Bull. Pure Appl. Sci. Sect. E Math. Stat.
43E(1), 32–36 (2024)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: https://doi.org/10.52710/bpas-math.4

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Fixed points in binary relation ordered dislocated quasi-metric spaces *

M. A. Ahmed[1], A. Kamal[2,†] and H. A. Nafadi[3]

1. Department of Mathematics, Faculty of Science, Majmaah University, Al-Zulfi, Saudi Arabia.
1. Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.
2,3. Department of Mathematics and Computer Science, Faculty of Science,
Port Said University, Port Said 42521, Egypt.
2. Department of Mathematics, College of Science,
Qassim University, Buraydah 51452, Saudi Arabia.
1. E-mail: moh.hassan@mu.edu.sa
2. E-mail: alaa_kamal@sci.psu.edu.eg , ak.ahmed@qu.edu.sa

3. E-mail: hatem9007@yahoo.com

Abstract In this paper, we prove some fixed point theorems in binary relation ordered dislocated quasi-metric spaces. These theorems generalize some previous results (see, J. J. Neito, and R. Rodriguez-Lopez, Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations, Order, Vol. 22(2005), 223–239, 2005).

Key words Complete dislocated quasi-metric, contraction, fixed point.

2020 Mathematics Subject Classification 47H10, 54H25.

1 Introduction

The study of fixed point theorems in partial ordered metric spaces was first investigated in 2004 by Ran and Reurings [5], and then by Nieto and Lopez [4]. Also, this study in dislocated metric spaces was under the name of metric domains in the context of domain theory [3]. In [7], Zeyada, Hassan and Ahmed generalized a result of Hizler and Seda [2] to dislocated quasi -metric spaces. Fixed point theory in partially ordered metric spaces has applications in the area of differential and matrix equations (see, e.g. [4,5]). Also, this theory in dislocated metric spaces has application in logic programming semantics (see, for example, [1]).
Throughout this paper, let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function. We need the following conditions:

(M_1) $d(x, x) = 0$,
(M_2) if $d(x, y) = d(y, x) = 0$, then $x = y$,

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received November 16, 2023 / Revised March 19, 2024 / Accepted March 27, 2024. Online First Published on June 15, 2024 at https://www.bpasjournals.com/.
†Corresponding author A. Kamal, E-mail: alaa_kamal@sci.psu.edu.eg , ak.ahmed@qu.edu.sa

(M_3) $d(x,y) = d(y,x)$,
(M_4) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$.
If d satisfies conditions (M_1)-(M_4), then it is called a **metric** on X. If d satisfies conditions (M_1), (M_2) and (M_4), it is called a **quasi-metric** on X. If d satisfies conditions (M_2) and (M_4), it is called a **dislocated metric** (or simply d-metric) (a **dislocated quasi-metric** (or simply dq-metric)) on X respectively.

Definition 1.1. A poset $(X, \preceq)$ (partially ordered set) is a nonempty set X and a binary relation $\preceq$ on X such that for all $x,y,z \in X$:

(1) $x \preceq x$ (reflexivity),
(2) if $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry),
(3) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity).

Two elements $x, y \in X$ are called comparable elements if either $x \preceq y$ or $y \preceq x$.

Definition 1.2. [4] Let $(X, \preceq)$ be a partially ordered set. It is said that $f : X \to X$ is **monotone nondecreasing** if $x, y \in X, x \preceq y \Longrightarrow f(x) \preceq f(y)$.

Definition 1.3. [7] A sequence $\{x_n\}$ in a dq-metric space (X,d) is called **dislocated quasi - convergent** (for short **dq- convergent)** if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

In this case, x is called a dq-limit of the sequence $\{x_n\}$.

Definition 1.4. [7] A sequence $\{x_n\}$ in a dq-metric space (X,d) is called **Cauchy** if $\forall\ \epsilon > 0, \exists\ n_0 \in \mathbb{N}$ such that $\forall\ m.n \geq n_0, d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$, where $\mathbb{N}$ is the set of all positive integers.

Replacing $d(x_m, x_n) < \epsilon$ and $d(x_n, x_m) < \epsilon$ in Definition 1.3 by $\max\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$, the sequence $\{x_n\}$ in a dq-metric space (X,d) is called **'bi' Cauchy** (see [6]).

Definition 1.5. [7] A dq-metric space (X,d) is called **complete** if every Cauchy sequence in it is dq-convergent.

Lemma 1.6. *[7] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.*

It is obvious that the converse of Lemma 1.6 may not be true.

Lemma 1.7. *[7] dq-limits in a dq-metric space are unique.*

In 2005, Nieto and Lopez [4] established a fixed point theorem in partially ordered complete metric spaces, stated below as Theorem 1.8, which analogues Banach's fixed point theorem.

Theorem 1.8. *[4] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $f : X \to X$ be a continuous and monotone nondecreasing mapping such that there exists $\alpha \in [0,1)$ with*

$$d(f(x), f(y)) \leq \alpha d(x,y) \quad \forall y \preceq x. \tag{*}$$

If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$, then f has a unique fixed point.
Also, the authors [4] stated and proved the following theorem.

Theorem 1.9. *[4] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Assume that X satisfies*

$$\textit{if a nondecreasing sequence } \{x_n\} \textit{ converges to } x, \textit{ then } x_n \preceq x\ \forall\ n \in \mathbb{N}. \tag{**}$$

Let $f : X \to X$ be a monotone nondecreasing mapping such that there exists $\alpha \in [0,1)$ with (*) holds. If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$, then f has a unique fixed point.
The purpose of this paper is to state and prove some fixed point theorems in binary relation ordered dq-metric spaces. These theorems generalize Theorem 3, Theorem 4 and some results in [4].

2 Main Results

First, we rewrite some definitions.

Definition 2.1. Let X be a nonempty set and $\preceq$ be a binary relation. A Bset $(X, \preceq)$ is a binary relation with a nonempty set.

Definition 2.2. Let $(X, \preceq)$ be a Bset. A mapping $f : X \to X$ is called **monotone nondecreasing** if $x, y \in X, x \preceq y \Longrightarrow f(x) \preceq f(y)$.

Definition 2.3. A sequence $\{x_n\}$ in a dq-metric space (X, d) is called **dislocated quasi -convergent** (for short **dq- convergent)** if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

Definition 2.4. A sequence $\{x_n\}$ in a dq-metric space (X, d) is called **Cauchy** if $\forall\ \epsilon > 0, \exists\ n_0 \in \mathbb{N}$ such that $\forall\ m.n \geq n_0, d(x_m, x_n) < \epsilon$, or $d(x_n, x_m) < \epsilon$, where $\mathbb{N}$ is the set of all positive integers.

Now, we are ready to state and prove the first theorem as follows:

Theorem 2.5. *Let $(X, \preceq)$ be Bset. Suppose that there exists a dq-metric d on X such that (X, d) is a complete dq-metric space. If $f : X \to X$ be a continuous monotone nondecreasing mapping such that*
(1) $\exists\ 0 \leq \alpha < 1$ such that $d(f(y), f(x)) \leq \alpha d(y, x)$ and $d(f(x), f(y)) \leq \alpha d(x, y) \quad \forall x, y \in X$ with $y \preceq x$,
(2) $\exists\ x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has a fixed point.

Proof. Since f is a monotone nondecreasing mapping and there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then by induction one can deduce that $f^n(x_0) \preceq f^{n+1}(x_0)\ \forall\ n \in \mathbb{N} \cup \{0\}$. Then, one can obtain from (1) that

$$d(f^{n+1}(x_0), f^n(x_0)) \leq \alpha^n d(f(x_0), x_0) \text{ and } d(f^n(x_0), f^{n+1}(x_0)) \leq \alpha^n d(x_0, f(x_0)),$$

for every $n \in \mathbb{N}$. If $m, n \in \mathbb{N}$ with $m > n$, we get that

$$\begin{aligned} d(f^m(x_0), f^n(x_0)) &\leq \sum_{i=n}^{m-1} d(f^{i+1}(x_0), f^i(x_0)) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) d(x_0, f(x_0)) \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_0, f(x_0)). \end{aligned}$$

Hence, $\lim\limits_{m,n\to\infty} d(f^m(x_0), f^n(x_0)) = 0$. Similarly, one can deduce that $\lim\limits_{m,n\to\infty} d(f^n(x_0), f^m(x_0)) = 0$. Therefore, the sequence $\{f^n(x_0)\}$ is Cauchy. Since X is complete, then there exists $y_0 \in X$ such that

$$\lim_{n\to\infty} d(f^n(x_0), y_0) = \lim_{n\to\infty} d(y_0, f^n(x_0)) = 0.$$

Since $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$, then $\{f^{n+1}(x_0)\}$ dq-converges to y_0. From the continuity of f, $\{f^{n+1}(x_0)\}$ dq-converges to $f(y_0)$. The uniqueness of dq-limits gives that $d(y_0, f(y_0)) = 0$; i.e., y_0 is a fixed point of f. □

Corollary 2.6. *Let $(X, \preceq)$ be a a partially ordered set. Suppose that there exists a d-metric d on X such that (X, d) is a complete d-metric space. If $f : X \to X$ be a continuous monotone nondecreasing mapping such that (1) and (2) hold as in Theorem 2.5, then f has a fixed point.*

Remark 2.7. Corollary 2.6 improves Theorem 2.5 since
(i) $(X, \preceq)$ is Bset instead of $(X, \preceq)$ is a partially ordered set,
(ii) d is a d-metric on X instead of d is a metric on X.

Theorem 2.8. *Let $(X, \preceq)$ be Bset. Suppose that there exists a dq-metric d on X such that (X, d) is a complete dq-metric space. Assume that X satisfies (**) as in Theorem 1.9. If $f : X \to X$ is a monotone nondecreasing mapping such that (1) and (2) hold as in Theorem 2.5, then f has a fixed point.*

Proof. In the proof of Theorem 2.5, we deduce that the sequence $\{f^n(x_0)\}$ converges to $y_0 \in X$. Then, given $\epsilon > 0$ there exists $n_1 \in N$ such that for all $n \geq n_1$,

$$d(f^n(x_0), y_0) < \frac{\epsilon}{2} \ \text{ and } \ d(y_0, f^n(x_0)) < \frac{\epsilon}{2}.$$

From (**), we obtain that $f^n(x_0) \preceq y_0$ for all $n \in \mathbb{N}$. Since $\{f^{n+1}(x_0)\}$ is a subsequence of the sequence $\{f^n(x_0)\}$, then we have from Lemma 1.7 that $\{f^{n+1}(x_0)\}$ dq-converges to $y_0 \in X$.
Thus, given $\epsilon > 0$ there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$,

$$d(f^{n+1}(x_0), y_0) < \frac{\epsilon}{2} \ \text{ and } \ d(y_0, f^{n+1}(x_0)) < \frac{\epsilon}{2}.$$

Therefore, given $\epsilon > 0$ there exists $m \in \mathbb{N}$ with $m = \max(n_1, n_2)$ such that for all $n \geq m$,

$$\begin{aligned} d(f(y_0), y_0) &\leq d(f(y_0), f(f^n(x_0))) + d(f^{n+1}(x_0), y_0) \\ &\leq \alpha d(y_0, f^n(x_0)) + d(f^{n+1}(x_0), y_0) \\ &< d(y_0, f^n(x_0)) + d(f^{n+1}(x_0), y_0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and

$$\begin{aligned} d(y_0, f(y_0)) &\leq d(y_0, f^{n+1}(x_0)) + d(f(f^n(x_0)), f(y_0)) \\ &\leq d(y_0, f^{n+1}(x_0)) + \alpha d(f^n(x_0), y_0) \\ &< d(y_0, f^{n+1}(x_0)) + d(f^n(x_0), y_0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ is an arbitrary, then one can deduce that y_0 is a fixed point of f. □

Corollary 2.9. *Let $(X, \preceq)$ be a partailly ordered set and suppose that there exists a d-metric d on X such that (X, d) is a complete d-metric space. Assume that X satisfies (**) as in Theorem 1.9. If $f : X \to X$ is a monotone nondecreasing mapping such that (1) and (2) hold as in Theorem 2.5, then f has a fixed point.*

Similarly, one can prove the following Theorem.

Theorem 2.10. *Let $(X, \preceq)$ be a Bset. Suppose that there exists a dq-metric d on X such that (X, d) is a complete dq-metric space. If $f : X \to X$ is a monotone nondecreasing mapping such that (1) holds as in Theorem 2.5. Assume that either f is continuous or X satisfies*

(3) f is a nonincreasing sequence $\{x_n\}$ which converges to $u \in X$, say, then $u \preceq x_n$ $\forall n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that $f(x_0) \preceq x_0$, then f has a fixed point.

Corollary 2.11. *Let $(X, \preceq)$ be a partailly ordered set. Suppose that there exists a d-metric d on X such that (X, d) is a complete d-metric space. If $f : X \to X$ is a monotone nondecreasing mapping such that (1) holds as in Theorem 2.5. Assume that either f is continuous or X satisfies (3) as in Theorem 2.10. If there exists $x_0 \in X$ such that $f(x_0) \preceq x_0$, then f has a fixed point.*

Remark 2.12. (i) In Theorems 2.5, 2.8 and 2.10, if for every $x, y \in X$, there exists an upper bound or a lower bound, then the fixed point is unique.

(ii) Corollary 2.9 (resp. Corollary 2.11) improves Theorem 4 (resp. Theorem 8 of [4]) for the same reasons as in Remark 2.7.

Acknowledgments The authors would like to thank the Deanship of Scientific Research at Majmaah University for the Financial support (project number R-1441-56). All the authors are also highly indebted to the Editor-in-Chief and the anonymous Referees for their painstaking efforts towards the improvement of the original version of this manuscript.

References

[1] Hitzler, P. (2001). Generalized metrics and topology in logic programming semantics, Ph. D. Thesis (2001), National University of Ireland (University College Cork).

[2] Hitzler, P. and Seda, A. K. (2003). Dislocated topologies, *J. Electr. Engin.*, 51, No. 12/s(2003), 3 –7.

[3] Matthews, S. G. (1986). Metric domains for completeness, Ph.D. Thesis, 1985 (Research Report 76), Department of Computer Science, University of Warwick, UK.

[4] Neito, J. J. and Rodriguez-Lopez, R. (2005). Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations, *Order*, Vol. 22(2005), 223–239.

[5] Ran, A. C. M. and Reurings, M. C. B. (2003). A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, 132(5), 1435–1443.

[6] Rutten, J. J. M. M. (1996). Elements of generalized ultrametric domain theory, *Theoretic. Comput. Sci.*, 170, 349–381.

[7] Zeyada, F. M., Hassan, G. H. and Ahmed, M. A. (2006). A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, *Arab J. Sci. Eng.*, 31(1A), 111–114.