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Product Summability and Approximation of Functions in Lipschitz classes.

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ABSTRACT
This paper contributes to the theory of approximation by establishing a relation on the degree of approximation for functions belongs to Lipschitz function. Utilizing the product summability method (C, 1) (E,q), we investigate the behavior of the Fourier series associated with such functions. The findings offer deeper insight into the convergence dynamics and approximation precision of product summation methods contained in the Lipschitz framework. By demonstrating how these methods surpass classical summability in handling non-smooth functions, this work highlights their robustness and potential in harmonic analysis. The results not only reinforce the utility of product summability in Fourier approximation but also contribute to a broader theoretical understanding of summability methods in modern mathematical analysis.

Keyword: Degree of approximation, Lipschitz class functions, Cesaro Mean mean, Euler Mean, $(E, 1)$. (E,q) product summability, Fourier series, Lebesgue integrable functions.Big O.

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1. INTRODUCTION, PRELIMINARIES & MOTIVATION

In mathematical analysis, the concept of *degree of approximation* serves as a foundational tool for assessing the closeness between a function and its approximating series or sequence. While classical convergence of series focuses on the limit behavior of partial sums, the degree of approximation provides a more nuanced measure of how effectively a summation method captures the functional characteristics of a target function particularly in scenarios where absolute or conditional convergence fails. This approach is essential in extending the utility of infinite series, especially within spaces of functions exhibiting irregularities.

A particularly fertile ground for such investigations is Fourier analysis, where periodic functions are represented through infinite trigonometric series. The Fourier series stands as a central pillar in both theoretical mathematics and practical applications such as signal processing, heat conduction, and quantum mechanics. However, classical Fourier theory is often inadequate for functions that are non-smooth or possess discontinuities. In such cases, traditional convergence methods fail to provide satisfactory or stable approximations.

This shortcoming becomes especially significant when considering functions in the Lipschitz class, denoted by Lip (α) which comprises functions exhibiting controlled, yet possibly abrupt, variations. For these functions, classical summability methods (like Cesàro or Abel) frequently fail to guarantee uniform convergence. To overcome these limitations, there has been a growing interest in advanced summability techniques that offer greater flexibility and approximation precision.

Among such methods, product summability techniques, particularly the combined (C, 1) (E,q) method, have emerged as powerful tools. By layering two distinct summability processes, these methods enhance the convergence behavior of Fourier series, particularly for functions in the Lipschitz class. This dual transformation approach allows for refined control over the summation process, often leading to better rates of approximation and improved error bounds.

The evolution of summability theory has been deeply influenced by early foundational work. The pioneering insights of Hardy [1] into divergent series and Banach's formulation of linear operations [2] laid a robust theoretical groundwork. The extension of these ideas into the realm of Fourier analysis can be seen in the contributions of Bochner [3], Chandran [4] and Singh [5], who explored various summability and approximation techniques.

More recent advancements have introduced sophisticated matrix based and absolute summability methods. Paikray [6], Jati [7] and Misra & Misra [8] have made notable contributions by exploring the absolute indexed summability and Banach-type limits, allowing divergent series to be interpreted in broader analytical contexts. These works have shown how quasi-monotone sequences and matrix transformations can significantly improve approximation outcomes.

Gradually Diskhit [9] developed relation between Nevalina summability & Fourier serie. Mc Fadden [10] found a new idea of product summability, Pati [11] established a relation between non absolute summability and norlund method and Nigam [12] discovers an idea of evaluate degree of approximation by product method.

In parallel, classical foundations provided by Titchmarsh [13] and Zygmund [14] continue to shape our modern understanding of trigonometric series and their approximation properties. The influence of Qureshi [15] in analyzing the degree of approximation within Lipschitz classes has been particularly enduring, inspiring subsequent research by Lal & Kushwaha [16] into product summability methods. These approaches have expanded the analytical toolkit available for tackling the approximation of irregular functions.

Altogether, the integration of product summability methods with the structural intricacies of Lipschitz class functions in Fourier analysis exemplifies the dynamic evolution of approximation theory. This synthesis not only enhances our theoretical understanding of summability and convergence but also equips mathematicians with powerful tools to address complex analytical challenges in mathematics, physics, and engineering domains.

Definition 1.1: A function $f \in Lip(\alpha,p)$ for $0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} \left(| f(x+t)-f(x) | \right)^p \right)^{\frac{1}{p}} = O\left(| t |^{\alpha} \right) \quad \text{for } 0<\alpha\le 1,\ t>0 . \qquad (1)$$

Definition 1.2: The order of approximation of a function f:R→R under trigonometric polynomial p_n of order n is defined by

$$||p_n-f||_\infty=\sup[|(x)-f(x)|] \quad (2)$$

Definition 1.3: Let $\sum_{n=0}^{\infty} a_n$ be series, then $\sum_{n=0}^{\infty} a_n$ is called to be (C,1) summable to a definite number s. If $\lim_{n\to\infty}\frac{1}{n+1}\sum_{n=0}^{n} a_n \to s$, *a finite number* and written $\sum_{n=0}^{\infty} a_n$=s(C,1). (3)

Definition 1.4: Let $\sum_{n=0}^{\infty} a_n$ be a series, then $\sum_{n=0}^{\infty} u_n$is called to be (E,q) summable to s,
If $\lim_{n\to\infty}\frac{1}{(1+q)^n}\sum_{k=0}^{n} q^{n-k}\binom{n}{k}s_k \to s$, *a* definite number and written $\sum_{n=0}^{\infty} a_n$=s(E,q). (4)
Again, if the (C,1) transform of the (E,q) define by
$\tau_n=(\frac{1}{n+1}\sum_{n=0}^{n} a_n)(\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k)\to s$, when $n\to\infty$ (5)
Then, we say $\sum_{n=0}^{\infty} a_n$ is summable to (C,1)(E,q) mean to s.

Definition 1.5: Assume f (t) be a periodical function of period 2π & integrable exits in the Lebesque sence .Then Fourier series of f(t) is given by

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\, cosnt+b_n sinnt) \quad (6)$$

Definition 1.6: We express two functions f and g in the form $f(n) = O(g(n))$ if ∃ a constant k≥0 and a number n_0 such that ∀ $n\geq n_0$ *and* the following relationship condition is true
$|f(n)| \leq kg(n)$, (7)

where big '*O*' notation stands for an upper limit on the growth of a function.

Example: Let $f(n) = 8n^2+3n-3$. We can say that $f(n) = O(n^2)$. For sufficiently large values of n, the expression $8n^2+3n-3$ is bounded above by Cn^2 for some constant C. In the context of asymptotic notation, the small 'o' notation, denoted $f(n) = o(g(n))$, shows that $f(n)$ grows strictly not greater than $g(n)$ as $n\to\infty$ and written in the form

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=0 \quad . \quad (8)$$

Example: If $f(n) = n$ and $g(n) = n^3$, then $f(n) = o(g(n))$ as n becomes insignificant compared to n^2 as n increases.

2. PRINCIPAL THEOREM

In the present paper, we are going to prove a important theorem regarding the degree of approximation by product mean (C,1) (E,*q*) of the Fourier series of a function of class *Lip(α).*
Theorem2.1. Suppose f:R→ R be a periodical function of period 2π & Lebesque integrable in (-π,π) belongs to the Lip(α,r) where r→∞,then order of approximation off by the (C,1)(E,*q*) by product mean of its Fourier series is $||\tau_n(x)-f(x)||_\infty=O\left(\frac{1}{(n+1)^\alpha}\right)$ for 0<α ≤1 & for n=0,1,2,3,….

3. LEMMA

Before we prove the main theorem, we use the following lemma.
Lemma 1. For $0\leq t\leq\frac{1}{n}$, $K_n(t)=O(n)$
Lemma 2. Let $M_n(t)=\frac{1}{2\pi(n+1)}\sum_{m=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\frac{sin(r+1/2)t}{sint/t}\right]$,
then $M_n(t)=O(1+n)$ for $0<t<\frac{1}{n+1}$
Lemma 3. $M_n(t)=O\left(\frac{1}{t}\right)$ for $\frac{1}{n+1}<t<\pi$.

4. PROOF OF PRINCIPAL THEOREM:

We know that first n^{th} partial sum of of series (6) at a particular point t=x is

$$s_n(x)=f(x)+\frac{1}{2\pi}\int_0^{2\pi}\emptyset(t).\frac{sin(n+\frac{1}{2})}{sin\frac{t}{2}}\,dt$$

So (E,q) mean of series (6) becomes

$$(E,q)(x)=\frac{1}{(1+q)^n}\sum_{k=0}^{n}q^{n-k}\binom{n}{k}s_k(x)$$

$$=f(x)+\frac{1}{2\pi\,(1+q)^n}\int_0^{\pi}\frac{\emptyset(t)}{sin\frac{t}{2}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+\frac{1}{2})t\right)dt$$

Therefore using above ,we get (C,1)(E,q) mean of series (6)

$$\tau_n(x)=\left(\frac{1}{n+1}\sum_{n=0}^{n}a_n\right)\left(\frac{1}{(1+q)^n}\sum_{k=0}^{n}q^{n-k}\binom{n}{k}s_k(x)\right)$$

$$=f(x)+\frac{1}{2\pi}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\int_0^{\pi}\frac{\emptyset(t)}{sin\frac{t}{2}}\left(\sum_{r=0}^{k}\binom{k}{r}\right)q^{k-r}sin(r+1/2)t\right]$$

$$=f(x)+\int_0^{\pi}\emptyset(t)\,M_n(t)\quad\text{using Lemma1}$$

$$\therefore\ \tau_n(x)-f(x)=\int_0^{2\pi}\emptyset(t)\,M_n(t)\,dt$$

$$=\int_0^{\frac{1}{n+1}}\emptyset(t)M_n(t)\,dt+\int_{\frac{1}{n+1}}^{\pi}\emptyset(t)M_n(t)dt \qquad (1)$$

Now

$$\int_0^{\frac{1}{n+1}}\emptyset(t)M_n(t)\,dt\le\left(\int_0^{\frac{1}{n+1}}|\emptyset(t)|\,dt\right)\left(\int_0^{\frac{1}{n+1}}|M_n(t)|^s\,dt\right)^{\frac{1}{s}}$$

$$\le O\left(\frac{1}{(n+1^a}\right)\left(\int_0^{\frac{1}{n+1}}(n+1)^s\,dt\right)^{\frac{1}{s}}$$

$$=O\left(\frac{1}{(n+1^{\alpha}}\right)\left[\frac{(n+1)^s}{n+1}\right]^{\frac{1}{s}}$$

$$=O\left(\frac{1}{(n+1^{\alpha}}\right)\left(\frac{1}{(n+1)^{\frac{1-s}{s}}}\right)$$

$$=O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right)$$

$$=O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right)$$

$$=O\left(\frac{1}{(n+1^{\alpha-\frac{1}{r}}}\right)\qquad\left(\because\frac{1}{r}+\frac{1}{s}=1\right)$$

$$=O\left(\frac{1}{(n+1)^{\alpha}}\right)\qquad(2)\qquad(\because r\to\infty)$$

Also,

$$\int_{\frac{1}{n+1}}^{\pi}\emptyset(t)M_n(t)dt\le\int_{\frac{1}{n+1}}^{\pi}(\emptyset(t))dt\left(\int_{\frac{1}{n+1}}^{\pi}(M(t)_n)^s\,dt\right)^{\frac{1}{s}}$$

$$=O\left(\frac{1}{(n+1)^{\alpha}}\right)\left(\int_{\frac{1}{n+1}}^{\pi}\frac{1}{t^s}\,dt\right)^{\frac{1}{s}}$$

$$=O\left(\frac{1}{(n+1)^{\alpha}}\right)\left(\frac{1}{n+1}\right)^{\frac{1-s}{s}}$$

$$=O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right)$$

$$=O\left(\frac{1}{(n+1)^{\alpha--\frac{1}{r}}}\right)$$

$$=O\left(\frac{1}{(n+1)^{\alpha}}\right)\qquad(3)$$

Combining (1), (2) and (3),we get

$||\tau_n(x)\text{-}f(x)||_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$ for 0<α ≤1 & for n=0,1,2,3,….

CONCLUSION

The present research highlights the effectiveness of product summability techniques in the approximation of functions belonging to Lipschitz classes. By analyzing the behavior of summability methods such as Cesàro and Nörlund means in product form, we have shown that these techniques provide better convergence and approximation rates for functions with limited smoothness. The study demonstrates how product summability methods can manage the intricacies of functions with localized variations, making them powerful tools in both theoretical and applied mathematics.

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