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Generalizations of Pythagoras Theorem to Polygons - Part 2

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ABSTRACT

The celebrated Greek mathematician Pythagoras, (circa 569 B.C.) re-discovered a legendary result, now known after him, as Pythagoras theorem:

Sum of squares of two mutually perpendicular sides in a right triangle equals the square of the hypotenuse, i.e. $b^2 + p^2 = h^2$, where b, p, h are the lengths of base, perpendicular and hypotenuse of the triangle. In the previous paper [3] several results were derived for quadrilaterals comprising of two right triangles expressing the left-hand expression, in above equation, as the sum of squares of 3 integers becoming the square of some fourth integer. The first fifteen Sections in [3] dealt with the direct sum of squares of some positive integers making the square of a fourth integer. Special choices for $d = c + k$, where k ranged over the set of natural numbers from 1 to 17 were discussed. Presently, we continue with the discussion for further integral powers of $k = 18$ onward.

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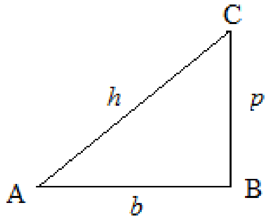
1. Introduction

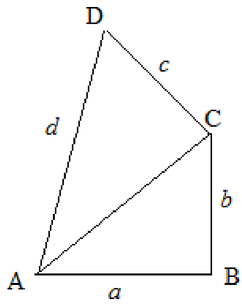


The celebrated Greek mathematician (more precisely a geometer) Pythagoras, born on the Island of Samos (Greece) during the period about 569 B.C., re-discovered a legendary result, now known after him, as Pythagoras theorem:

Sum of squares of two mutually perpendicular sides in a right triangle equals the square of the hypotenuse, i.e. $b^2 + p^2 = h^2$, where b, p, h are the lengths of base, perpendicular and hypotenuse of the triangle. In the previous paper several results were derived by extending above result to quadrilaterals comprising of two right triangles so that the left-hand expression in above equation had the sum of squares of 3 integers equaling with the square of the fourth integer. In terms of symbols, we found solutions of the quadratic equation

$$a^2 + b^2 + c^2 = d^2.$$

Interpreting the result geometrically the integers a, b, c, d satisfying above relation gave the lengths of consecutive sides AB, BC, CD and DA respectively of a quadrilateral ABCD composed of 2 right triangles ABC and ACD with right angles at their vertices B and C. The first fifteen Sections in [3] dealt with the direct sum of squares of some positive integers making the square of a fourth integer. Special choices for $d = c + k$, where k ranged over the set of natural numbers from 1 to 17 were discussed. Presently, we continue with the discussion for further integral powers of $k = 18$ onward.



2. Identities of the type $a^2 + n^2 + b^2 = (b + 18)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 18^2) / 36 = (a^2 + n^2) / 36 - 9. \quad (1.1)$$

To get integral values of b the sum $a^2 + n^2$ must be divisible by 36, which is possible only when both a and n must be integral multiples of 6: say $a = 6\,l$ and $n = 6\,m$, where both l, m are integers making

$$(a^2 + n^2) / 36 = l^2 + m^2 \Rightarrow b = l^2 + m^2 - 9 \quad (1.2)$$

Thus, the different integral values of l, m yield different integral values to b and, therefore, to b + 18. The following tables depict such choices and yield the corresponding identities of the desired form:

Theorem 1.1. For $l = 1$, so that $a = 6$, there hold the identities for integral values of n, b. etc.

m	n	b	$b + 18$	Identity	Refer [3]
1	6	– 7	11	$6^2 + 6^2 + (-7)^2 = 11^2$	Th. 5.3
2	12	– 4	14	$6^2 + 12^2 + (-4)^2 = 14^2$	Th. 3.2
				i.e. $3^2 + 6^2 + (-2)^2 = 7^2$	Th. 2.2
3	18	1	19	$6^2 + 18^2 + 1^2 = 19^2$	Th. 2.1
4	24	8	26	$6^2 + 24^2 + 8^2 = 26^2$	Th. 3.4
				i.e. $3^2 + 12^2 + (-4)^2 = 13^2$	Th. 2.3
5	30	17	35	$6^2 + 30^2 + 17^2 = 35^2$	Th. 6.6
6	36	28	46	$6^2 + 36^2 + 28^2 = 46^2$	Th. 11.3
				i.e. $3^2 + 18^2 + 14^2 = 23^2$	Th. 6.3
7	42	41	59	$6^2 + 42^2 + 41^2 = 59^2$	Th. 18.6
8	48	56	74	$6^2 + 48^2 + 56^2 = 74^2$, i.e. $3^2 + 24^2 + 28^2 = 37^2$	Th. 10.1
9	54	73	91	$6^2 + 54^2 + 73^2 = 91^2$	
10	60	92	110	$6^2 + 60^2 + 92^2 = 110^2$	
				i.e. $3^2 + 30^2 + 46^2 = 55^2$	Th. 10. 1

Theorem 1.2. For $l = 2$ so that $a = 12$, there hold the identities for integral values of n, b. etc.

m	*n*	*b*	*b* + 18	Identity	Refer [3]
2	12	– 1	17	$12^2 + 12^2 + (-1)^2 = 17^2$	Th. 6.1
3	18	4	22	$12^2 + 18^2 + 4^2 = 22^2$	Th. 5.2
				i.e. $6^2 + 9^2 + 2^2 = 11^2$	Th. 3.1
4	24	11	29	$12^2 + 24^2 + 11^2 = 29^2$	
5	30	20	38	$12^2 + 30^2 + 20^2 = 38^2$	Th. 9.3
				i.e. $6^2 + 15^2 + 10^2 = 19^2$	Th. 5.3
6	36	31	49	$12^2 + 36^2 + 31^2 = 49^2$	Th. 14.12
7	42	44	62	$12^2 + 42^2 + 44^2 = 62^2$	
				i.e. $6^2 + 21^2 + 22^2 = 31^2$	Th. 10.2
8	48	59	77	$12^2 + 48^2 + 59^2 = 77^2$	
9	54	76	94	$12^2 + 54^2 + 76^2 = 94^2$	
				i.e. $6^2 + 27^2 + 38^2 = 47^2$	Th. 10.2
10	60	95	113	$12^2 + 60^2 + 95^2 = 113^2$	

It may be noted that we have dropped the case $a = 12$, $n = 6$ leading to same value of $b = -4$ already considered in Theorem 1.1.

Theorem 1.3. Taking $l = 3$ so that $a = 18$, and $b = m^2$ by Eq. (1.2). Hence, there hold the following identities for integral values of *n*, *b*. etc.

m	*n*	*b*	*b* + 18	Identity	Refer [3]
3	18	9	27	$18^2 + 18^2 + 9^2 = 27^2$	Th. 10.3
				i.e. $2^2 + 2^2 + 1^2 = 3^2$	Th. 2.1
4	24	16	34	$18^2 + 24^2 + 16^2 = 34^2$	
				i.e. $9^2 + 12^2 + 8^2 = 17^2$	Th. 6.8
5	30	25	43	$18^2 + 30^2 + 25^2 = 43^2$	
6	36	36	54	$18^2 + 36^2 + 36^2 = 54^2$	
				i.e. $1^2 + 2^2 + 2^2 = 3^2$	Th. 2.1
7	42	49	67	$18^2 + 42^2 + 49^2 = 67^2$	
8	48	64	82	$18^2 + 48^2 + 64^2 = 82^2$	
				i.e. $9^2 + 24^2 + 32^2 = 41^2$	Th. 10.3
9	54	81	99	$18^2 + 54^2 + 81^2 = 99^2$, i.e. $2^2 + 6^2 + 9^2 = 11^2$	Th. 3.1
10	60	100	118	$18^2 + 60^2 + 100^2 = 118^2$, i.e. $9^2 + 30^2 + 50^2 = 59^2$	Th. 10. 3

Here also, the pairs $(a, n) = (18, 6)$ and $(18, 12)$ have been dropped as they are already considered in Theorems 1.1 1nd 1.2 respectively.

Theorem 1.4. Taking l = 4 so that a = 24, and $b = m^2 + 7$ by Eq. (1.2). Hence, there hold the following identities for integral values of n, b.

m	n	b	b + 18	Identity	Refer [3]
4	24	23	41	$24^2 + 24^2 + 23^2 = 41^2$	
5	30	32	50	$24^2 + 30^2 + 32^2 = 50^2$	
				i.e. $12^2 + 15^2 + 16^2 = 25^2$	Th. 10.4
6	36	43	61	$24^2 + 36^2 + 43^2 = 61^2$	
7	42	56	74	$24^2 + 42^2 + 56^2 = 74^2$	
				i.e. $12^2 + 21^2 + 28^2 = 37^2$	Th. 10.4
8	48	71	89	$24^2 + 48^2 + 71^2 = 89^2$	
9	54	88	106	$24^2 + 54^2 + 88^2 = 106^2$	
				i.e. $12^2 + 27^2 + 44^2 = 53^2$	Th. 10.4
10	60	107	125	$24^2 + 60^2 + 107^2 = 125^2$	

Here also, the values of a and n in the pairs (a, n) = (24, 6), (24, 12) and (24, 18) have been dropped as they are already included in previous Theorems.

Theorem 1.5. Taking l = 5 so that a = 30, and $b = m^2 + 16$ by Eq. (1.2). Hence, there hold the following identities for integral values of n, b.

m	n	b	b + 18	Identity	Refer [3]
5	30	41	59	$30^2 + 30^2 + 41^2 = 59^2$	
6	36	52	70	$30^2 + 36^2 + 52^2 = 70^2$	
				i.e. $15^2 + 18^2 + 26^2 = 35^2$	Th. 10.4
7	42	65	83	$30^2 + 42^2 + 65^2 = 83^2$	
8	48	80	98	$30^2 + 48^2 + 80^2 = 98^2$	
				i.e. $15^2 + 24^2 + 40^2 = 49^2$	Th. 10.4
9	54	97	115	$30^2 + 54^2 + 97^2 = 115^2$	
10	60	116	134	$30^2 + 60^2 + 116^2 = 134^2$	
				i.e. $15^2 + 30^2 + 58^2 = 67^2$	Th. 10.4

3. Identities of the type $a^2 + n^2 + b^2 = (b + 19)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 19^2) / 38 = \{ (a - 19).(a + 19) + n^2 \} / 38 = (a^2 + n^2 - 19) / 38 - 9 \qquad (3.1)$$

To get integral values of b the sum $a^2 + n^2 - 19$ must be divisible by 38, i.e. it should be an integral multiple of 38. The possibility for vanishing of the sum $a^2 + n^2 - 19$ is nil as no integral values of a make n integer. Same is the situation for $a^2 + n^2 = 38k + 19$ for some integer k. We explore the following situations arising out from Eq. (3.1):

Table 3.1. Conclusively, b becomes integer when a takes *odd* integral multiples of 19 together with corresponding n as *even* integral multiples of 19 and vice-versa, i.e. when a assuming even but n taking odd integral multiples of 19. Thus, we have the following theorems. Symmetry of a and n in the sum $a^2 + n^2$ is also noticeable hence, the pairs (a, n) and (n, a) shall yield the same values of b.

a	b	Remark
19	$n^2 / 38$	b is integer when n is an even integral multiple of 19.
38	$(n^2 + 19 \times 57) / 38 = (n^2 + 19) / 38 + 28$	b is integer when n is an odd integral multiple of 19.
57	$(n^2 + 38 \times 76) / 38 = n^2 / 38 + 76$	As for $a = 19$.
76	$(n^2 + 57 \times 95) / 38 = (n^2 + 19) / 38 + 142$	As for $a = 38$.
95	$(n^2 + 76 \times 114) / 38 = n^2 / 38 + 228$	As for $a = 19$.
114	$(n^2 + 95 \times 133) / 38 = (n^2 + 19) / 38 + 332$	As for $a = 38$.
133	$(n^2 + 114 \times 152) / 38 = n^2 / 38 + 456$	As for $a = 19$.
152	$(n^2 + 133 \times 171) / 38 = (n^2 + 19) / 38 + 598$	As for $a = 38$.
171	$(n^2 + 152 \times 190) / 38 = n^2 / 38 + 760$	As for $a = 19$.
190	$(n^2 + 171 \times 209) / 38 = (n^2 + 19) / 38 + 940$	As for $a = 38$.

Theorem 3.1. For $a = 19$, $b = n^2 / 38$, by Table 3.1. Hence, $n = 38, 76, 114, 152$, etc. yield the following identities.

n	b	$b + 19$	Identity	Refer [3]
38	38	57	$19^2 + 38^2 + 38^2 = 57^2$, i.e. $1^2 + 2^2 + 2^2 = 3^2$	Th. 2.1
76	152	171	$19^2 + 76^2 + 152^2 = 171^2$, i.e. $1^2 + 4^2 + 8^2 = 9^2$	″
114	342	361	$19^2 + 114^2 + 342^2 = 361^2$, i.e. $1^2 + 6^2 + 18^2 = 19^2$	″
152	608	627	$19^2 + 152^2 + 608^2 = 627^2$, i.e. $1^2 + 8^2 + 32^2 = 33^2$	″
190	950	969	$19^2 + 190^2 + 950^2 = 969^2$, i.e. $1^2 + 10^2 + 50^2 = 51^2$	″
228	1368	1387	$19^2 + 228^2 + 1368^2 = 1387^2$, i.e. $1^2 + 12^2 + 72^2 = 73^2$	″
266	1862	1881	$19^2 + 266^2 + 1862^2 = 1881^2$, i.e. $1^2 + 14^2 + 98^2 = 99^2$	″
304	2432	2451	$19^2 + 304^2 + 2432^2 = 2451^2$, i.e. $1^2 + 16^2 + 128^2 = 129^2$	″
342	3078	3097	$19^2 + 342^2 + 3078^2 = 3097^2$, i.e. $1^2 + 18^2 + 162^2 = 163^2$	″
380	3800	3819	$19^2 + 380^2 + 3800^2 = 3819^2$, i.e. $1^2 + 20^2 + 200^2 = 201^2$	″

Theorem 3.2. For a = 38, b = $(n^2 + 19)/38 + 28$, by Table 3.1. Hence, for n = 57, 95, 133 etc., we derive the following identities.

n	b	$b + 19$	Identity	Refer [3]
57	114	133	$38^2 + 57^2 + 114^2 = 133^2$	
			i.e. $2^2 + 3^2 + 6^2 = 7^2$	Th. 2.2
95	266	285	$38^2 + 95^2 + 266^2 = 285^2$	
			i.e. $2^2 + 5^2 + 14^2 = 15^2$	Th. 2.2
133	494	513	$38^2 + 133^2 + 494^2 = 513^2$	
			i.e. $2^2 + 7^2 + 26^2 = 27^2$	Th. 2.2
171	798	817	$38^2 + 171^2 + 798^2 = 817^2$	
			i.e. $2^2 + 9^2 + 42^2 = 43^2$	Th. 2.2
209	1178	1197	$38^2 + 209^2 + 1178^2 = 1197^2$	
			i.e. $2^2 + 11^2 + 62^2 = 63^2$	Th. 2.2
247	1634	1653	$38^2 + 247^2 + 1634^2 = 1653^2$	
			i.e. $2^2 + 13^2 + 86^2 = 87^2$	Th. 2.2
285	2166	2185	$38^2 + 285^2 + 2166^2 = 2185^2$	
			i.e. $2^2 + 15^2 + 114^2 = 115^2$	Th. 2.2
323	2774	2793	$38^2 + 323^2 + 2774^2 = 2793^2$	
			i.e. $2^2 + 17^2 + 146^2 = 147^2$	Th. 2.2
361	3458	3477	$38^2 + 361^2 + 3458^2 = 3477^2$	
			i.e. $2^2 + 19^2 + 182^2 = 183^2$	Th. 2.2

The choice of pair (a, n) = (38, 19) is dropped as (a, n) = (19, 38) yielding the same value of b is already considered in Theorem 3.1.

Theorem 3.3. For a = 57, $b = n^2/38 + 76$, by Table 3.1. Hence, n = 76, 114, 152, 190, etc. yield the following identities.

n	b	$b + 19$	Identity	Refer [3]
76	228	247	$57^2 + 76^2 + 228^2 = 247^2$, i.e. $3^2 + 4^2 + 12^2 = 13^2$	Th. 2.1
114	418	437	$57^2 + 114^2 + 418^2 = 437^2$, i.e. $3^2 + 6^2 + 22^2 = 23^2$	″
152	684	703	$57^2 + 152^2 + 684^2 = 703^2$, i.e. $3^2 + 8^2 + 36^2 = 37^2$	″
190	1026	1045	$57^2 + 190^2 + 1026^2 = 1045^2$, i.e. $3^2 + 10^2 + 54^2 = 55^2$	″
228	1444	1463	$57^2 + 228^2 + 1444^2 = 1463^2$, i.e. $3^2 + 12^2 + 76^2 = 77^2$	″
266	1938	1957	$57^2 + 266^2 + 1938^2 = 1957^2$, i.e. $3^2 + 14^2 + 102^2 = 103^2$	″
304	2508	2527	$57^2 + 304^2 + 2508^2 = 2527^2$, i.e. $3^2 + 16^2 + 132^2 = 133^2$	″

342	3154	3173	$57^2 + 342^2 + 3154^2 = 3173^2$, i.e. $3^2 + 18^2 + 166^2 = 167^2$	″
380	3876	3895	$57^2 + 380^2 + 3876^2 = 3895^2$, i.e. $3^2 + 20^2 + 204^2 = 205^2$	″

The choice of pair (*a*, *n*) = (57, 38) is dropped as (*a*, *n*) = (38, 57) is already considered in Theorem 3.2.

Theorem 3.4. For $a = 76$, $b = (n^2 + 19) / 38 + 142$, by Table 3.1. Hence, n = 95, 133, 171, 209, 247, etc. yield the following identities.

n	*b*	*b* + 19	Identity	Refer [3]
95	380	399	$76^2 + 95^2 + 380^2 = 399^2$, i.e. $4^2 + 5^2 + 20^2 = 21^2$	Th. 2.2
133	608	627	$76^2 + 133^2 + 608^2 = 627^2$, i.e. $4^2 + 7^2 + 32^2 = 33^2$	
171	912	931	$76^2 + 171^2 + 912^2 = 931^2$, i.e. $4^2 + 9^2 + 48^2 = 49^2$	
209	1292	1311	$76^2 + 209^2 + 1292^2 = 1311^2$, i.e. $4^2 + 11^2 + 68^2 = 69^2$	
247	1748	1767	$76^2 + 247^2 + 1748^2 = 1767^2$, i.e. $4^2 + 13^2 + 92^2 = 93^2$	
323	2888	2907	$76^2 + 323^2 + 2888^2 = 2907^2$, i.e. $4^2 + 17^2 + 152^2 = 153^2$	
361	3572	3591	$76^2 + 361^2 + 3572^2 = 3591^2$, i.e. $4^2 + 19^2 + 188^2 = 189^2$	

The choices (*a*, *n*) = (76, 19) and (76, 57) are dropped as the corresponding results are already considered in Theorems 3.1 and 3.3 respectively.

4. Identities of the type $a^2 + n^2 + b^2 = (b + 20)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 20^2) / 40 = (a^2 + n^2) / 40 - 10. \qquad (3.1)$$

To get integral values of b the sum $a^2 + n^2$ must be divisible by 40. The choice of paired values (*a*, *n*) = (2*k*, 6*k*), where k is some integer make the possibility; for which Eq. (3.1) yields $b = k^2 - 10$. Thus, we have the following Theorem.

Theorem 4.1. For k = 1, 2, 3, … there hold the identities

a = 2*k*	*n* = 6*k*	*b*	*b* + 20	Identity	Refer [3]
2	6,	– 9	11	$2^2 + 6^2 + (-9)^2 = 11^2$	
4	12	– 6	14	$4^2 + 12^2 + (-6)^2 = 14^2$, i.e. $2^2 + 6^2 + (-3)^2 = 7^2$	Th. 2.1
6	18	– 1	19	$6^2 + 18^2 + (-1)^2 = 19^2$	″
8	24	6	26	$8^2 + 24^2 + 6^2 = 26^2$, i.e. $4^2 + 12^2 + 3^2 = 13^2$	″
10	30	15	35	$10^2 + 30^2 + 15^2 = 35^2$, i.e. $2^2 + 6^2 + 3^2 = 7^2$	″
12	36	26	46	$12^2 + 36^2 + 26^2 = 46^2$, i.e. $6^2 + 18^2 + 13^2 = 23^2$	″
14	42	39	59	$14^2 + 42^2 + 39^2 = 59^2$	
16	48	54	74	$16^2 + 48^2 + 54^2 = 74^2$, i.e. $8^2 + 24^2 + 27^2 = 37^2$	Th. 2.1
18	54	71	91	$18^2 + 54^2 + 71^2 = 91^2$	

20	60	90	110	$20^2 + 60^2 + 90^2 = 110^2$, i.e. $2^2 + 6^2 + 9^2 = 11^2$	Th. 2.1

5. Identities of the type $a^2 + n^2 + b^2 = (b + 21)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 21^2) / 42 = \{ (a - 21).(a + 21) + n^2 \} / 42 = (a^2 + n^2 - 21) / 42 - 10. \quad (5.1)$$

To get integral values of b the sum $a^2 + n^2 - 21$ must be divisible by 42. The possibility for vanishing of the sum $a^2 + n^2 - 21$ is nil as no integral values of a make n integer. Same is the situation for $a^2 + n^2 = 42k + 21$ for some integer k. We explore the following situations arising out from Eq. (5.1):

Table 5.1.

a	*b*	Remark
21	$n^2 / 42$	b is integer when n is an even integral multiple of 21.
42	$(n^2 + 21 \times 63) / 42 = (n^2 + 21) / 42 + 31$	b is integer when n is an odd integral multiple of 21.
63	$(n^2 + 42 \times 84) / 42 = n^2 / 42 + 84$	As for $a = 21$.
84	$(n^2 + 63 \times 105) / 42 = (n^2 + 21) / 42 + 157$	As for $a = 42$.
105	$(n^2 + 84 \times 126) / 42 = n^2 / 42 + 252$	As for $a = 21$.
126	$(n^2 + 105 \times 147) / 42 = (n^2 + 21) / 42 + 367$	As for $a = 42$.
147	$(n^2 + 126 \times 168) / 42 = n^2 / 42 + 504$	As for $a = 21$.
168	$(n^2 + 147 \times 189) / 42 = (n^2 + 21) / 42 + 661$	As for $a = 42$.
189	$(n^2 + 168 \times 210) / 42 = n^2 / 42 + 840$	As for $a = 21$.
210	$(n^2 + 189 \times 231) / 42 = (n^2 + 21) / 42 + 1039$	As for $a = 42$.

Conclusively, b becomes integer when a takes *odd* integral multiples of 21 together with corresponding n as *even* integral multiples of 21 and vice-versa, i.e. when a assuming even but n taking odd integral multiples of 21. Thus, we have the following theorems. Symmetry of a and n in the sum $a^2 + n^2$ is also noticeable hence, the pairs (a, n) and (n, a) shall yield the same values of b.

Theorem 5.1. For $a = 21$, $b = n^2 / 42$, by Table 5.1. Hence, $n = 42, 84, 126, 168$, etc. yield the following identities.

n	*b*	*b* + 21	Identity	Refer [3]
42	42	63	$21^2 + 42^2 + 42^2 = 63^2$	
			i.e. $1^2 + 2^2 + 2^2 = 3^2$	Th. 2.1
84	168	189	$21^2 + 84^2 + 168^2 = 189^2$	
			i.e. $1^2 + 4^2 + 8^2 = 9^2$	Th. 2.1
126	378	399	$21^2 + 126^2 + 378^2 = 399^2$	
			i.e. $1^2 + 6^2 + 18^2 = 19^2$	Th. 2.1

168	672	693	$21^2 + 168^2 + 672^2 = 693^2$	
			i.e. $1^2 + 8^2 + 32^2 = 33^2$	Th. 2.1
210	1050	1071	$21^2 + 210^2 + 1050^2 = 1071^2$	
			i.e. $1^2 + 10^2 + 50^2 = 51^2$	Th. 2.1
252	1512	1533	$21^2 + 252^2 + 1512^2 = 1533^2$	
			i.e. $1^2 + 12^2 + 72^2 = 73^2$	Th. 2.1
294	2058	2079	$21^2 + 294^2 + 2058^2 = 2079^2$	
			i.e. $1^2 + 14^2 + 98^2 = 99^2$	Th. 2.1
336	2688	2709	$21^2 + 336^2 + 2688^2 = 2709^2$	
			i.e. $1^2 + 16^2 + 128^2 = 129^2$	Th. 2.1
378	3402	3423	$21^2 + 378^2 + 3402^2 = 3423^2$	
			i.e. $1^2 + 18^2 + 162^2 = 163^2$	Th. 2.1
420	4200	4221	$21^2 + 420^2 + 4200^2 = 4221^2$	
			i.e. $1^2 + 20^2 + 200^2 = 201^2$	Th. 2.1

Theorem 5.2. For $a = 42$, $b = (n^2 + 21)/42 + 31$, by Table 5.1. Hence, for $n = 63, 105, 147$ etc., we derive the following identities.

n	*b*	*b* + 21	Identity	Refer [3]
63	126	147	$42^2 + 63^2 + 126^2 = 147^2$	
			i.e. $2^2 + 3^2 + 6^2 = 7^2$	Th. 2.2
105	294	315	$42^2 + 105^2 + 294^2 = 315^2$	
			i.e. $2^2 + 5^2 + 14^2 = 15^2$	Th. 2.2
147	546	567	$42^2 + 147^2 + 546^2 = 567^2$	
			i.e. $2^2 + 7^2 + 26^2 = 27^2$	Th. 2.2
189	882	903	$42^2 + 189^2 + 882^2 = 903^2$	
			i.e. $2^2 + 9^2 + 42^2 = 43^2$	Th. 2.2
231	1302	1323	$42^2 + 231^2 + 1302^2 = 1323^2$	
			i.e. $2^2 + 11^2 + 62^2 = 63^2$	Th. 2.2
273	1806	1827	$42^2 + 273^2 + 1806^2 = 1827^2$	
			i.e. $2^2 + 13^2 + 86^2 = 87^2$	Th. 2.2
315	2394	2415	$42^2 + 315^2 + 2394^2 = 2415^2$	
			i.e. $2^2 + 15^2 + 114^2 = 115^2$	Th. 2.2
357	3066	3087	$42^2 + 357^2 + 3066^2 = 3087^2$	
			i.e. $2^2 + 17^2 + 146^2 = 147^2$	Th. 2.2

399	3822	3843	$42^2 + 399^2 + 3822^2 = 3843^2$	
			i.e. $2^2 + 19^2 + 182^2 = 183^2$	Th. 2.2

The choice of pair (a, n) = (42, 21) is dropped as (a, n) = (21, 42) yielding the same value of b is already considered in Theorem 5.1.

Theorem 5.3. For $a = 63$, $b = n^2 / 42 + 84$, by Table 5.1. Hence, n = 84, 126, 168, 210, etc. yield the following identities.

n	***b***	***b* + 21**	**Identity**	**Refer [3]**
84	252	273	$63^2 + 84^2 + 252^2 = 273^2$	
			i.e. $3^2 + 4^2 + 12^2 = 13^2$	Th. 2.1
126	462	483	$63^2 + 126^2 + 462^2 = 483^2$	
			i.e. $3^2 + 6^2 + 22^2 = 23^2$	Th. 2.1
168	756	777	$63^2 + 168^2 + 756^2 = 777^2$	
			i.e. $3^2 + 8^2 + 36^2 = 37^2$	Th. 2.1
210	1134	1155	$63^2 + 210^2 + 1134^2 = 1155^2$	
			i.e. $3^2 + 10^2 + 54^2 = 55^2$	Th. 2.1
252	1596	1617	$63^2 + 252^2 + 1596^2 = 1617^2$	
			i.e. $3^2 + 12^2 + 76^2 = 77^2$	Th. 2.1
294	2142	2163	$63^2 + 294^2 + 2142^2 = 2163^2$	
			i.e. $3^2 + 14^2 + 102^2 = 103^2$	Th. 2.1
336	2772	2793	$63^2 + 336^2 + 2772^2 = 2793^2$	
			i.e. $3^2 + 16^2 + 132^2 = 133^2$	Th. 2.1
378	3486	3507	$63^2 + 378^2 + 3486^2 = 3507^2$	
			i.e. $3^2 + 18^2 + 166^2 = 167^2$	Th. 2.1
420	4284	4305	$63^2 + 420^2 + 4284^2 = 4305^2$	
			i.e. $3^2 + 20^2 + 204^2 = 205^2$	Th. 2.1

The choice of pair (a, n) = (63, 42) is dropped as (a, n) = (42, 63) is already considered in Theorem 5.2.

Theorem 5.4. For $a = 84$, $b = (n^2 + 21) / 42 + 157$, by Table 5.1. Hence, n = 105, 147, 189, 231, 273, etc. yield the following identities.

n	***b***	***b* + 19**	**Identity**	**Refer [3]**
105	420	441	$84^2 + 105^2 + 420^2 = 441^2$	
			i.e. $4^2 + 5^2 + 20^2 = 21^2$	Th. 2.2
147	672	693	$84^2 + 147^2 + 672^2 = 693^2$	

			i.e. $4^2 + 7^2 + 32^2 = 33^2$	Th. 2.2
189	1008	1029	$84^2 + 189^2 + 1008^2 = 1029^2$	
			i.e. $4^2 + 9^2 + 48^2 = 49^2$	Th. 2.2
231	1428	1449	$84^2 + 231^2 + 1428^2 = 1449^2$	
			i.e. $4^2 + 11^2 + 68^2 = 69^2$	Th. 2.2
273	1932	1953	$84^2 + 273^2 + 1932^2 = 1953^2$	
			i.e. $4^2 + 13^2 + 92^2 = 93^2$	Th. 2.2
315	2520	2541	$84^2 + 315^2 + 2520^2 = 2541^2$	
			i.e. $4^2 + 17^2 + 120^2 = 121^2$	Th. 2.2
357	3192	3213	$84^2 + 357^2 + 3192^2 = 3213^2$	
			i.e. $4^2 + 17^2 + 152^2 = 153^2$	Th. 2.2

The choices $(a, n) = (84, 21)$ and $(84, 63)$ are dropped as the corresponding results are already considered in Theorems 5.1 and 5.3 respectively.

6. General case: identities of the type

$$a^2 + n^2 + b^2 = (b + k)^2, \tag{6.1}$$

where k is an integer. Above type of identities require:

$$b = (a^2 + n^2 - k^2) / 2k = \{ (a - k).(a + k) + n^2 \} / 2k. \tag{6.2}$$

We examine the following choices of a and n:

(i) When both a, n are some integral multiples of k, say $a = a_1 k$ and $n = n_1 k$, where a_1, n_1 are any integers. Such choice of a, n reduces Eq. (6.2) to

$$b = (a_1^2 + n_1^2 - 1)\, k / 2 \quad \Rightarrow b + k = (a_1^2 + n_1^2 + 1)\, k/2, \tag{6.3}$$

giving integral values of b and $b + k$ only when k is *even*, say $2k_1$, k_1 being an integer. As such, Eq. (6.3) further reduces to

$$b = (a_1^2 + n_1^2 - 1)\, k_1, \quad \text{and} \quad b + k = (a_1^2 + n_1^2 + 1)\, k_1. \tag{6.3b}$$

Thus, there exist identities of above type and we have the:

Theorem 6.1. For $a = 2a_1 k_1$ and $n = 2n_1 k_1$, there hold the identities

$$(2a_1)^2 + (2n_1)^2 + (a_1^2 + n_1^2 - 1)^2 = (a_1^2 + n_1^2 + 1)^2, \tag{6.4}$$

where division by the common factor k_1^2 has been made throughout.

Note 6.1. Giving non-zero integral values to a_1, n_1 the identities discussed earlier when k takes even values 2, 4, 6, 8, etc. are deducible from the identity (6.4).

(ii) When both a, n are odd integral multiples of k, say $a = (2a_1 + 1)\, k$ and $n = (2n_1 + 1)\, k$, where a_1, n_1 are any integers. Such choice reduces Eq. (6.2) to

$$b = 2\,(a_1^2 + n_1^2 + a_1 + n_1)\,k + k/2, \qquad (6.5)$$

which is integer only when k is even. Such case is already covered above in Part (i).

(iii) When both a, n are even integral multiples of k, say $a = 2a_1\,k$ and $n = 2n_1\,k$, where a_1, n_1 are any integers. Such choice reduces Eq. (6.2) to

$$b = 2\,(a_1^2 + n_1^2)\,k - k/2, \qquad (6.6)$$

which is integer only when k is even. Such case is also already covered above in Part (i).

(iv) When one of a, n say a is an odd integral multiple of k, i.e. $a = (2a_1 + 1)\,k$, but $n = 2n_1 k$ is even multiple of k. This choice of a, n reduces Eq. (6.2) to

$$b = 2\,(a_1^2 + n_1^2 + a_1)\,k \quad (6.7); \quad \Rightarrow \quad b + k = \{2\,(a_1^2 + n_1^2 + a_1) + 1\}\,k, \qquad (6.8)$$

giving integral values. Thus, there exist identities of above type and we have the:

Theorem 6.2. For $a = (2a_1 + 1)\,k$, $n = 2n_1 k$ there hold the identities:

$$(2a_1 + 1)^2 + (2n_1)^2 + \{2\,(a_1^2 + n_1^2 + a_1)^2 = \{2\,(a_1^2 + n_1^2 + a_1) + 1\}^2, \qquad (6.9)$$

where the common factor k^2 is divided throughout.

Note 6.2. Giving non-zero integral values to a_1, n_1 the identities discussed earlier when k takes any integral value are deducible from the identity (6.9).

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