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A New Approach for Solving Nonlinear Class Time-Fractional Partial Differential Equations

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ABSTRACT

This article introduces the fractional Residual Power Series Method and the Elzaki transform tool as a way to solve nonlinear time fractional partial differential equations (ERPSM). Two physical models are solved to show the approach utilized. The first model consists of the Rosenau-Hyman equation and the nonlinear time-fractional Burgers equation with the required initial data.

Keywords: Caputo-derivative, Elzaki transform, Residual power series method, Burger's equation and Rosenau-Hyman equation.

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1. Introduction

In recent decades, many mathematical models have been reformulated using the idea of fractional calculusbecause it was found to reflect the modeled phenomenon in a more accurate and realistic way by replacingthe ordinary derivative with a fractional derivative (FD) in the model [1-8].

In 2013, Abu Arqub originally presented the residual power-series (RPS) method, a potent technique for resolving both linear and nonlinear issues.

Tarig Elzaki introduced the Elzaki transform at the beginning of 2011 [6, 12], which is specified for the function of exponential order. Consider the set's function defined as:

$$S = \{\mathrm{B}(\tau) : \exists M, k_1, k_2 > 0, |\mathrm{B}(\tau)| < M\, e^{\frac{|\tau|}{k_j}}, if\ \ \tau \in (-1)^j \times [0,\infty)\}$$

For a given function $\mathrm{B}(\tau)$ in the set S , the constant M must be finite; number k_1, k_2 may be finite or infinite. The Elzaki transform denoted by the operator E is defined as

$$E\left[\mathrm{B}(\tau)\right]=T(\rho)=\rho\int_{0}^{\infty}e^{-\frac{\tau}{\rho}}\mathrm{B}(\tau)d\tau \tag{1}$$

The variable ρ in this transforms is used to factorize the variable τ.

This article's general organization is as follows: The ERPSM is proposed with the essential fundamental technique and posed with the essential fundamental technique and examples of its application to solve nonlinear time fractional partial differential equations, followed by conclusions. First, we start with some fundamental concepts of fractional calculus.

2. Fundamental Concepts of Fractional Calculus

Definition: 2.1 [13]

Let $\mathrm{B}(\kappa)$ be a continuous function, but not necessarily differentiable, then

i- Let us presume that $\mathrm{B}(\kappa)=\lambda$ where λ is a constant, thus α- derivative of the function $\mathrm{B}(\kappa)$ is

$$\begin{cases} \mathrm{D}_{\kappa}^{\alpha}\lambda=\lambda\dfrac{\kappa^{\alpha}}{\Gamma(1+\alpha)} & , \quad \alpha>0 \\ =0 & , \text{otherwise} \end{cases}$$

On the other hand, when $\mathrm{B}(\kappa)\neq\lambda$ hens

$\mathrm{B}(\kappa)=\mathrm{B}(0)+(\mathrm{B}(\kappa)-\mathrm{B}(0))$,

And fractional derivative of the function $\mathrm{B}(\kappa)$ will be known as

$D^{\alpha}\mathrm{B}(\kappa)=D_{\kappa}^{\alpha}\mathrm{B}(0)+D_{\kappa}^{\alpha}(\mathrm{B}(\kappa)-\mathrm{B}(0))$,

ii- at any positive α, $(\alpha>0)$ one has

$$D^{-\alpha}\mathrm{B}(\kappa)=J^{\alpha}\mathrm{B}(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{0}^{\kappa}(\kappa-\tau)^{\alpha-1}\mathrm{B}(\kappa)d\tau,\ \alpha>0. \tag{2}$$

Definition: 2.2[12, 19]

Caputo fractional derivative of the left sided $\mathrm{B}\in C_{-1}^{n}, n\in N\cup\{0\}$ is known as, accordingly

$$\mathrm{D}^{\alpha}\mathrm{B}(\tau)=\frac{\partial^{\alpha}\mathrm{B}(\tau)}{\partial\tau^{\alpha}}=J^{m-\alpha}\left[\frac{\partial^{m}\mathrm{B}(\tau)}{\partial\tau^{m}}\right],\ m-1<\alpha\leq m\ ,m\in N \tag{3}$$

We hold properties of the operator **[**5, 7, 18, 19**]**

i- $\mathrm{D}^{\alpha}J^{\alpha}\mathrm{B}(\tau)=\mathrm{B}(\tau)$,

ii- $J^{\alpha}\tau^{\mu}=\dfrac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}\tau^{\alpha+\mu}$, $\alpha>0,\ \mu>-1,\tau>0$

iii- $J^{\alpha}(\mathrm{D}^{\alpha}\mathrm{B}(\tau))=\mathrm{B}(\tau)-\displaystyle\sum_{m=0}^{n-1}\mathrm{B}^{(m)}(0)\frac{\tau^{m}}{m!}$,

Definition: 2.3[15, 16**]**

The power series

$$\sum_{r=0}^{\infty}c_{r}\left(\tau-\tau_{0}\right)^{rs}=c_{0}+c_{1}\left(\tau-\tau_{0}\right)+c_{2}\left(\tau-\tau_{0}\right)^{2}+\ldots$$

is called a fractional power series about $\tau=\tau_{0}$, where τ is a variable and $c_{r}(r=0,1,2,\ldots)$ are the coefficients of the series, $s\in R^{+}$.

Definition: 2.4 [12]

If $m-1<\alpha \le m$, $m \in N$;
then the Elzaki transform of the fractional derivative is,

$$E\left[D^{\alpha}B(\kappa,\tau)\right]=\rho^{-\alpha}B(\kappa,\rho)-\sum_{i=0}^{m-1}\rho^{2-\alpha+i}B^{(i)}(\kappa,0) \quad , m-1<\alpha<m , \tag{4}$$

3. Elzaki Transform Residual Power Series (ERPS) Method

In this section, we establish the general form of a nonlinear inhomogeneous fractional partial differential equation:

$$\frac{\partial^{\alpha}B(\kappa,\tau)}{\partial\tau^{\alpha}}=LB(\kappa,\tau)+NB(\kappa,\tau)+\sigma B(\kappa,\tau), \tag{5}$$

With the initial conditions

$$B_i(\kappa,\tau)\big|_{\tau=0}=g_m(\kappa),\ m=0,1,2,...,n-1 \tag{6}$$

Where L denotes a linear fraction differential operator, N is general nonlinear fraction differential operator, and σ is a known function.
Using the Elzaki transform on both sides of Eqs (5) and (6), to obtain:

$$E\left[\frac{\partial^{\alpha}B(\kappa,\tau)}{\partial\tau^{\alpha}}\right]+E\left[LB(\kappa,\tau)+NB(\kappa,\tau)+\sigma B(\kappa,\tau)\right], \tag{7}$$

Using the properties of Elzaki transform, to obtain:

$$\rho^{-\alpha}E\left[B(\kappa,\tau)\right]=\sum_{r=0}^{m-1}\rho^{2-m+k}\frac{\partial^{r}T(\kappa,0)}{\partial\tau^{r}}+E\left[LB(\kappa,\tau)+NB(\kappa,\tau)+\sigma B(\kappa,\tau)\right], \tag{8}$$

Operating the inverse transform on both sides of Eq. (8), to get:

$$B(\kappa,\tau)=G(\kappa,\tau)-E^{-1}\left(\rho^{\alpha}E\left[LB(\kappa,\tau)+NB(\kappa,\tau)+\sigma B(\kappa,\tau)\right]\right), \tag{9}$$

Where $G(\kappa,\tau)$ represents the term arising from the source term and the prescribed initial conditions.
The converted function is written as follows in the second phase of the residual power series method:

$$B(\kappa,\tau)=\sum_{r=0}^{\infty}\Phi_r(\kappa,\tau)\frac{\tau^{r\alpha}}{\Gamma(1+r\alpha)} \tag{10}$$

To obtain the approximate value of (11), the form of $B_m(\kappa,\tau)$ can be written as

$$B_m(\kappa,\tau)=\sum_{r=0}^{m}B_r(\kappa,\tau)=\Phi_0(\kappa,\tau)+\sum_{r=1}^{m}\Phi_r(\kappa,\tau)\frac{\tau^{r\alpha}}{\Gamma(1+r\alpha)} \tag{11}$$

We combine (9) with (10), we can attain

$$\mathrm{Res}_m(\kappa,\tau)=B_m(\kappa,\tau)-\left\{G(\kappa,\tau)+E^{-1}\left(\rho^{\alpha}E\left[LB_m(\kappa,\tau)+NB_m(\kappa,\tau)+\sigma B_m(\kappa,\tau)\right]\right)\right\}, \tag{12}$$

Substitute the m^{th} -truncated series (11) into (12), multiply the resulting equation by $\tau^{-m\alpha}$ and then solve the equation

$$\tau^{-m\alpha}\,\mathrm{Res}_m(\kappa,\tau)\big|_{\tau=0}=0,\ m=1,2,3,.. \quad ,$$

Here, ERPSM will give the m^{th} - order approximate solutions with

$$\sum_{i=0}^{m}B_i(\kappa,\tau)=B_0+B_1+B_2+...+B_m$$

$$B_0=\Phi_0(\kappa,\tau) \quad , \quad B_1=\Phi_1(\kappa,\tau)\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}$$

$$B_2=\Phi_2(\kappa,\tau)\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} \ , \ . \ . \ . \ . \ , \ B_m=\Phi_m(\kappa,\tau)\frac{\tau^{m\alpha}}{\Gamma(1+m\alpha)}.$$

4. Application of Elzaki Transform Residual Power Series (ERPS) Method for Fpdes

In this part, we use Rosenau-Hyman equation models with adequate beginning data and the Elzaki residual power series (ERPSM) approach to solve the nonlinear time-fractional Burgers' equation.
Example 4.1:
Take into account the nonlinear time-fractional Burgers' equation below,

$$D_\tau^\alpha B(\kappa,\tau) - D_\kappa^2 B(\kappa,\tau) + B(\kappa,\tau) D_\kappa B(\kappa,\tau) = 0\ ,\ \ 0<\alpha\le 1, \tag{13}$$

With the initial condition:

$$B(\kappa,0) = 2\kappa \tag{14}$$

For the standard case when $\alpha = 1$, the exact solution of (1) is $B(\kappa,\tau) = \dfrac{2\kappa}{1+2\tau}$.

Now, to begin the ERPS methodology stages, we transform (13) using the Elzaki transform and (14) using the initial condition to obtain:

$$B(\kappa,\tau) = 2\kappa + E^{-1}\left(\rho^\alpha E\left(D_\kappa^2\left[B(\kappa,\tau)\right] - \left(\left[B(\kappa,\tau)\right]\left[D_\kappa B(\kappa,\tau)\right]\right)\right)\right) . \tag{15}$$

Then, we write that the transformed function of (15) is of the form:

$$B(\kappa,\tau) = \sum_{r=0}^{\infty}\Phi_r(\kappa)\frac{\tau^{r\alpha}}{\Gamma(r\alpha+1)} . \tag{16}$$

and the m^{th} -truncated series of (4) is

$$B_m(\kappa,\tau) = 2\kappa + \sum_{r=1}^{m}\Phi_r(\kappa)\frac{\tau^{r\alpha}}{\Gamma(r\alpha+1)} . \tag{17}$$

Accordingly, the m^{th} - Elzaki residual function takes the form:

$$\begin{aligned}\mathrm{Re}s_m(\kappa,\tau) &= B_m(\kappa,\tau) - 2\kappa - \\ &\left(E^{-1}\left(\rho^\alpha E\left(D_\kappa^2 B_m(\kappa,\tau) - \left[B_m(\kappa,\tau)\right]\left[D_\kappa B_m(\kappa,\tau)\right]\right)\right)\right)\end{aligned} \tag{18}$$

Substitute the m^{th} -truncated series (17) into (18), multiply the resulting equation by $\tau^{-m\alpha}$ and then solve the equation:

$$\tau^{-m\alpha}\,\mathrm{Re}s_m(\kappa,\tau)\Big|_{\tau=0} = 0,\ m = 1,2,3,..,$$

To define $B_1(\kappa,\tau)$ we consider $m = 1$ in (18)

$$\begin{aligned}\mathrm{Re}s_1(\kappa,\tau) = B_1(\kappa,\tau) - 2\kappa - E^{-1}\left\{\rho^\alpha D_\kappa^2 E\left[2\kappa + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right\} + \\ E^{-1}\left\{\rho^\alpha E\left(\left[2\kappa + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right] D_\kappa\left[2\kappa + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right)\right\}\end{aligned} \tag{19}$$

for $B_m(\kappa,\tau)$ gives

$$B_1(\kappa,\tau) = \Phi_0(\kappa) + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}$$

with $B_0(\kappa,\tau) = B(\kappa,0) = \Phi_0(\kappa) = 2\kappa$

and $\mathrm{B}_2(\kappa,\tau)=\Phi_0(\kappa)+\Phi_1(\kappa)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\Phi_2(\kappa)\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)},...$ (20)

$$\Phi_1(\kappa)=-4\kappa \quad , \quad \Phi_2(\kappa)=16\kappa \quad , \quad \Phi_3(\kappa)=-16\kappa\left(4+\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right),$$

$$\Phi_4(\kappa)=64\kappa\left(4+\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+\frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right),$$

$$\Phi_5(\kappa)=-256\kappa\left(4+\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+\frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right)-$$
$$128\kappa\left(8+\left[\frac{2}{\Gamma^2(2\alpha+1)}+\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}\right]\Gamma(4\alpha+1)\right). \tag{21}$$

Therefore, the closed form solution of (1) -(2) is

$$\mathrm{B}(\kappa,\tau)= 2\kappa-\frac{4\kappa}{\Gamma(1+\alpha)}\tau^{\alpha}+\frac{16\kappa}{\Gamma(1+2\alpha)}\tau^{2\alpha}-\frac{16\kappa\left(4+\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)}{\Gamma(1+3\alpha)}\tau^{3\alpha}+... \tag{22}$$

Now, if we substitute $\alpha=1$ in (21) and (22), it gives

$$\mathrm{B}(\kappa,\tau)= 2\kappa\left(1-2\tau+(2\tau)^2-(2\tau)^3+...\right). \tag{23}$$

In (23) agrees with the Maclaurin series of

$$\mathrm{B}(\kappa,\tau)=\frac{2\kappa}{1+2\tau}$$

Example 4.2:
Consider the following nonlinear time-fractional Rosenau-Hyman equation

$$D_{\tau}^{\alpha}\mathrm{B}(\kappa,\tau)-\mathrm{B}(\kappa,\tau)D_{\kappa}^{3}\mathrm{B}(\kappa,\tau)-\mathrm{B}(\kappa,\tau)D_{\kappa}\mathrm{B}(\kappa,\tau)-3D_{\kappa}\mathrm{B}(\kappa,\tau)D_{\kappa}^{2}\mathrm{B}(\kappa,\tau)=0\ ,\ 0<\alpha\le 1, \tag{24}$$

with the initial condition:

$$\mathrm{B}(\kappa,0)=-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4}\ , \tag{25}$$

For the standard case when $\alpha=1$, the exact solution of (24) is $\mathrm{B}(\kappa,\tau)=-\frac{8\lambda}{3}\cos^2\left(\frac{\kappa-\lambda\tau}{4}\right)$.

As we begin the ERPS technique stages, we first apply the Elzaki transform to (24) and use the initial condition (25) to obtain:

$$\mathrm{B}(\kappa,\tau)=\frac{8\lambda}{3}\cos^2\frac{\kappa}{4}+E^{-1}\left(\rho^{\alpha}E\left(\left[\mathrm{B}(\kappa,\tau)\right]D_{\kappa}^{3}\left[\mathrm{B}(\kappa,\tau)\right]\right)\right)+$$
$$E^{-1}\left(\rho^{\alpha}E\left(\left[\mathrm{B}(\kappa,\tau)\right]D_{\kappa}\left[\mathrm{B}(\kappa,\tau)\right]\right)\right)+3E^{-1}\left(\rho^{\alpha}E\left(\left[\mathrm{B}(\kappa,\tau)\right]D_{\kappa}^{2}\left[\mathrm{B}(\kappa,\tau)\right]\right)\right), \tag{26}$$

The transformed function of (26) is then written as follows:

$$\mathrm{B}(\kappa,\tau)=\sum_{r=0}^{\infty}\Phi_r(\kappa)\frac{\tau^{r\alpha}}{\Gamma(r\alpha+1)}. \tag{27}$$

and the m^{th} -truncated series of (27) is

$$\mathrm{B}_m(\kappa,\tau) = \sum_{r=0}^{m}\mathrm{B}_r(\kappa,\tau) = -\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \sum_{r=1}^{m}\Phi_r(\kappa)\frac{\tau^{r\alpha}}{\Gamma(r\alpha+1)}. \tag{28}$$

Accordingly, the m^{th} - Elzaki residual function takes the form

$$\mathrm{Re}s_m(\kappa,\tau) = \mathrm{B}_m(\kappa,\tau) + \frac{8\lambda}{3}\cos^2\frac{\kappa}{4} - E^{-1}\left(\rho^\alpha\left(E\left[\mathrm{B}_m(\kappa,\tau)\right]D_\kappa^3\left[\mathrm{B}_m(\kappa,\tau)\right]\right)\right) -$$
$$E^{-1}\left(\rho^\alpha\left(E\left[\mathrm{B}_m(\kappa,\tau)\right]\mathrm{D}_\kappa\left[\mathrm{B}_m(\kappa,\tau)\right]\right)\right) - 3E^{-1}\left(\rho^\alpha E\left(\left[\mathrm{B}_m(\kappa,\tau)\right]\mathrm{D}_\kappa^2\left[\mathrm{B}_m(\kappa,\tau)\right]\right)\right). \tag{29}$$

Substitute the m^{th} -truncated series into (29), multiply the resulting equation by $\tau^{-m\alpha}$ and then solve the equation

$$\tau^{-m\alpha}\,\mathrm{Re}s_m(\kappa,\tau)\Big|_{\tau=0} = 0,\ m = 1,2,3,..,$$

To define $\mathrm{B}_1(\kappa,\tau)$ we consider $m = 1$ in (29)

$$\mathrm{Re}s_1(\kappa,\rho) = -\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{8\lambda}{3}\cos^2\frac{\kappa}{4} -$$
$$E^{-1}\left\{\rho^\alpha E\left(\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]D_\kappa^3\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right)\right\} -$$
$$E^{-1}\left\{\rho^\alpha E\left(\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]D_\kappa\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right)\right\} -$$
$$3E^{-1}\left\{\rho^\alpha E\left(D_\kappa\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]D_\kappa^2\left[-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right)\right\} \tag{30}$$

for $\mathrm{B}_m(\kappa,\tau)$ gives

$$\mathrm{B}_1(\kappa,\tau) = \Phi_0(\kappa) + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)}$$

with $\mathrm{B}_0(\kappa,\tau) = \mathrm{B}(\kappa,0) = \Phi_0(\kappa) = -\frac{8\lambda}{3}\cos^2\frac{\kappa}{4}$

and $\mathrm{B}_2(\kappa,\tau) = \Phi_0(\kappa) + \Phi_1(\kappa)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \Phi_2(\kappa)\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)},...$

$$\Phi_1(\kappa) = \frac{2\lambda^2}{3}\sin\frac{\kappa}{2}\ ,\quad \Phi_2(\kappa) = \frac{\lambda^3}{3}\cos\frac{\kappa}{2}\ ,\quad \Phi_3(\kappa) = \frac{\lambda^4}{6}\sin\frac{\kappa}{2}\ ,$$
$$\Phi_4(\kappa) = -\frac{\lambda^5}{12}\cos\frac{\kappa}{2}\ ,\quad \Phi_5(\kappa) = -\frac{\lambda^6}{24}\sin\frac{\kappa}{2}, ... \tag{31}$$

which leads to

$$\mathrm{B}(\kappa,\tau) = -\frac{8\lambda}{3}\cos^2\frac{\kappa}{4} - \frac{2\lambda^2}{3}\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sin\frac{\kappa}{2} + \frac{\lambda^3}{3}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\cos\frac{\kappa}{2} + \frac{\lambda^4}{6}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}\sin\frac{\kappa}{2} -$$

$$\frac{\lambda^5}{12}\frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)}\cos\frac{\kappa}{2}-\frac{\lambda^6}{24}\frac{\tau^{5\alpha}}{\Gamma(5\alpha+1)}\sin\frac{\kappa}{2}+\ldots \tag{32}$$

If we replace term $\left(-\frac{8\lambda}{3}\cos^2\frac{\kappa}{4}\right)$ by the tem $\left(-\frac{4\lambda}{3}-\frac{4\lambda}{3}\cos\frac{\kappa}{2}\right)$ in (32) and perform some algebra iterations

$$\mathrm{B}\,(\kappa,\tau) = -\frac{4\lambda}{3}\left(1+\sin\frac{\kappa}{2}\left(\frac{\lambda}{2}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}-\left(\frac{\lambda}{2}\right)^3\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}\ldots\right)\right)-$$
$$\frac{4\lambda}{3}\cos\frac{\kappa}{2}\left(1-\left(\frac{\lambda}{2}\right)^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\left(\frac{\lambda}{2}\right)^4\frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)}+\ldots\right)$$

Therefore, the closed form solution of (24), (25) is,

$$\mathrm{B}\,(\kappa,\tau) = -\frac{4\lambda}{3}\left(1+\sin\frac{\kappa}{2}\left(\sum_{i=0}^{\infty}(-1)^i\left(\frac{\lambda\tau^{\alpha}}{2}\right)^{2i+1}\frac{1}{\Gamma((2i+1)\alpha+1)}\right)\right)-$$
$$\frac{4\lambda}{3}\cos\frac{\kappa}{2}\left(\sum_{i=0}^{\infty}(-1)^i\left(\frac{\lambda\tau^{\alpha}}{2}\right)^{2i}\frac{1}{\Gamma(2i\,\alpha+1)}\right) \tag{33}$$

Now, if we substitute $\alpha=1$ in (32) and (33), it gives

$$\mathrm{B}\,(\kappa,\tau) = -\frac{4\lambda}{3}\left(1+\sin\frac{\kappa}{2}\left(\sum_{i=0}^{\infty}(-1)^i\left(\frac{\lambda\tau}{2}\right)^{2i+1}\frac{1}{(2i+1)!}\right)\right)-$$
$$\frac{4\lambda}{3}\cos\frac{\kappa}{2}\left(\sum_{i=0}^{\infty}(-1)^i\left(\frac{\lambda\tau}{2}\right)^{2i}\frac{1}{(2i)!}\right) \tag{34}$$

This is fully in agreement with the Maclaurin series expansion of the exact solution:

$$\mathrm{B}\,(\kappa,\tau) = -\frac{8\lambda}{3}\cos^2\left(\frac{\kappa-\lambda\tau}{4}\right)$$

5. Conclusions

In this article, a new scheme constructed by a combination of the Elzaki transform tool with the fractional residual power series (ERPSM) is presented to solve some important nonlinear time-fractional Burgers' type equations and the Rosenau-Hyman equation.

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