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On the Triple Laplace- Sumudu-Elzaki Transform and Their Properties

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ABSTRACT

In this work, we have discuses and prove the different properties of the triple Laplace-Sumudu-Elzaki transform like linearity property, convolution theorems property, first shitting property, periodic function property, and the some applications to solve partial differential equations in three- dimensions.

KEYWORDS: Triple Laplace-Sumudu-Elzaki transform, Inverse triple Laplace-Sumudu-Elzaki transform, Partial differential equations.

1. Introduction

In the past two centuries, the integrated transformations have been widely applied as a tool to solve various problems in pure and applied mathematics.[1,3] Several integrated transformations such as the P. S. Laplace (1749–1827) introduced the idea of the Laplace transform in 1782 [5,6],

Definition 1.1[5-6] The Laplace transform denoted by the operator L is defined as

$$L\left[f(x):\sigma)\right]=\int_0^\infty e^{-\sigma x}f(x)dx=F(\sigma)\,,\quad x>0$$

In the early 1990's, Watugala, provided Sumudu transform. [3]

Definition 1.2.[3] The Sumudu transform denoted by the operator S is defined as

$$S\left[f(\hbar):\rho)\right]=\frac{1}{\rho}\int_0^\infty e^{-\frac{\hbar}{\rho}}f(\hbar)d\hbar=F(\rho)\,,\quad \hbar>0$$

In the early 2011's, Tarig Elzaki introduced the Elzaki transform. [7-13]

Definition 1.3 [7-13] The Elzaki transform denoted by the operator E is defined as

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$$E\left[f(\tau):\delta)\right]=\delta\int_0^\infty e^{-\frac{\tau}{\delta}}f(\tau)d\tau=F(\delta)\,,\quad \tau>0$$

Definition 1.4[2-6]the triple Lapl a ce – Sumudu – Elzaki transform is denoted by:

$$L_xS_\hbar E_\tau\left[f(x,\hbar,\tau):(\sigma,\rho,\delta)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau \qquad (1)$$

The structure of this paper is organized as follows: First, we begin with some basic definitions and the use of Linearity property, Convolution theorem property, First shitting property, Periodic function property of $f(x,\hbar,\tau)$, $g(x,\hbar,\tau)$ and the examples application to solve partial differential equations in three-dimensions.

2. Theorems and Properties of Triple Laplace – Sumudu – Elzaki Transform

Theorem 1: (linearity of the triple Laplace – Sumudu – Elzakitransform)

Let $f(x,\hbar,\tau)$ and $g(x,\hbar,\tau)$ be functions who's the triple Laplace – Sumudu - Elzaki transform exists then

$$L_xS_\hbar E_\tau\left[\alpha f(x,\hbar,\tau)+\beta g(x,\hbar,\tau)\right]=\alpha L_xS_\hbar E_\tau\left[f(x,\hbar,\tau)\right]+\beta L_xS_\hbar E_\tau\left[g(x,\hbar,\tau)\right]$$

Where a and b are constants

Proof:

$$L_xS_\hbar E_\tau\left[\alpha f(x,\hbar,\tau)+\beta g(x,\hbar,\tau)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty\left[\alpha f(x,\hbar,\tau)+\beta g(x,\hbar,\tau)\right]e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}dx\,d\hbar\,d\tau$$

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty\left[\alpha f(x,\hbar,\tau)\right]e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}dx\,d\hbar\,d\tau+\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty\left[\beta g(x,\hbar,\tau)\right]e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}dx\,d\hbar\,d\tau$$

$$=\frac{\delta}{\rho}\alpha\int_0^\infty\int_0^\infty\int_0^\infty\left[f(x,\hbar,\tau)\right]e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}dx\,d\hbar\,d\tau+\frac{\delta}{\rho}\beta\int_0^\infty\int_0^\infty\int_0^\infty\left[g(x,\hbar,\tau)\right]e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}dx\,d\hbar\,d\tau$$

$$=\alpha L_xS_\hbar E_\tau\left[f(x,\hbar,\tau)\right]+\beta L_xS_\hbar E_\tau\left[g(x,\hbar,\tau)\right]$$

Theorem 2: **(**first shifting**)**

If $L_xS_\hbar E_\tau\left[f(x,\hbar,\tau)\right]=T(\sigma,\rho,\delta)$

Then $L_xS_\hbar E_\tau\left[e^{-\alpha\sigma x-\frac{\beta\hbar}{\rho}-\frac{\kappa\tau}{\delta}}f(x,\hbar,\tau)\right]=T(1+\alpha,1+\beta,1+\kappa)$

Proof: let

$$L_xS_\hbar E_\tau\left[f(x,\hbar,\tau):(\sigma,\rho,\delta)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

Then

$$L_xS_\hbar E_\tau\left[e^{-\alpha\sigma x-\frac{\beta\hbar}{\rho}-\frac{\kappa\tau}{\delta}}f(x,\hbar,\tau)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}\left[e^{-\alpha\sigma x-\frac{\beta\hbar}{\rho}-\frac{\kappa\tau}{\delta}}\right]f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(1+\alpha)\sigma x-\frac{(1+\beta)\hbar}{\rho}-\frac{(1+\kappa)\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

$$=\int_0^\infty e^{-(1+\alpha)\sigma x}\left\{\frac{\delta}{\rho}\int_0^\infty\int_0^\infty e^{-\frac{(1+\beta)\hbar}{\rho}-\frac{(1+\kappa)\tau}{\delta}}f(x,\hbar,\tau)d\hbar\,d\tau\right\}dx$$

$$=\int_0^\infty e^{-(1+\alpha)\sigma x}f(x,1+\beta,1+\kappa)dx=T(1+\alpha,1+\beta,1+\kappa)$$

Theorem 3: if $f(x,\hbar,\tau)$ is periodic function of periods a , b , k
$f(x+\alpha,\hbar+\beta,\tau+\kappa)=f(x,\hbar,\tau)$ and if $L_xS_\hbar E_\tau[f(x,\hbar,\tau)]$ exits
then

$$L_xS_\hbar E_\tau[f(x,\hbar,\tau)]=T(\sigma,\rho,\delta)=\frac{\delta}{\rho\left[1-e^{-\alpha\sigma-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

Proof: let

$$L_xS_\hbar E_\tau[f(x,\hbar,\tau)]=T(\sigma,\rho,\delta)$$

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

$$=\frac{\delta}{\rho}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau+\frac{\delta}{\rho}\int_\alpha^\infty\int_\beta^\infty\int_\kappa^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

Putting $x=u+\alpha$, $\hbar=v+\beta,\tau=w+\kappa$ in second triple integral

$$=\frac{\delta}{\rho}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau+$$

$$\frac{\delta}{\rho}\int_\alpha^\infty\int_\beta^\infty\int_\kappa^\infty e^{-(u+\alpha)\sigma-\frac{(v+\beta)}{\rho}-\frac{(w+\kappa)}{\delta}}f(u+\alpha,v+\beta,w+\kappa)dudvdw$$

$$T(\sigma,\rho,\delta)=\frac{\delta}{\rho}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau+$$

$$\frac{\delta}{\rho}\left[e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]\int_\alpha^\infty\int_\beta^\infty\int_\kappa^\infty e^{-\sigma u-\frac{v}{\rho}-\frac{w}{\delta}}f(u+\alpha,v+\beta,w+\kappa)dudvdw$$

$$=\frac{\delta}{\rho}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau+\left[e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]T(\sigma,\rho,\delta)$$

$$\therefore\frac{\delta}{\rho}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau=T(\sigma,\rho,\delta)-\left[e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]T(\sigma,\rho,\delta)$$

$$T(\sigma,\rho,\delta)=\frac{\delta}{\rho\left[1-e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]}\int_0^\alpha\int_0^\beta\int_0^\kappa e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

Theorem 4:
If, $L_xS_\hbar E_\tau[f(x,\hbar,\tau)]=T(\sigma,\rho,\delta)$ then,

$$L_xS_\hbar E_\tau[f(x-\alpha,\hbar-\beta,\tau-\kappa)H(x-\alpha,\hbar-\beta,\tau-\kappa)]=e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}T(\sigma,\rho,\delta)$$

Where, $H(x,\hbar,\tau)$ is the Heaviside unit step function defined by
And, $\tau>\kappa$ $H(x-\alpha,\hbar-\beta,\tau-\kappa)=1$ when $x>\alpha$, $\hbar>\beta$, And
$H(x-\alpha,\hbar-\beta,\tau-\kappa)=0$ when $x<\alpha,\hbar<\beta$ and $\tau<\kappa$

Proof:

Let $L_xS_\hbar E_\tau[f(x,\hbar,\tau)]=T(\sigma,\rho,\delta)=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f(x,\hbar,\tau)dx\,d\hbar\,d\tau$

Then

$$L_x S_\hbar E_\tau\left[f\,(x-\alpha,\hbar-\beta,\tau-\kappa)H\,(x-\alpha,\hbar-\beta,\tau-\kappa)\right]=$$

$$\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}} f\,(x-\alpha,\hbar-\beta,\tau-\kappa)H\,(x-\alpha,\hbar-\beta,\tau-\kappa)dx\,d\hbar\,d\tau=$$

$$\frac{\delta}{\rho}\int_\alpha^\infty\int_\beta^\infty\int_\kappa^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}} f\,(x-\alpha,\hbar-\beta,\tau-\kappa)dx\,d\hbar\,d\tau$$

by putting, $x-\alpha=u$, $\hbar-\beta=v$, $\tau-\kappa=w$ we get,

$$=\left[e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma u-\frac{v}{\rho}-\frac{w}{\delta}} f(u,v,w)dudvdw=\left[e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}\right]T(\sigma,\rho,\delta)$$

Theorem 5: (Convolution theorem)

If $L_x S_\hbar E_\tau\left[F(x,\hbar,\tau)\right]=f\,(\sigma,\rho,\delta)$, $L_x S_\hbar E_\tau\left[G(x,\hbar,\tau)\right]=g\,(\sigma,\rho,\delta)$

And, $L_x S_\hbar E_\tau\left[\left(F***G\right)(x,\hbar,\tau)\right]=\int_0^x\int_0^\hbar\int_0^\tau F(x-\alpha,\hbar-\beta,\tau-\kappa)G(\alpha,\beta,\kappa)dx\,d\hbar\,d\tau$ then,

$$L_x S_\hbar E_\tau\left[\left(F***G\right)(x,\hbar,\tau)\right]=L_x S_\hbar E_\tau\left[F(x,\hbar,\tau)\right].L_x S_\hbar E_\tau\left[G(x,\hbar,\tau)\right]=\frac{\rho}{\delta}f\,(\sigma,\rho,\delta).g\,(\sigma,\rho,\delta)$$

Proof:

From the definition we have,

$$L_x S_\hbar E_\tau\left[\left(F***G\right)(x,\hbar,\tau)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}\left(F***G\right)(x,\hbar,\tau)dx\,d\hbar\,d\tau$$

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{\frac{-x}{\sigma}-\frac{y}{\rho}-\frac{t}{\delta}}\left[\frac{\delta}{\rho}\int_0^x\int_0^\hbar\int_0^\tau F(x-\alpha,\hbar-\beta,\tau-\kappa)G(\alpha,\beta,\kappa)d\alpha\,d\beta\,d\kappa\right]dx\,d\hbar\,d\tau$$

using the Heaviside unit step function, to get,

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty G(\alpha,\beta,\kappa)d\alpha\,d\beta\,d\kappa\left[\frac{\delta}{\rho}\int_0^x\int_0^\hbar\int_0^\tau e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}F(x-\alpha,\hbar-\beta,\tau-\kappa)H\,(x-\alpha,\hbar-\beta,\tau-\kappa)\right]dxdyd\tau$$

And by using theorem 4 , we have:

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}f(\sigma,\rho,\delta)G(\alpha,\beta,\kappa)d\alpha\,d\beta\,d\kappa$$

$$=\frac{\delta}{\rho}f(\sigma,\rho,\delta)\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma\alpha-\frac{\beta}{\rho}-\frac{\kappa}{\delta}}G(\alpha,\beta,\kappa)d\alpha\,d\beta\,d\kappa=f(\sigma,\rho,\delta).g(\sigma,\rho,\delta)$$

Property 6:

If $f\,(x,\hbar,\tau)=\frac{\partial^3 f\,(x,\hbar,\tau)}{\partial x\,\partial\hbar\partial\tau}$ then,

$$L_x S_\hbar E_\tau\left[\frac{\partial^3 f\,(x,\hbar,\tau)}{\partial x\,\partial\hbar\partial\tau}:(\sigma,\rho,\delta)\right]=-\frac{\delta}{\rho}f\,(0,0,0)+\frac{\sigma\delta}{\rho}L_x\,(x,0,0)+\frac{\delta}{\rho}S_\hbar(0,\hbar,0)-$$

$$\frac{\sigma\delta}{\rho^2}L_x S_\hbar(x,\hbar,0)+\frac{1}{\rho^2\delta}E_\tau(0,0,\tau)-\frac{\sigma}{\rho^2}L_x E_\tau(x,0,\tau)-\frac{1}{\rho^2\delta}S_\hbar E_\tau(0,\hbar,\tau)+\frac{\sigma}{\rho^2\delta}L_x S_\hbar E_\tau(x,\hbar,\tau) \quad (2)$$

Proof: let

$$L_x S_\hbar E_\tau\left[\frac{\partial^3 f(x,\hbar,\tau)}{\partial x\partial\hbar\partial\tau}:(\sigma,\rho,\delta)\right]=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}f_{x\hbar\tau}(x,\hbar,\tau)dxd\hbar d\tau$$

$$=\frac{\delta}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}}\left[\int_0^\infty e^{\frac{-\tau}{\delta}}f_{x\hbar\tau}(x,\hbar,\tau)d\tau\right]dxd\hbar$$

$$=\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}}\left[\delta\left[e^{\frac{-\tau}{\delta}}f_{x\hbar}(x,\hbar,\tau)\Big|_0^\infty+\frac{1}{\delta}\int_0^\infty e^{\frac{-\tau}{\delta}}f_{x\hbar}(x,\hbar,\tau)d\tau\right]\right]dxd\hbar$$

$$=\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}}\left[\delta\left[\left(0-f_{x\hbar}(x,\hbar,0)\right)+\frac{1}{\delta}\int_0^\infty e^{-\frac{\tau}{\delta}}f_{x\hbar}(x,\hbar,\tau)d\tau\right]\right]dxd\hbar$$

$$=-\frac{\delta}{\rho}\int_0^\infty e^{-\sigma x}\left[\int_0^\infty e^{-\frac{\hbar}{\rho}}f_{x\hbar}(x,\hbar,0)d\hbar\right]+\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\tau}{\delta}}\left[\int_0^\infty e^{-\frac{\hbar}{\rho}}f_{x\hbar}(x,\hbar,\tau)d\hbar\right]dxd\tau$$

$$=-\frac{\delta}{\rho}\int_0^\infty e^{-\sigma x}\left[\left(0-f_x(x,0,0)\right)+\frac{1}{\rho}\int_0^\infty e^{-\frac{\hbar}{\rho}}f_x(x,\hbar,0)dx\right]$$

$$+\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\tau}{\delta}}\left[\int_0^\infty e^{-\frac{\hbar}{\rho}}f_{x\hbar}(x,\hbar,\tau)d\hbar\right]dxd\tau$$

$$=\frac{\delta}{\rho}\int_0^\infty e^{-\sigma x}f_x(x,0,0)dx-\frac{\delta}{\rho^2}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\hbar}{\rho}}f_x(x,\hbar,0)dxd\hbar$$

$$+\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\sigma x-\frac{\tau}{\delta}}\left[\left(0-f_x(x,0,\tau)\right)+\frac{1}{\rho}\int_0^\infty e^{-\frac{\hbar}{\rho}}f_x(x,\hbar,\tau)dx\right]d\hbar$$

$$=\frac{\delta}{\rho}\left[\left(0-f(0,0,0)\right)+\sigma\int_0^\infty e^{-\sigma x}f(x,0,0)dx\right]-\frac{\delta}{\rho^2}\int_0^\infty e^{-\frac{\hbar}{\rho}}\left[\left(0-f(0,\hbar,0)\right)+\sigma\int_0^\infty e^{-\sigma x}f(x,\hbar,0)dx\right]d\hbar-$$

$$\frac{1}{\rho}\int_0^\infty e^{-\frac{\tau}{\delta}}\left[\int_0^\infty e^{-\sigma x}f_x(x,0,\tau)dx\right]d\tau+\frac{1}{\rho^2}\int_0^\infty\int_0^\infty e^{-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}\left[\int_0^\infty e^{-\sigma x}f_x(x,\hbar,\tau)dx\right]d\hbar d\tau$$

$$=-\frac{\delta}{\rho}f(0,0,0)+\frac{\sigma\delta}{\rho}L_x(x,0,0)+\frac{\delta}{\rho}S_\hbar(0,\rho,0)-\frac{\sigma\delta}{\rho}L_xS_\hbar(x,\hbar,0)$$

$$-\frac{1}{\rho}\int_0^\infty e^{-\frac{\tau}{\delta}}\left[\left(0-f(0,0,\tau)\right)+\sigma\int_0^\infty e^{-\sigma x}f(x,0,\tau)dx\right]d\tau+$$

$$\frac{1}{\rho^2}\int_0^\infty\int_0^\infty e^{-\frac{\hbar}{\rho}-\frac{\tau}{\delta}}\left[\left(0-f(0,\hbar,\tau)\right)+\sigma\int_0^\infty e^{-\sigma x}f(x,\hbar,\tau)dx\right]d\hbar d\tau$$

$$=-\frac{\delta}{\rho}f(0,0,0)+\frac{\sigma\delta}{\rho}L_x(x,0,0)+\frac{\delta}{\rho}S_\hbar(0,\hbar,0)-\frac{\sigma\delta}{\rho}L_xS_\hbar(x,\hbar,0)+\frac{1}{\rho\delta}E_\tau(0,0,\tau)-\frac{\sigma}{\rho\delta}L_xE_\tau(x,0,\tau)$$

$$-\frac{1}{\rho\delta}S_\hbar E_\tau(0,\hbar,\tau)+\frac{\sigma}{\rho\delta}L_xS_\hbar E_\tau(x,\hbar,\tau)$$

3. Application of Triple Laplace – Sumudu - Elzaki

Example 3.1: consider the following third-order partial differential equation,

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial t}+u(x,y,t)=0 \qquad (3)$$

With the initial conditions:

$$u(x,y,0)=e^{x+y},\ u(x,0,t)=e^{x-t},\ u(0,y,t)=e^{y-t},\ u(x,0,0)=e^{x},$$
$$u(0,y,0)=e^{y},\ u(0,0,t)=e^{-t},\ u(0,0,0)=1 \quad (4)$$

Solution: Using the triple Laplace – Sumudu - Elzaki transform of both sides to Eq. (3), to obtain,

$$L_x S_\hbar E_\tau\left[\frac{\partial^3}{\partial x\,\partial\hbar\partial\tau}u(x,\hbar,\tau)\right]+L_x S_\hbar E_\tau\left[u(x,\hbar,\tau)\right]=L_x S_\hbar E_\tau\left[0\right] (5)$$

Also we considering the double transform of Eq. (4) to get,

$$T(\sigma,\rho,0)=\frac{1}{[\sigma-1][1-\rho]},\ T(\sigma,0,\delta)=\frac{\delta^2}{[\sigma-1][1+\delta]},T(0,\rho,\delta)=\frac{\delta^2}{[1-\rho][1+\delta]}$$

$$T(\sigma,0,0)=\frac{1}{\sigma-1},\ T(0,\rho,0)=\frac{1}{1-\rho},\ T(0,0,\delta)=\frac{\delta^2}{1+\delta},T(0,0,0)=1 \quad (6)$$

Substituting Eq. (6) into Eq. (5), to get

$$\left[\frac{\sigma}{\rho\delta}+1\right]T(\sigma,\rho,\delta)=\frac{\delta}{\rho}\left[\frac{\sigma}{[\sigma-1][1-\rho]}\right]-\frac{\delta}{\rho}\left[\frac{1}{1+\delta}\right]+\frac{\delta}{\rho}\left[\frac{\sigma}{[\sigma-1][1+\delta]}\right]-\frac{\delta}{\rho}\left[\frac{1}{1-\rho}\right]+$$
$$\frac{\delta}{\rho}\left[\frac{1}{[1-\rho][1+\delta]}\right]-\frac{\delta}{\rho}\left[\frac{\sigma}{\sigma-1}\right]+\frac{\delta}{\rho}\left[\frac{[\sigma-1][1-\rho][1+\delta]}{[\sigma-1][1-\rho][1+\delta]}\right]$$

$$T(\sigma,\rho,\delta)\left[\frac{\sigma+\rho\delta}{\rho\delta}\right]=\frac{\delta}{\rho}\left[\frac{\sigma+\rho\delta}{[\sigma-1][1-\rho][1+\delta]}\right]$$

$$T(\sigma,\rho,\delta)=\frac{\delta^2}{[\sigma-1][1-\rho][1+\delta]} (7)$$

Applying the inverse triple Laplace – Sumudu - Elzaki transform of equation (7) to get,

$$u(x,y,t)=e^{x+y-t}$$

Example 3.2: consider the following third-order partial differential equation

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial t}+u(x,y,t)=\cos x\,\cos y\,\cos(-t)+\sin x\,\sin y\,\sin(-t) \quad (8)$$

With the initial conditions:

$$u(x,y,0)=\cos x\,\cos y,\ u(x,0,t)=\cos x\,\cos(-t), u(0,y,t)=\cos y\,\cos(-t),$$
$$u(x,0,0)=\cos x,\ u(0,y,0)=\cos y,\ u(0,0,t)=\cos(-t),\ u(0,0,0)=1 \qquad (9)$$

Solution: taking the triple Laplace – Sumudu - Elzaki transform of both sides to Eq. (8), to obtain,

$$L_x S_\hbar E_\tau\left[\frac{\partial^3}{\partial x\,\partial\hbar\partial\tau}u(x,\hbar,\tau)\right]+L_x S_\hbar E_\tau\left[u(x,\hbar,\tau)\right]=L_x S_\hbar E_\tau\left[\cos x\,\cos y\,\cos(-t)+\sin x\,\sin y\,\sin(-t)\right] \quad (10)$$

And by applying the double transform of Eq. (9) we get,

$$T(\sigma,\rho,0)=\frac{\sigma}{\left[1+\sigma^2\right]\left[1+\rho^2\right]},\ T(\sigma,0,\delta)=\frac{\sigma\delta^2}{\left[1+\sigma^2\right]\left[1-\delta^2\right]},\ T(0,\rho,\delta)=\frac{\delta^2}{\left[1+\rho^2\right]\left[1-\delta^2\right]}$$

$$T(\sigma,0,0)=\frac{\sigma}{1+\sigma^2},\ T(0,\rho,0)=\frac{1}{1+\rho^2},\ T(0,0,\delta)=\frac{\delta^2}{1-\delta^2},\ T(0,0,0)=1 \quad (11)$$

Substituting Eq. (11) into Eq. (10), to get

$$\left[\frac{\sigma+\rho\delta}{\rho\delta}\right]T(\sigma,\rho,\delta)=\frac{\delta}{\rho}\left[\frac{\sigma\rho\delta+\rho^2\delta^2-\rho^2\delta^2+\sigma^2}{\left[1+\sigma^2\right]\left[1+\rho^2\right]\left[1-\delta^2\right]}\right]=\frac{\delta}{\rho}\left[\frac{\sigma\left(\rho\delta+\sigma\right)}{\left[1+\sigma^2\right]\left[1+\rho^2\right]\left[1-\delta^2\right]}\right]$$

$$T(\sigma,\rho,\delta)=\frac{\delta}{\rho}\left[\frac{\sigma\left(\rho\delta+\sigma\right)}{\left[1+\sigma^2\right]\left[1+\rho^2\right]\left[1-\delta^2\right]}\right]\left[\frac{\rho\delta}{\sigma+\rho\delta}\right]$$

$$T(\sigma,\rho,\delta)=\frac{\sigma\delta^2}{\left[1+\sigma^2\right]\left[1+\rho^2\right]\left[1-\delta^2\right]} \quad (12)$$

Appling the inverse triple Laplace – Sumudu – Elzaki transform of both sides of equation (12), to find the exact solution in the form,

$$u(x,y,t)=\cos x\ \cos y\ \cos(-t)$$

4. Conclusions

This work dealt with the Laplace transform and some definitions, important theorems and properties related to it, then after that we use this transform to find the solutions for three dimensions partial differential equations of third-order under the initial conditions, the triple Laplace – Sumudu – Elzaki transform study succeeded in achieving solutions.

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