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Degree of Approximation of a Class of Functions using Weighted Product Means of Fourier Series

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ABSTRACT
The degree of approximation plays a crucial role in various mathematical discipline, particularly in mathematical modelling and scientific computing, where accurate approximations are essential for reliable predictions and simulations, presently , we have discussing the concept of weighted product means & establish a new results concerning the degree of approximation of periodic Lebesque integral functions.The current developed theorem provide a broader class of summability technique that gives better result than previous results by researchers.

Keyword: Degree of approximation, Lipschitz class functions, Weighted Product summability, Fourier series; Lebesgue integrable functions, Big O.

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1. INTRODUCTION

The investigation of approximation of a function has long been a central theme in mathematical analysis, particularly in the context of Fourier series. Approximation theory not only enriches pure mathematics but also finds practical applications in areas such as signal processing, numerical computation, and data analysis. Recent developments in summability methods have significantly improved the convergence behavior of Fourier series, offering more accurate representations of complex functions. Among these, the product summability technique provides a powerful approach for handling slowly convergent or divergent series. In the present article, we investigate the degree of approximation of Lipschitz class functions using a weighted product mean (C,1)w(E,q) This generalized framework introduces a weight function to enhance flexibility and control over approximation errors. The proposed method extends earlier summability results and provides sharper convergence estimates, thereby contributing to a deeper understanding of the approximation of periodic Lebesgue integrable functions.

2. LITERATURE REVIEW

Many mathematician have worked in the area of approximation theory and its application. Earlier efforts of Hardy and Banach who proposed more general approaches to convergence [1,2]. These frameworks consider the behavior of a series in the context of more general limiting processes instead of considering the question of whether a series converges or not in the traditional sense. This new view enables us to give some of the divergent series a meaning, and use it as an approximation. Subsequently, this change has brought significant changes in the theory of summability. Among the contributions, Waterman contributed to the summability of Fourier series [3]. Later, Nigam & Sharma examined order of approximation of a function by using product means of Fourier series [4]. Singh examined Norlund methods, which advance the knowledge of approximation [5]. Researchers like Paikray, Jati and Misra & Misra have in recent years come up with new matrix based summability techniques. These tend to be quasi-monotone sequences and other special tools to enhance the approximation by fourier series of functions [6,7,8]. Then Dikshit, McFadden, Pati are investigating properties and implication of absolutely convergence and non-absolute convergence of series [9,10,11]. Other significant work is the work by Nigam and Sharma, who examined the way product means could help to improve approximation results [12]. Meanwhile the classical of Titchmarsh] and Zygmund still remains a textbook in the theory of trigonometric series [13,14]. An important point of development in contemporary approximation theory is a study by Qureshi [who introduced spaces of Lipschitz whose study has led to the innovation of new directions [15]. Based on this, Nigam, have given more insights by applying the Lipschitz function in order to get a clear insight on the level of approximation in mathematical analysis [16]. Later Lal, Khan [18] give some new idea regarding weighted summability[17] .Subsequently Mitttal, Mishra introduced some new thoughts regarding weighted summmability and conjugate fourier series.This continued evolution is what makes sure that summability is an effective tool in pure and applied mathematics [19,20].

3. PRELIMINARIES & DEFINITIONS

Definition 3.1. Let $\sum a_n$ be a series , then $\sum a_n$ is called Euler summable to a definite s if $\lim_{n\to\infty} \frac{1}{(1+q)^n} \sum_{k=0}^{n} q^{n-k} \binom{n}{k} s_k \to s$ and written $\sum a_n = s(E,q)$. (1)

Definition 3.2. A function f ∈ Lip(α) for $0<x\leq 2\pi$

If $|f(x+t)-f(x)| = O(|t|^{\alpha})$ for $0<\alpha\leq 1$ and (2)

f∈Lip(α,r) if $\left(\int_0^{2\pi} (|\, f(x+t)-f(x)\,|)^r dx\right)^{\frac{1}{r}} = O(|\,t\,|^{\alpha})$ for $0<\alpha\leq 1$,r>0

(3)

Definition 3.3. The accuracy of approximation of a function $f\in(L^p(0,2\pi)$ using a trigonometric polynomial P_n(f;x) is defined by $E_n(f) = \|P_n(f;x) - f\|_p = O(\phi(n))$. (4)

where $\phi(n)$ denotes the rate function shows approximation error.

Definition3.4. A function f belongs to weighted Lipschitz class $Lip_w(a,p)$ if

$\|f(x+t)-f(x)\|_p = O(w(t).t^{\alpha})$ for $0<\alpha\leq 1$ where w(t) is a increasing function such that $\lim_{t\to 0} w(t) = 1$

(5)

Definition 3.5. Let $(w_n)>0$ be a monotonic bounded sequence such that $\sum_{n=0}^{\infty} w_n = W<\infty$,then Weighted product mean(C,1)w(E,q) of a Fourier seriers $\sum a_n$ is given by

$$T_n(f;x) = \frac{1}{W_n}\sum_{k=0}^{n} w_k \left[\frac{1}{k+1}\binom{k}{r} q^r (1-q)^{k-r} S_r(f;x)\right] \quad (6)$$

where $S_r(f;x)$ denotes r-th partial sum of the given Fourier series.

Definition 3.6. Let f(t) be a periodic function of period 2π & integrable in the Lebesque sense .Then Fourier series f(t) is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\, cosnt + b_n sinnt) \quad (7)$$

and conjugate Fourier series of (8) is given by

f(t)= $\sum_{n=1}^{\infty} (b_n sinnt\text{-}a_n cosnt)$ (8)

Definition 3.7. We write two functions by relation $f(n) = O(g(n))$, it means∃ $n \geq n_0$& positive constant k such that $|f(n)| \leq kg(n)$ *(9)*

where big '*O*' notation represents for an upper limit on the growth of a function.

Example: Let $f(n) = 3n^2+5n+1$

$\leq 3n^2+ 5n^2+n^2$

$= 9n^2$ (for n≥1)

$=kn^2$.

where k=9 and n_0=1 .

Then, we can say that $f(n) = O(n^2)$.

4. PRINCIPAL THEOREM.

In the present article, we proved degree of approximation of a Class of Functions using weighted product means of Fourier Series

Theorem 4.1. For a periodic function f belonging to the weighted Lipschitz class $Lip_w(a,p)$, $0<a \leq 1$.Then degree of approximation of f by weighted product mean (C,1)w(E,q) of its Fourier series satisfies $E_n(f)= O\left(w_n\, n^{-a}\right)$

5. KNOWN RESULT

Before we established the main theorem, we shall use the following Lemma.

Lemma 5.1 .For f∈$Lip(a,p)$, then $|s_n(f,x)\text{-}f(x)|= O\left(\frac{1}{n^a}\right)$ (10)

Lemma 5.2.If w_n is a bounded and monotonic, then $\sum_{k=0}^{n} w_k = O(nw_n)$ (11)

Lemma 5.3. If f∈$Lip_w(a,p)$, then $E_n(f) \leq k w_n n^{-a}$ where k is a constant (12)

7. Proof of the MainTheorem.

Since we know from definition of weighted product mean,

$$|T_n(f;x)\text{-}f(x)|=\left|\frac{1}{W_n}\sum_{k=0}^{n} w_k \left(f(x)\text{-}S_K(f;x)\right)\right|$$

$$\leq \frac{1}{W_n}\sum_{k=0}^{n} w_k \left|\left(f(x)\text{-}S_K(f(x))\right)\right|$$

(13)

Applying Lemma (5.1) in (12),we obtain

$$|T_n(f;x)\text{-}f(x)| \leq \frac{1}{W_n}\sum_{k=0}^{n} w_k.k^{-a} \quad (14)$$

Again using Lemma (5.2) and monotonic properties of w_n in (13), we find

$$|T_n(f;x)\text{-}f(x)| = O\left(w_n n^{-a}\right)$$

Corollary1. If w_n=1, then results reduces to unweighted product mean (C,1).(E,q) .

Corollary2. If q =0 , then results reduces to a weighted Cesaro mean $(C, 1)_w$.

Corollary3. If $w_n=n^{\beta}$ for some $\beta>0$, then $|T_n(f;x)\text{-}f(x)|= O\left(n^{\beta-a}\right)$.

6. RESULT AND DISCUSSION

The obtained results demonstrate that the order of approximation of periodic Lebesgue integrable functions using the weighted product mean of Cesaro and Euler of Fourier series provides a significantly improved rate of convergence compared to traditional summability methods. The inclusion of weight functions enhances flexibility and control over approximation errors, especially for functions in the weighted Lipschitz class. The theorems established in this study extend earlier results by introducing broader applicability and sharper bounds. These findings highlight that the proposed method not only refines convergence behavior but also strengthens the practical efficiency of Fourier approximation techniques.

7. CONCLUSION

In this study, we investigated the approximation degree of a class of periodic functions using the weighted product mean of Fourier series. The results revealed that introducing suitable weight functions significantly enhances the convergence rate and precision of approximations, especially for functions satisfying the weighted Lipschitz class. This generalized approach provides a more powerful and flexible framework than classical summability methods, making it particularly useful for handling complex or rapidly varying functions. The results demonstrate that weighted product summability bridges theoretical mathematics with practical computation, improving efficiency in fields such as signal processing, numerical analysis, and data modeling. Overall, the study establishes the weighted product mean as a robust and effective tool in modern approximation theory.

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