzzy nil radical $\sqrt{\mu}$ of μ is the fuzzy subset of R defined by : $\sqrt{\mu}(x) = \sup_{n\geq 1} \mu(x^n)$, Or equivalently can be written as

$$\sqrt{\mu}(x) = \bigvee_{n\geq 1} \mu(x^n)$$

Definition 2.8: An object of the form $\left\{\left(x, \mu_A(x)\, , \lambda_A(x)\right) : x \in R\right\}$

Where$\mu_A : X \rightarrow L$ *and* $\lambda_A : X \rightarrow L$
define the degree of membership and non-membership of the element x of R ,respectively and for each $x \in R$, satisfaying , $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ is called intuitionistic fuzzy set (for short IFS) A in R.

For simplicity sake ,the symbol $A = (\mu_A, \lambda_A)$is used to denote the IFS A in R .
Clearly, for every fuzzy set μ , we can have an IFS : $A = \{(x, \mu(x), 1 - \mu(x)) : x \in X\}$.
For every two intuitionistic fuzzy sets, $A = (\mu_A, \lambda_A)$, B =(μ_B, λ_B),
We have the following definitions:

(a) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ *and* $\lambda_A(x) \geq \lambda_B(x)\ \forall \in X$.
(b) $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$.
(c) $A \cap B = (\mu_A \cap \mu_B, \lambda_A \cap \lambda_B)$
(d) $A \cup B = (\mu_A \cup \mu_B, \lambda_A \cup \lambda_B)$

Definition 2.9: An intuitionistic fuzzy set A =(μ_A, λ_A) is said to be an intuitionistic fuzzy left ideal of R if :
(i). $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$

(ii). $\lambda_A(x+y) \leq \lambda_A(x) \vee \lambda_A(y)$
(iii). $\mu_A(x\,y) \geq \mu_A(y)$
(iv). $\lambda_A(xy) \leq \lambda_A(y)\ \forall\, x, y \in R$.

Similar definition can be given for the right case.

An intuitionistic fuzzy ideal of R is one which is both intuitionistic fuzzy right and left ideal.

3. **Intuitionistic nil radicals of an intuitionistic fuzzy ideal of hemirings**

Definition 3.1: Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy ideals of R. The intuitionistic nil radicals of $A = (\mu_A, \lambda_A) A = (\mu_A, \lambda_A)$, denoted by $\sqrt{A} = (\mu_{\sqrt{A}}, \lambda_{\sqrt{A}})$, is defined by

$$\mu_{\sqrt{A}}(x) = \bigvee_{n\geq 1} \mu_A(x^n), \qquad \lambda_{\sqrt{A}}(x) = \bigwedge_{n\geq} \lambda_A(x^n)$$

for all $x \in R$ and for some $n \in N$.

Or equivalently, the above definition can be written as

$$\mu_{\sqrt{A}}(x) = \sup_{n\geq 1} \mu_A(x^n), \qquad \lambda_{\sqrt{A}}(x) = \inf_{n\geq 1} \lambda_A(x^n)$$

for all $x \in R$ and for some $n \in N$.

In the sequel, we will use both interchangeably.
The following lemma is the direct consequence of Definition 3.1 and can easily be shown.

Lemma 3.2:.For all intuitionistic fuzzy ideals $A = (\mu_A, \lambda_A)$, &$B = (\mu_B, \lambda_B)$ of R, we have

(i) $A \subseteq \sqrt{A}$

(ii) $A \subseteq B$ *implies* $\sqrt{A} \subseteq \sqrt{B}$

(iii) $\sqrt{\sqrt{A}} = \sqrt{A}$

Theorem 3.3: $\sqrt{A} = (\mu_{\sqrt{A}}, \lambda_{\sqrt{A}})$is an intuitionistic fuzzy ideal of .

Proof: First let us show that

$$\mu_{\sqrt{A}}(x+y) \geq min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \,\&\, \lambda_{\sqrt{A}}(x+y) \leq \max\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\}$$

Let $x, y \in R$. Then we have the following .

$$min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} = min\left\{\sup_{n\geq 1} \mu_A(x^m), \sup_{n\geq 1} \mu_A(y^n)\right\}$$

$$= \sup_{m\geq 1}(\sup_{n\geq 1} \min\{\mu_A(x^m), \mu_A(y^n)\} \ldots\ldots\ldots.(1)$$

$$\max\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\} = \max\left\{\inf_{m\geq 1} \lambda_A(x^m), \inf_{n\geq 1} \lambda_A(y^n)\right\}$$

$$= \inf_{m\geq 1}(\inf_{n\geq 1} \max\{\lambda_A(x^m), \lambda_A(y^n)\} \ldots\ldots\ldots..(2)$$

As R is commutative ,for any two positive integers m and n ,each term in the binomial expansion of $(x+y)^{m+n}$ contains either x^m *or* y^n as a factor .

Thus ,$(x+y)^{m+n} = tx^m + ry^n$, for some t , r $\in R$. Then we have

$$\min\{\mu_A(x^m),\mu_A(y^n)\} \le \min\{max\{\mu_A(x^m),\mu_A(t)\}, max\{\mu_A(y^n),\mu_A(r)\}\}$$
$$\le \min\{\mu_A(tx^m),\mu_A(ry^n)\}\ (as\ \mu_A\ is\ intuitionistic\ fuzzy\ ideal)$$
$$\le \mu_A(tx^m+ry^n) = \mu_A(x+y)^{m+n} \le \sup_{k\ge1}\mu_A(x+y)^k = \mu_{\sqrt{A}}(x+y)\ \ldots\ldots\ldots(3)$$

and

$$max\left\{\lambda_A(x^m), \lambda_A(y^n)\right\}$$
$$\ge \max\{\min\left\{\lambda_A(x^m), \lambda_A(t)\right\}, \min\left\{\lambda_A(y^n), \lambda_A(r)\right\}\}$$
$$\ge \max\left\{\lambda_A(tx^m), \lambda_A(ry^n)\right\}\ ((as\ \mu_A\ is\ intuitionistic\ fuzzy\ ideal)$$
$$\ge \lambda_A(tx^m+ry^n) = \lambda_A(x+y)^{m+n} \ge \inf_{k\ge1}\lambda_A((x+y)^k)$$
$$= \lambda_{\sqrt{A}}(x+y).\ \ \ldots\ldots\ldots\ldots\ldots\ (4)$$

Hence, from (1) & (2) we have that

$$\mu_{\sqrt{A}}(x+y) \ge min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \& \lambda_{\sqrt{A}}(x+y) \le \max\left\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\right\}$$

Secondly to show that:

$$\mu_{\sqrt{A}}(xy) \ge max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \& \lambda_{\sqrt{A}}(xy) \le \min\left\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\right\}$$

$$max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} = max\left\{\sup_n \mu_A(x^n), \sup_n \mu_A(y^n)\right\} = \sup_n \max\{\mu_A(x^n),\mu_A(y^n)\}\ ..(5)$$

$$\min\left\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\right\} = \min\left\{\inf_n \lambda_A(x^n), \inf_n \lambda_A(y^n)\right\} = \inf_n \min\{\lambda_A(x^m), \lambda_A(y^n)\}\ ..(6)$$

Now for any positive integer n, we have

$$\max\{\mu_A(x^n), \mu_A(y^n)\} \le \mu_A(x^ny^n) = \mu_A(xy)^n \le \sup_{k\ge1}\mu_A(xy)^k = \mu_{\sqrt{A}}(xy)$$

and

$$min\left\{\lambda_A(x^n), \lambda_A(y^n)\right\} \ge \lambda_A(x^ny^n) = \lambda_A(xy)^n \ge \inf_{k\ge1}\lambda_A(xy)^k = \lambda_{\sqrt{A}}(xy)$$

Thus , from (5)& (6), we have that

$$\mu_{\sqrt{A}}(xy) \ge max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \& \lambda_{\sqrt{A}}(xy) \le \min\left\{\lambda_{\sqrt{A}}(x), \lambda_{\sqrt{A}}(y)\right\}$$

Hence $\sqrt{A} = (\mu_{\sqrt{A}}, \lambda_{\sqrt{A}})$ is an intuitionistic fuzzy ideal of R.

Definition 3.4: Let $A = (\mu_A, \lambda_A)$ & $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy subsets in ahemiring R (not necessarily commutative).The intuitionistic intrinsic product of A =(μ_A, λ_A),& B =(μ_B, λ_B) is defined to be the intuitionisticfuzzy set $A*B = (\mu_{A*B}, \lambda_{A*B})$ in R given by

$$\mu_{A*B}(x) = \bigvee\begin{cases}\bigwedge_{1\le i\le k}\mu_A(a_i)\wedge\mu_B(b_i) & , if\ x=\sum_{i=1}^{k}a_ib_i\ , k\in N\ , \\ 0 & otherwise\end{cases}$$

$$\lambda_{A*B}(x) = \bigwedge \begin{cases} \bigvee_{1\le i\le k} \lambda_A(a_i) \vee \lambda_B(b_i) & , if\ x = \sum_{i=1}^{k} a_i b_i \ \ , k \in N \\ 1 & otherwise \end{cases}$$

Clearly the product $A * B$ is commutative if R is a commutative hemiring.

Theorem 3.5: If $A = (\mu_A, \lambda_A)$,&$B = (\mu_B, \lambda_B)$ are intuitionistic fuzzyideals of R , then their product $A * B = (\mu_{A*B}, \lambda_{A*B})$ is also an intuitionistic fuzzy ideal of R.

Proof: let $x, y \in R$, then
We want to show
(i) $\mu_{A*B}(x+y) \geq min\{\mu_{A*B}(x), \mu_{A*B}(y)\}$ and $\lambda_{A*B}(x+y) \leq max\{\lambda_{A*B}(x), \lambda_{A*B}(y)\}$
(ii) $\mu_{A*B}(xy) \geq \mu_{A*B}(x) \left(\mu_{A*B}(xy) \geq \mu_{A*B}(y) \right)$ and $\lambda_{A*B}(xy) \leq \lambda_{A*B}(x) \left(\lambda_{A*B}(xy) \leq \lambda_{A*B}(y) \right)$
To show the first assertion:

$$\mu_{A*B}(x+y) = \bigvee \left\{ \bigwedge_{i\le 1\le\le k} \mu_A(\alpha_i) \wedge \mu_B(\beta_i) \quad : \ x + y = \sum_{i=1}^{k} \alpha_i \beta_i \ , k \in N \right\}$$
$$\geq \bigvee \left\{ \left(\bigwedge_{1\le i\le m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \left(\bigwedge_{1\le i\le n} \mu_A(c_i) \wedge \mu_B(d_i) \right) \ : \ x = \sum_{i=1}^{m} a_i b_i \ , y = \sum_{i=1}^{n} c_i d_i \quad , m, n \in N \right\} = \bigvee \left\{ \bigwedge_{1\le i\le m} \mu_A(a_i) \wedge \mu_B(b_i) : x = \sum_{i=1}^{m} a_i b_i \ \in N \right\} \wedge$$
$$\bigvee \left\{ \bigwedge_{1\le i\le n} \mu_A(c_i) \wedge \mu_B(d_i) : y = \sum_{i=1}^{n} c_i d_i \ n \in N \right\} = \mu_{A*B}(x) \wedge \mu_{A*B}(y)$$

and

$$\lambda_{A*B}(x+y) = \bigwedge \left\{ \bigvee_{1\le i\le k} \lambda_A(\alpha_i) \vee \lambda_B(\beta_i) : x + y = \sum_{i=1}^{k} \alpha_i \beta_i \ , k \in N \right\}$$
$$\leq \bigwedge \left\{ \left(\bigvee_{1\le i\le m} \lambda_A(a_i) \vee \lambda_B(b_i) \right) \vee \left(\bigvee_{1\le i\le n} \lambda_A(c_i) \vee \lambda_B(d_i) \right) \ : \ x = \sum_{i=1}^{m} a_i b_i \ , \ y = \sum_{i=1}^{n} c_i d_i \ , m, n \in N \right\}$$
$$= \bigwedge \left\{ \bigvee_{1\le i\le m} \lambda_A(a_i) \vee \lambda_B(b_i) : \ x = \sum_{i=1}^{m} a_i b_i \ , m \in N \right\} \vee \bigwedge \left\{ \bigvee_{1\le k\le n} \lambda_A(c_i) \vee \lambda_B(d_i) : \ y = \sum_{i=1}^{n} c_i d_i \ , n \in N \right\} = \lambda_{A*B}(x) \vee \lambda_{A*B}(y)$$

Hence,$\mu_{A*B}(x+y) \geq min\{\mu_{A*B}(x), \mu_{A*B}(y)\}$ and $\lambda_{A*B}(x+y) \leq max\{\lambda_{A*B}(x), \lambda_{A*B}(y)\}$

To show the second assertion:

$$\mu_{A*B}(y) = \bigvee \left\{ \bigwedge_{1 \le i \le n} \mu_A(a_i) \wedge \mu_B(b_i) : \ y = \sum_{i=1}^{n} a_i b_i \ , n \in N \right\}$$
$$\le \bigvee \left\{ \bigwedge_{1 \le i \le n} \mu_A(a_i x) \wedge \mu_B(b_i) : \ xy = \sum_{i=1}^{n} (a_i x) b_i \ , n \in N \right\}$$
$$\le \bigvee \left\{ \bigwedge_{1 \le i \le k} \mu_A(\alpha_i) \wedge \mu_B(\beta_i) : \ xy = \sum_{i=1}^{k} \alpha_i \beta_i \ , k \in N \right\} = \mu_{A*B}(xy)$$

Similarly

$$\mu_{A*B}(x) = \bigvee \left\{ \bigwedge_{1 \le i \le n} \mu_A(a_i) \wedge \mu_B(b_i) : \ x = \sum_{i=1}^{n} a_i b_i \ , n \in N \right\}$$
$$\le \bigvee \left\{ \bigwedge_{1 \le i \le n} \mu_A(a_i) \wedge \mu_B(b_i y) : \ xy = \sum_{i=1}^{n} a_i (b_i y) \ , n \in N \right\}$$
$$\le \bigvee \left\{ \bigwedge_{1 \le i \le k} \mu_A(\alpha_i) \wedge \mu_B(\beta_i) : \ xy = \sum_{i=1}^{k} \alpha_i \beta_i \ , k \in N \right\} = \mu_{A*B}(xy)$$

and

$$\lambda_{A*B}(y) = \bigwedge \left\{ \bigvee_{1 \le k \le m} \lambda_A(a_i) \vee \lambda_B(b_i) : y = \sum_{i=1}^{m} a_i b_i \ \ , m \in N \right\}$$
$$\ge \bigwedge \left\{ \bigvee_{1 \le k \le m} \lambda_A(a_i x) \vee \lambda_B(b_i) : xy = \sum_{i=1}^{m} (a_i x) b_i \ , m \in N \right\}$$
$$\ge \bigwedge \left\{ \bigvee_{1 \le k \le k} \lambda_A(\alpha_i) \vee \lambda_B(\beta_i) : xy = \sum_{i=1}^{k} (\alpha_i) \beta_i \ , k \in N \right\} = \lambda_{A*B}(xy)$$

Similarly

$$\lambda_{A*B}(x) = \bigwedge \left\{ \bigvee_{1 \le k \le m} \lambda_A(a_i) \vee \lambda_B(b_i) : x = \sum_{i=1}^{m} a_i b_i \ \ , m \in N \right\}$$
$$\ge \bigwedge \left\{ \bigvee_{1 \le k \le m} \lambda_A(a_i) \vee \lambda_B(b_i y) : xy = \sum_{i=1}^{m} a_i (b_i y) \ , m \in N \right\}$$
$$\ge \bigwedge \left\{ \bigvee_{1 \le k \le k} \lambda_A(\alpha_i) \vee \lambda_B(\beta_i) : xy = \sum_{i=1}^{k} (\alpha_i) \beta_i \ , k \in N \right\} = \lambda_{A*B}(xy)$$

Therefore, $\mu_{A*B}(xy) \ge \mu_{A*B}(x) \left(\ \mu_{A*B}(xy) \ge \mu_{A*B}(y) \ \right)$
and $\lambda_{A*B}(xy) \le \lambda_{A*B}(x) \left(\ \lambda_{A*B}(xy) \le \lambda_{A*B}(y) \ \right)$

Hence , $A * B = (\mu_{A*B}, \lambda_{A*B})$ is an intuitionistic fuzzy ideal of R.

Theorem 3.6: If $A = \left(\mu_A, \lambda_A \right)$, & $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy ideals of R , then $\sqrt{A * B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

Claim 1 $\sqrt{A * B} = \sqrt{A \cap B}$

Let $x \in R$ and let $x = \sum_1^m a_i b_i = a_1 b_1 + a_2 b_2 + \cdots a_m b_m$ where $a_i b_i \ne 0$ are in R
Then
$min\{\mu_A(a_i), \mu_B(b_i)\} \le \mu_A(a_i) \le \mu_A(a_i b_i)$ and

$$\max\{\lambda_A(a_i) \vee \lambda_B(b_i)\} \geq \lambda_A(a_i) \geq \lambda_A(a_ib_i) \ , 1 \leq i \leq m$$

Then we have,

$$\min\{\mu_A(a_1), \dots, \mu_A(a_m), \mu_B(b_1), \dots, \mu_B(b_m)\} \leq \min\{\mu_A(a_1b_1), \dots, \mu_A(a_mb_m)\}$$
$\leq \mu_A(a_1b_1 + \dots + a_mb_m) = \mu_A(x)$…………………………(7)

and
$\max\left\{\lambda_A(a_1), \dots, \lambda_A(a_m), \lambda_B(b_1), \dots, \lambda_B(b_m)\right\} \geq \max\left\{\lambda_A(a_1b_1), \dots \lambda_A(a_mb_m)\right\} \geq \lambda_A(a_1b_1 + \dots + a_mb_m) = \lambda_A(x)$ ……………………………………(8)

Taking the supremum and infimum respectively, overall expressions, we get

$\mu_{A*B}(x) \leq \mu_A(x)$, and $\lambda_{A*B}(x) \geq \lambda_A(x)$ Similarly $\mu_{A*B}(x) \leq \mu_B(x)$, and $\lambda_{A*B}(x) \geq \lambda_B(x)$

$\Rightarrow \mu_{A*B}(x) \leq \min\{\mu_A(x), \mu_B(x)\}$and , $\lambda_{A*B}(x) \geq \max\{\lambda_A(x), \lambda_B(x)\}$ for all $x \in R$.
$\Rightarrow A * B \subseteq A \cap B \Rightarrow \sqrt{A * B} \subseteq \sqrt{A \cap B}$ (by lemma 3.2 ii) ………..(8)

Now,let $x \in R$. Then we have that
$\mu_{\sqrt{A*B}}(x) = \sup_k \mu_{A*B}(x^k) \geq \mu_{A*B}(x^{2n}) \geq \min\{\mu_A(x^n), \mu_B(x^n)\} = \mu_{A\cap B}(x^n)$and
$\lambda_{\sqrt{A*B}}(x) = \inf_k \lambda_{A*B}(x^k) \leq \lambda_{A*B}(x^{2n}) \leq \max\left\{\lambda_A(x^n), \lambda_B(x^n)\right\} = \lambda_{A\cap B}(x^n)$,for all $n \geq 1$.

Taking the supremum and infimum , respectively over all $n \geq 1$, we get

$\mu_{\sqrt{A\cap B}}(x) \leq \mu_{\sqrt{A*B}}(x)$and $\lambda_{\sqrt{A\cap B}}(x) \geq \lambda_{\sqrt{A*B}}(x)$

$\Rightarrow \sqrt{A \cap B} \subseteq \sqrt{A * B}$ ………………………(9)

From (8) and (9) we have that $\sqrt{A * B} = \sqrt{A \cap B}$.

Claim 2 $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$

From lemma 3.2i, we have that $\sqrt{A \cap B} \subseteq \sqrt{A}$ and $\sqrt{A \cap B} \subseteq \sqrt{B}$
Which implies $\sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$ ………….. (10)

To prove: $\sqrt{A} \cap \sqrt{B} \subseteq \sqrt{A \cap B}$,let $x \in R$

$$(\mu_{\sqrt{A}} \wedge \mu_{\sqrt{B}})(x) = \min\left\{\sup_m \mu_A(x^m), \sup_n \mu_B(x^n)\right\} = \sup_m(\sup_n \min\{\mu_A(x^m), \mu_B(y^n)\})$$
$$(\lambda_{\sqrt{A}} \vee \lambda_{\sqrt{B}})(x) = \max\left\{\inf_m \lambda_A(x^n), \inf_n \lambda_B(x^n)\right\} = \inf_m(\inf_n \max\{\lambda_A(x^m), \lambda_A(y^n)\}$$

Now, for any positive integers m & n, we have

$\min\{\mu_A(x^m), \mu_B(x^n)\} \leq \min\{\mu_A(x^{mn}), \mu_B(x^{mn})\} = (\mu_A \wedge \mu_B)(x^{mn}) \leq \sup_{k\geq 1}(\mu_A \wedge \mu_B)(x^k) = \mu_{\sqrt{A\cap B}}(x)$, and

$$\max\{\lambda_A(x^n), \lambda_B(x^n)\} \geq \max\{\lambda_A(x^{mn}), \lambda_B(x^{mn})\} = (\lambda_A \vee \lambda_B)(x^{mn})$$
$$\geq \inf_{k\geq 1} (\lambda_A \vee \lambda_B)(x^k) = \lambda_{\sqrt{A\cap B}}(x)$$

Thus, $\sqrt{A} \cap \sqrt{B} \subseteq \sqrt{A \cap B}$ which implies the second equality follows.

Therefore, from claim 1 &2 , we have that $\sqrt{A * B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$

Corollary 3.7: If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy ideal of , then $\sqrt{A^n} = \sqrt{A}$ for all $n \geq 1$, where

$A^n = A * A * A * \ldots * A$ (n-times)

Proof: We prove this by induction.

For $n = 1$,clearly follows
For $n = 2$, put $A = B$ in theorem 3.6 , we get $\sqrt{A * A} = \sqrt{A} \Rightarrow \sqrt{A^2} = \sqrt{A}$

Assume it is true for $n = k$

Now , $\sqrt{A^{k+1}} = \sqrt{A^k * A} = \sqrt{A^k} \cap \sqrt{A} = \sqrt{A} \cap \sqrt{A} = \sqrt{A}$ (by theorem 3.6)
$\Rightarrow \sqrt{A^{k+1}} = \sqrt{A}$

Thus , $\sqrt{A^n} = \sqrt{A}$, $n \geq 1$

Corollary 3.8 If $A = (\mu_A, \lambda_A)$,& $B = (\mu_B, \lambda_B)$ are intuitionisticfuzzyideals of R with $A^k \subseteq B$,for some $k \geq 1$, then $\sqrt{A} \subseteq \sqrt{B}$,

Proof: Since $A^k \subseteq B$, we have that $\sqrt{A^k} \subseteq \sqrt{B}$ …. (by lemma3.2 ii) ………(11)

But , $\sqrt{A^k} = \sqrt{A}$ ……. by corollary 3.7 ………………..(12)

From (11) and (12) we get that $\sqrt{A} \subseteq \sqrt{B}$.

Definition 3.9: Let $A = (\mu_A, \lambda_A)$,& $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in ahemiring R (not necessarily commutative).The intuitionistic sum of $A = (\mu_A, \lambda_A)$,& $B = (\mu_B, \lambda_B)$ is defined to be the intuitionisticfuzzy set $A + B = (\mu_{A+B}, \lambda_{A+B})$ in R , given by

$$\mu_{A+B}(x) = \begin{cases} \bigvee_{x=y+z} \{\mu_A(y) \wedge \mu_B(y)\}, & if\ x = y + z \\ 0 & otherwise \end{cases}$$

$$\lambda_{A+B}(x) = \begin{cases} \bigwedge_{x=y+z} \{\lambda_A(y) \vee \lambda_B(z)\}, & if\ x = y + z \\ 1 & otherwise \end{cases}$$

Or equivalently the definition can be written as

$$\mu_{A+B}(x) = \begin{cases} \sup_{x=y+z} \min\{\mu_A(y), \mu_B(z)\} & if\ x = y+z \\ 0 & otherwise \end{cases}$$

$$\lambda_{A+B}(x) = \begin{cases} \inf_{x=y+z} \max\{\lambda_A(y), \lambda_B(z)\} & if\ x = y+z \\ 1 & otherwise \end{cases}$$

Theorem 3.10: If $A = (\mu_A, \lambda_A)$, & $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy ideals of R, then their sum $A + B = (\mu_{A+B}, \lambda_{A+B})$ is also an intuitionistic fuzzy ideal of R.

Proof: Let $x, y \in R$,

$$\begin{aligned} \mu_{A+B}(x) \wedge \mu_{A+B}(y) &= \bigvee\{\mu_A(a) \wedge \mu_B(b): x = a+b\} \wedge \bigvee\{\mu_A(c) \wedge \mu_B(d): y = c+d\} \\ &= \bigvee\{(\mu_A(a) \wedge \mu_B(b)) \wedge (\mu_A(c) \wedge \mu_B(d)): x = a+b: y = c+d\} \\ &= \bigvee\{(\mu_A(a) \wedge \mu_A(c)) \wedge (\mu_B(b) \wedge \mu_B(d)): x = a+b: y = c+d\} \\ &\leq \bigvee\{(\mu_A(a+c) \wedge \mu_B(b+d)): x+y = a+c+b+d\} = \mu_{A+B}(x+y) \end{aligned}$$

and

$$\begin{aligned} \lambda_{A+B}(x) \vee \lambda_{A+B}(y) &= \bigwedge\left\{\lambda_A(a) \vee \lambda_B(b): x = a+b\right\} \vee \bigwedge\{\lambda_A(c) \vee \lambda_B(d): y = c+d\} \\ &= \bigvee\{\left(\lambda_A(a) \vee \lambda_B(b)\right) \vee ((c) \vee \lambda_B(d)): x = a+b, y = c+d\} \\ &= \bigvee\{\left(\lambda_A(a) \vee \lambda_B(b)\right) \vee ((c) \vee \lambda_B(d)): x+y = a+b+c+d\} \\ &\geq \bigvee\left\{\lambda_A(a+c) \vee \lambda_B(b+d): x+y = a+b+c+d\right\} = \lambda_{A+B}(x+y) \end{aligned}$$

Hence, $\mu_{A+B}(x+y) \leq \mu_{A+B}(x) \wedge \mu_{A+B}(y)$ and $\lambda_{A+B}(x+y) \geq \lambda_{A+B}(x) \vee \lambda_{A+B}(y)$

Secondly

$$\begin{aligned} \mu_{A+B}(x) &= \bigvee\{\mu_A(a) \wedge \mu_B(b): x = a+b\} \leq \bigvee\{\mu_A(ay) \wedge \mu_B(by): xy = ay+by\} \\ &\leq \bigvee\{\mu_A(\alpha) \wedge \mu_B(\beta): xy = \alpha+\beta\} = \mu_{A+B}(xy) \end{aligned}$$

and

$$\begin{aligned} \lambda_{A+B}(x) &= \bigwedge\left\{\lambda_A(a) \vee \lambda_B(b): x = a+b\right\} \\ &\geq \bigwedge\left\{\lambda_A(ay) \vee \lambda_B(by): xy = ay+by\right\} \geq \bigwedge\left\{\lambda_A(\alpha) \vee \lambda_B(\beta): xy = \alpha+\beta\right\} \\ &= \lambda_{A+B}(xy) \end{aligned}$$

Thus

$\mu_{A+B}(xy) \geq \mu_{A+B}(x)$ and $\lambda_{A+B}(xy) \leq \lambda_{A+B}(x)$

Similarly it can be shown that
$\mu_{A+B}(xy) \geq \mu_{A+B}(y)$ and $\lambda_{A+B}(xy) \leq \lambda_{A+B}(y)$

Therefore, $A+B=(\mu_{A+B},\lambda_{A+B})$ is an intuitionistic fuzzy ideal of$R$.

Theorem 3.11: If $A=(\mu_A,\lambda_A)$,&$B=(\mu_B,\lambda_B)$ are intuitionistic fuzzy ideals of R , then

$$\sqrt{A}+\sqrt{B}\subseteq\sqrt{\sqrt{A}+\sqrt{B}}=\sqrt{A+B}$$

Proof:

The inclusion part follows from lemma 3.2 i ,

To show the equality part:

Since $A\subseteq\sqrt{A}$ & $B\subseteq\sqrt{B}$ (by lemma 3.2 i), we have that $A+B\subseteq\sqrt{A}+\sqrt{B}\Rightarrow\sqrt{A+B}\subseteq\sqrt{\sqrt{A}+\sqrt{B}}$ (by lemma 3.2 ii)

To show the other way round , let $x\in R$, then

$$\mu_{\sqrt{A}+\sqrt{B}}(x)=\sup_{x=y+z}\min\{\mu_{\sqrt{A}}(a),\mu_{\sqrt{B}}(b)\}=$$

$$\sup_{x=y+z}\min\left\{\sup_{m}\mu_A(a^m),\sup_{n}\mu_B(b^n)\right\}=\sup_{x=y+z}(\sup_{m}(\sup_{n}\min\{\mu_A(a^m),\mu_B(b^n)\}\ ..(13)$$

and

$$\lambda_{\sqrt{A}+\sqrt{B}}(x)=\inf_{x=y+z}\max\{\lambda_{\sqrt{A}}(a),\lambda_{\sqrt{B}}(b)\}=\inf_{x=y+z}\max\left\{\inf_{m}\lambda_A(x^n),\inf_{n}\lambda_B(x^n)\right\}=$$

$$\inf_{x=y+z}\inf_{m}(\inf_{n}\max\{\lambda_A(x^m),\lambda_A(y^n)\}\ldots\ldots\ldots\ldots(14)$$

Now, let $x=a+b$, a & $b\in R:m\&n$ are any positive integers. Since R commutative, we have that

$x^{m+n}=ta^m+rb^n$, for some , $r\in R$.

Hence we get

$$\min\{\mu_A(a^m),\mu_B(b^n)\}\le\min\{\mu_A(ta^m),\mu_B(rb^n)\}\le\mu_{A+B}(ta^m+rb^n)=\mu_{A+B}(x^{m+n})\le\sup_{k\ge1}\mu_{A+B}(x^k)$$
$$=\mu_{\sqrt{A+B}}(x)$$

and

$$\max\{\lambda_A(a^m),\lambda_A(b^n)\}\ge\max\{\lambda_A(ta^m),\lambda_A(rb^n)\}\ge\lambda_{A+B}(ta^m+rb^n)=\lambda_{A+B}(x^{m+n})$$
$$\ge\inf_{k\ge1}\lambda_{A+B}(x^k)\ge\lambda_{\sqrt{A+B}}(x)$$

Thus , from (13) and (14) , we get that

$\mu_{\sqrt{A+B}}(x)\ \ge\mu_{\sqrt{A}+\sqrt{B}}(x)$and $\lambda_{\sqrt{A+B}}(x)\le\lambda_{\sqrt{A}+\sqrt{B}}(x)$, for all $x\in R$

$\Rightarrow \sqrt{\sqrt{A}+\sqrt{B}} \subseteq \sqrt{A+B}$

Hence the theorem.

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