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Hyers-Ulam Stability of Quadratic Functional Equation in Modular Spaces

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ABSTRACT
In this research paper, we introduce a new quadratic functional equation and determine its general solution. Under suitable assumptions on the modular, we establish the Hyers–Ulam stability of this equation in modular spaces by direct method. These results extend classical stability theorems for quadratic mappings to the modular-spaces. Additionally, we present a counter example demonstrating the instability of the equation in certain cases.

Keyword: Hyer-Ulam stability, Modular spaces, Quadratic functional equation.
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1. INTRODUCTION

In 1940, S.M. Ulam was the first to propose the stability of homomorphisms between groups [1]. The first affirmative answer to this question was given by Hyers in 1941 [2]. Hyers' theorem was generalized by Aoki and Th. M. Rassias independently [3,4]. The functional equation

$$\wp(x_1 + x_2) + \wp(x_1 - x_2) = 2\wp(x_1) + 2\wp(x_2) \tag{1}$$

is said to be the quadratic functional equation. A function which satisfies (1) is called quadratic function. Some quadratic type functional equations has been introduced by several authors and also established many results concerning the Hyers-Ulam stability ([5], [6], [7], [8] [9], [10], [11], [12], [13], [14]. [15]).
Nakano [16] studied the theory of modular linear spaces in 1950, since then this theory have been thoroughly established by many authors, referred as ([17], [18], [19], [20]). Recently Aboutaib et al. and Tamilvanan et al. established stability results for different type of functional equations in Modular Spaces [21].

In the present paper, we use the direct method to investigate the stability of new quadratic functional equation

$$\sum_{i=1}^{n} \wp\left(-2x_i + \sum_{j=1, i\neq j}^{n} x_j\right) = (n-6)\sum_{1\leq i<j\leq n} \wp(x_i + x_j) - (n^2 - 8n + 3)\sum_{i=1}^{n} \wp(x_i) \tag{2}$$

where $n \neq 1$ is a finite natural number in modular spaces.

We first recall some basic notions and properties in modular spaces, as in ([22], [23], [24]).

Definition 1.1 Let Ξ is a vector space over K, a generalized functional $\varpi : \Xi \to [0, \infty)$ is termed as a modular if, for every $u, x \in \Xi$ (where $K = \mathbb{C}$ or $\mathbb{R}$)

(a) $\varpi(u) = 0$ if and only if $u = 0$

(b) $\varpi(\beta u) = \varpi(u)$ for all $\beta \in K$ with $|\beta| = 1$

(c) $\varpi(\beta u + \gamma x) \leq \varpi(u) + \varpi(x)$, when $\beta, \gamma \geq 0$ and $\beta + \gamma = 1$

(c′) $\varpi(\beta u + \gamma x) \leq \beta\varpi(u) + \gamma\varpi(x)$, for all $\beta, \gamma \geq 0$ and $\beta + \gamma = 1$.

If we use (c′) at the place of (c) then, the modular ϖ is called convex modular. If ϖ is a modular on a vector space Ξ then the set

$$\Xi\varpi = \{u \in \Xi : \lim_{t\to 0}\varpi(tu) = 0\}.$$

is modular space. Let us denote $\Xi\varpi$ is a linear subspace of Ξ.

Definition 1.2 Let $\Xi\varpi$ be modular space and $\{x_n\}$ be a sequence in $\Xi\varpi$, then

(i) $\{x_n\}$ is ϖ-convergent to a point $x \in \Xi\varpi$ and write $x_n \to x$ if $\varpi(x_n - x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is termed as ϖ-Cauchy if for given $\epsilon > 0$ we have $\varpi(x_m - x_n) < \epsilon$ for sufficiently large $m, n \in N$.

(iii) A subset $A \subseteq \Xi\varpi$ is termed as ϖ-complete if any ϖ-Cauchy sequence is ϖ-convergent in A.

2. SOLUTION OF EQUATION (2)

The general solution of the functional equation (2) is found in this section. Now onwards we assume that Ξ and W are real vector spaces.

Theorem 2.1 If a function $\wp : \Xi \to W$ satisfies the functional equation (2) for all $x_1, x_2, \ldots, x_n \in \Xi$, then the functional equation (2) is quadratic, that is $\wp$ satisfies the equality,

$$\wp(x_1 + x_2) + \wp(x_1 - x_2) = 2\wp(x_1) + 2\wp(x_2) \tag{3}$$

for all $x_1, x_2 \in \Xi$.

Proof: Suppose $\wp : \Xi \to W$ satisfies (2).

Putting $x_1 = x_2 = \cdots = x_n = 0$ in (2), we have $\wp(0) = 0$.

Replacing $(x_1, x_2, \ldots, x_n)$ by $(x, 0, \ldots, 0)$ in (2), we get

$$\wp(-2x) + (n-1)\wp(x) = (n-6)(n-1)\wp(x) - (n^2 - 8n + 3)\wp(x) \tag{4}$$

$$\wp(-2x) = (n^2 - 7n + 6 - n^2 + 8n - 3 - n + 1)\wp(x) \tag{5}$$

$$\wp(-2x) = 4\wp(x) \tag{6}$$

Now taking $x_1 = x_2 = x$ and $x_3 = \cdots = x_n = 0$ in (2), we have

$$2\wp(-x) + (n-2)\wp(2x) = (n-6)\wp(2x) + 2(n-6)(n-2)\wp(x) - 2(n^2 - 8n + 3)\wp(x)$$

$$2\wp(-x) + 4\wp(2x) = 18\wp(x) \tag{7}$$

Replacing x by $-x$ in (6), we get

$$\wp(2x) = 4\wp(-x) \tag{8}$$

From (7) and (8), we have

$$\wp(-x) = \wp(x) \tag{9}$$

So, $\wp$ is an even mapping, hence by (6) we have

$$\wp(2x) = 2^2\wp(x)$$

for all $x \in \Xi$. Now, replace x by $2x$, we have

$$\wp(2^2x) = 2^4\wp(x)$$

Continue like this, we generalize

$$\wp(2^n x) = 2^{2n}\wp(x) \tag{10}$$

for all $x \in \Xi$ and for any $n \geq 0$.

Similarly, we have

$$\wp(2^{-n}x) = 2^{-2n}\wp(x) \tag{11}$$

If we take $x_1 = x_2 = x_3 = x$ and $x_4 = \cdots = x_n = 0$ in (2) we have

$$(n-3)\wp(3x) = (n-6)[3\wp(2x) + 3(n-3)\wp(x)] - 3(n^2 - 8n + 3)\wp(x)$$
$$(n-3)\wp(3x) = (12n - 72 + 3n^2 - 27n + 54 - 3n^2 + 24n - 9)\wp(x)$$
$$(n-3)\wp(3x) = (9n - 27)\wp(x) \tag{12}$$

if $n \neq 3$

$$\wp(3x) = 9\wp(x)$$

if $n = 3$

$$\wp(2x) = 4\wp(x) \tag{13}$$

We claim that $\wp(kx) = k^2\wp(x)$.
Now, we shall prove it by the principle of mathematical induction.
assume that

$$\wp(mx) = m^2\wp(x) \tag{14}$$

for all $m < k$.
Now taking $x_1 = x_2 = x_3 = \cdots x_k = x$ and $x_{k+1} = \cdots = x_n = 0$ in (2) we have

$$k\wp\big((k-3)x\big) + (n-k)\wp(kx) = (n-6)[\frac{k(k-1)}{2}\wp(2x) + k(n-k)\wp(x)] - k(n^2 - 8n + 3)\wp(x)$$
$$k(k-3)^2\wp(x) + (n-k)\wp kx) = (n-6)[2k(k-1)\wp(x) + k(n-k)\wp(x)] - k(n^2 - 8n + 3)\wp(x)$$
$$(n-k)\wp(kx) = k(2nk - 2n - 12k + 12 + n^2 - nk - 6n$$
$$+6k - n^2 + 8n - 3 - k^2 + 6k - 9)\wp(x)$$
$$(n-k)\wp(kx) = k(nk - k^2)\wp x)$$

So,

$$\wp(kx) = k^2\wp(x) \tag{15}$$

Also, by taking $x_3 = x_4 = x_5 = \ldots = x_n = 0$ in (2) we get

$$\wp(2x_1 - x_2) + \wp(x_1 - 2x_2) = 9\wp(x_1) + 9\wp(x_2) - 4\wp(x_1 + x_2) \tag{16}$$

Replacing x_2 by $-x_2$ in (16) and using (8), we get

$$\wp(2x_1 + x_2) + \wp(x_1 + 2x_2) = 9\wp(x_1) + 9\wp(x_2) - 4\wp(x_1 - x_2) \tag{17}$$

Replacing x_1 by $x_1 + x_2$ in (16), we get

$$\wp(2x_1 + x_2) + \wp(x_1 - x_2) = 9\wp(x_1 + x_2) + 9\wp(x_2) - 4\wp(x_1 + 2x_2)$$
$$\wp(2x_1 + x_2) = 9\wp(x_1 + x_2) + 9\wp(x_2) - 4\wp(x_1 + 2x_2) - \wp(x_1 - x_2)$$

(18)

Using (18) in (17), we get

$$\wp(x_1 + 2x_2) = -3\wp(x_1) + \wp(x_1 - x_2) + 3\wp(x_1 + x_2) \tag{19}$$

Interchanging x_1 and x_2 in (19) and using (9), we get

$$\wp(2x_1 + x_2) = -3\wp(x_2) + \wp(x_1 - x_2) + 3\wp(x_1 + x_2) \tag{20}$$

By using (19) and (16) in (13), we have

$$-3\wp(x_2) + \wp(x_1 - x_2) + 3\wp(x_1 + x_2) - 3\wp(x_1) + \wp(x_1 - x_2) + 3\wp(x_1 + x_2)$$
$$= 9\wp(x_1) + 9\wp(x_2) - 4\wp(x_1 - x_2) \tag{21}$$

on solving this we obtain that

$$\wp(x_1 + x_2) + \wp(x_1 - x_2) = 2\wp(x_1) + 2\wp(x_2) \tag{22}$$

which is equation (1). Hence functional equation (2) is quadratic.

Result 2.2 Let F be a linear space, if a function $\xi: E \to F$ satisfies the functional equation (2), then the following results will hold:
(1) $\xi(q^t s) = q^{2t}\xi(s)$ foll all $s \in E, q \in Q, t \in Z$
(2) If ξ is continuous, $\xi(s) = s\xi(1)$ foll all $s \in E$.

3. HYERS-ULAM STABILITY OF (2)

In this section, we take Ξ be a real vector space and $W\varpi$ be a complete modular space. Here, we present the Hyers−Ulam Stability of the functional equation (2). For the mapping $\wp : \Xi \to W\varpi$, we define:
$D\wp(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} \wp(-2x_i + \sum_{j=1, i\neq j}^{n} x_j)$

$$-(n-6)\sum_{1\le i<j\le n}\wp(x_i+x_j)+(n^2-8n+3)\sum_{i=1}^{n}\wp(x_i) \quad (23)$$

for all $x_1, x_2, \dots, x_n \in \Xi$.

Theorem 3.1 Let $\psi: \Xi^n \to [0,\infty)$ be a function such that

$$\lim_{m\to\infty}\frac{1}{2^{2m}}\psi(2^m x_1, 2^m x_2, \dots, 2^m x_n)=0$$

and

$$\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0)<\infty \quad (24)$$

for all $x_1, x_2, \dots . x_n \in \Xi$. If a mapping $\wp: \Xi \to W\varpi$ with $\wp(0)=0$ and

$$\varpi(D\wp(x_1, x_2, \cdots, x_n)) \le \psi(x_1, x_2, \cdots, x_n). \quad (25)$$

for all $x_1, x_2, \cdots, x_n \in \Xi$, then, there exists a unique quadratic mapping $Q_2: \Xi \to W\varpi$ satisfying

$$\varpi(\wp(x)-Q_2(x)) \le \frac{1}{2^2}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0) \quad (26)$$

for all $x \in \Xi$.

Proof: Replace $(x_1, x_2, \dots, x_n)$ by $(x, 0, \dots, 0)$ in (25)

$$\varpi(\wp(2x)-2^2\wp(x)) \le \psi(x, 0, \cdots, 0)$$

$$\varpi\left(\wp(x)-\frac{\wp(2x)}{2^2}\right) \le \frac{1}{2^2}\psi(x, 0, \cdots, 0)$$

and by convexity of ϖ and $\sum_{i=0}^{m-1}\frac{1}{2^{2(i+1)}} \le 1$, we have

$$\varpi\left(\wp(x)-\frac{\wp(2^m x)}{2^{2m}}\right)=\varpi\left(\sum_{i=0}^{m-1}\left(\frac{\wp(2^i x)}{2^{2i}}-\frac{\wp(2^{i+1}x)}{2^{2(i+1)}}\right)\right)$$

$$\le \sum_{i=0}^{m-1}\varpi\left(\frac{\wp(2^i x)}{2^{2i}}-\frac{\wp(2^{i+1}x)}{2^{2(i+1)}}\right)$$

$$\le \frac{1}{2^2}\sum_{i=0}^{m-1}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0) \quad (27)$$

for all x in Ξ. So,

$$\varpi\left(\frac{\wp(2^m x)}{2^{2m}}-\frac{\wp(2^t x)}{2^{2t}}\right)=\frac{1}{2^{2t}}\varpi\left(\frac{\wp(2^{m-t}2^t x)}{2^{2(m-t)}}-\wp(2^t x)\right)$$

$$\le \frac{1}{2^2}\sum_{i=0}^{m-t-1}\frac{1}{2^{2i}}\psi(2^i 2^t x, 0, \dots, 0)$$

$$\le \frac{1}{2^2}\sum_{i=t}^{m-1}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0) \quad (28)$$

for all x in Ξ and all non negative integers m, t with $m>t$. Thus $\left\{\frac{\wp(2^m x)}{2^{2m}}\right\}$ is a Cauchy sequence in complete modular space $W\varpi$, so there exists a $Q_2: \Xi \to W\varpi$ as

$$\lim_{m\to\infty}\frac{\wp(2^m x)}{2^{2m}}=Q_2(x) \quad (29)$$

for all x in Ξ. Now,

$$\varpi\left(\frac{2^2 Q_2(x)-Q_2(2x)}{4^2}\right)=\varpi\left(\frac{1}{4^2}\left(\frac{\wp(2^{m+1}x)}{2^{2m}}-Q_2(2x)\right)+\frac{1}{2^2}\left(Q_2(x)-\frac{\wp(2^{m+1}x)}{2^{2(m+1)}}\right)\right)$$

$$\le \frac{1}{4^2}\varpi\left(\frac{\wp(2^{m+1}x)}{2^{2m}}-Q_2(2x)\right)+\frac{1}{2^2}\varpi\left(Q_2(x)-\frac{\wp(2^{m+1}x)}{2^{2(m+1)}}\right) \quad (30)$$

for all x in Ξ, then by (29) right hand side of (30) tends to 0 as $m \to \infty$. Thus

$$Q_2(2x)=2^2 Q_2(x) \quad (31)$$

Now,

$$\varpi(\wp(x)-Q_2(x)) \le \varpi\left(\sum_{i=0}^{m-1}\left(\frac{\wp(2^i x)}{2^{2i}}-\frac{\wp(2^{i+1}x)}{2^{2(i+1)}}\right)+\frac{\wp(2^m x)}{2^{2m}}-\frac{Q_2(2x)}{2^2}\right)$$

$$\le \frac{1}{2^2}\sum_{i=0}^{m-1}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0)+\frac{1}{2^2}\varpi\left(\frac{\wp(2^{m-1}2x)}{2^{2(m-1)}}-Q_2(2x)\right)$$

$$\le \frac{1}{2^2}\sum_{i=0}^{m-1}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0)+\frac{1}{2^2}\varpi\left(\frac{\wp(2^{m-1}2x)}{2^{2(m-1)}}-Q_2(2x)\right)$$

(32)

for all integer $m>1$ and for all x in Ξ. Applying $m \to \infty$, we get the required result.

Replacing $(x_1, x_2, \cdots, x_n)$ by $(2^m x_1, 2^m x_2, \cdots, 2^m x_n)$ in (25), we get

$$\varpi(D\wp(2^m x_1, 2^m x_2, \cdots, 2^m x_n)) \le \psi(2^m x_1, 2^m x_2, \cdots, 2^m x_n).$$

for all $x_1, x_2, \dots, x_n \in \Xi$.

Therefore

$$\varpi\left(\frac{1}{2^{2m}}D\wp(2^m x_1, 2^m x_2, \cdots, 2^m x_n)\right) \le \frac{1}{2^{2m}}\psi(2^m x_1, 2^m x_2, \cdots, 2^m x_n). \quad (33)$$

Taking the limit $m \to \infty$, we get

$$DQ_2(x_1, x_2, \dots, x_n) = 0 \tag{34}$$

for all $x_1, x_2, \dots, x_n \in \Xi$.

To prove the uniqueness of Q_2, let $T_2: \Xi \to W\varpi$ be another quadratic mapping satisfying (26).

$$\varpi\left(\frac{1}{2}Q_2(x) - \frac{1}{2}T_2(x)\right) \le \frac{1}{2}\varpi\left(\frac{Q_2(2^m x)}{2^{2m}} - \frac{\wp(2^m x)}{2^{2m}}\right) + \frac{1}{2}\varpi\left(\frac{\wp(2^m x)}{2^{2m}} - \frac{T_2(2^m x)}{2^{2m}}\right)$$
$$\le \frac{1}{2^{2m}}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\psi(2^i 2^m x, 0, \dots, 0)$$
$$= \sum_{i=m}^{\infty}\frac{1}{2^{2i}}\psi(2^i x, 0, \dots, 0)$$

By taking $m \to \infty$, we have $Q_2 = T_2$

Therefore, the function Q_2 is unique. This completes the proof.

Corollary 3.2 If a mapping $\xi: \Xi \to W\varpi$ with $\xi(0) = 0$ and

$$\varpi(D\xi(x_1, x_2, \cdots, x_n) \le \sum_{i=1}^{n} ||x_i||^p. \tag{35}$$

for all $x_1, x_2, \cdots, x_n \in x$, then there exists a unique quadratic mapping $Q_2: \Xi \to W\varpi$ satisfying

$$\varpi(\xi(x) - Q_2(x)) \le \frac{\|x\|^p}{2^2 - 2^p} \tag{36}$$

for all $x \in \Xi$.

Corollary 3.3 If a mapping $\xi: \Xi \to W\varpi$ with $\xi(0) = 0$ and

$$||D\xi(x_1, x_2, \cdots, x_n)|| \le \sum_{i=1}^{n} ||x_i||^p \tag{37}$$

for all $x_1, x_2, \cdots, x_n \in \Xi$, then there exists a unique quadratic mapping $Q_2: \Xi \to W\varpi$ satisfying

$$||Q_2(x) - \xi(x)|| \le \frac{||x||^p}{2^2 - 2^p} \tag{38}$$

for all $x \in \Xi$.

Now we discuss the non-stability to the functional equation (2), with the help of an example (motivated from [7]) as follows:

Example 3.4 Let a mapping $\xi: R \to R$ be defined as

$$\xi(x) = \sum_{t=0}^{\infty}\frac{g(2^t x)}{2^{2t}} \tag{39}$$

where $g(x) = \begin{cases} \theta x^2, & |x| < 1 \\ \theta, & \text{else} \end{cases}$

then the mapping $\xi: R \to R$ satisfies the inequality

$$|D\xi(x_1, x_2, \cdots, x_n)| \le (3n^3 - 25n^2 + 28n - 12)\frac{16}{3}\theta\sum_{i=1}^{n}|x_i|^2 \tag{40}$$

for all $x_1, x_2, \cdots, x_n \in R, n \ge 8$, but there does not exist a quadratic mapping $Q_2: R \to R$, which satisfies

$$|\xi(x) - Q_2(x)| \le \varepsilon|x|^2 \tag{41}$$

for all $x \in R$.

Proof: Now $|\xi(x)| \le \sum_{n=0}^{\infty}\left|\frac{g(2^n x)}{2^{2n}}\right| = \sum_{n=0}^{\infty}\frac{\theta}{2^{2n}} = \frac{4}{3}\theta$.

Thus ξ is bounded. Now we will prove that ξ satisfies (41). If $x_i = 0$ for $i = 1,2, \dots n$ then (41) is obvious. If $\sum_{i=1}^{n}|x_i|^2 \ge \frac{1}{2^2}$ then L.H.S of (41) is less than $(3n^3 - 25n^2 + 28n - 12)\frac{4}{3}\theta$. Now suppose, $\sum_{i=1}^{n}|x_i|^2 < \frac{1}{2^2}$ then there exists a positive integer m such that

$$\frac{1}{2^{2(m+1)}} \le \sum_{i=1}^{n}|x_i|^2 \le \frac{1}{2^{2m}}$$

so, that

$$2^{2(m-1)}|x_1|^2 < \frac{1}{2^2}, 2^{2(m-1)}|x_2|^2 < \frac{1}{2^2}, \dots,$$

$2^{2(m-1)}|x_n|^2 < \frac{1}{2^2}$ and $2^m|x_1| < 1, 2^m|x_2| < 1, \dots, 2^m|x_n| < 1$

So, consequently

$$\sum_{i=1}^{n}\left(-2^{m+1}x_i + \sum_{j=1, i\ne j}^{n}2^m x_j\right), \sum_{1\le i<j\le n}2^m(x_i + x_j), \sum_{i=1}^{n}2^m x_i \in (-1,1)$$

Therefore for each $t = 0,1,2, \dots, m-1$, we have

$$\sum_{i=1}^{n}\left(-2^{t+1}x_i + \sum_{j=1, i\ne j}^{n}2^t x_j\right), \sum_{1\le i<j\le n}2^t(x_i + x_j), \sum_{i=1}^{n}2^t x_i \in (-1,1).$$

And so, $D\xi(2^t x_1, 2^t x_2, \dots, 2^t x_n) = 0$ for $t = 0,1,2, \dots, m-1$.

Now,

$$|D\xi(x_1, x_2, \dots, x_n)| \le \sum_{t=0}^{\infty}\frac{1}{2^{2t}}|D\xi(2^t x_1, 2^t x_2, \dots, 2^t x_n)|$$

$$\leq \sum_{t=m}^{\infty} \frac{1}{2^{2t}} (\sum_{i=1}^{n} |g(-2^{t+1}x_i + \sum_{j=1,i\neq j}^{n} 2^t x_j|$$
$$+(n-6)\sum_{1\leq i<j\leq n} |g(2^t x_i + 2^t x_j)| + (n^2 - 8n + 3)\sum_{i=1}^{n} |g(2^t x_i)|$$
$$\leq \sum_{t=m}^{\infty} \frac{1}{2^{2t}} \left(\frac{3n^3 - 25n^2 + 28n - 12}{2}\right)\theta$$
$$= \frac{8(3n^3 - 25n^2 + 28n - 12)}{3}\theta \times \frac{1}{2^{2(m+1)}}$$
$$\leq \frac{16(3n^3 - 25n^2 + 28n - 12)}{3}\theta \sum_{i=1}^{n} |x_i|^2.$$

Thus ξ satisfies (41) . If possible, we assume that there exists a quadratic solution $Q_2: R \to R$ satisfies (41). For every $x \in R$, since ξ is a continuously bounded function, Q_2 is bounded on every open interval containing the origin and continuous at the origin. Q_2 must be of the form $Q_2(x) = cx^2$ for all x. So, $|\xi(x)| \leq (\varepsilon + |c|)|x|^2$ for all $x \in R$. We can find $s > 0$ with $s\theta > \varepsilon + |c|$. If $x \in \left(0, \frac{1}{2^{s-1}}\right)$, then $2^t x \in (0,1)$ for all $t = 0,1,2, \dots, s-1$, we have

$$\xi(x) = \sum_{t=0}^{\infty} \frac{g(2^t x)}{2^{2t}}$$
$$\geq \sum_{t=0}^{s-1} \frac{\theta(2^t x)^2}{2^{2t}} = s\theta x^2 > (\varepsilon + |c|)x^2,$$
$$|\xi(x) - Q_2(x)| > \varepsilon|x|^2.$$

which is contradiction.

CONCLUSION

In this research paper we introduced a new quadratic functional equation in n-variables and discussed it's Hyers-Ulam stability using direct method in modular spaces. We have also given a counter example for non-stability to the functional equation.

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