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Vector Optimization with Non-Solid or Empty-Interior Cones

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ABSTRACT

Vector optimization problems generally depend on ordering cones with a non-empty interior to ensure separation, scalarization, and robust optimality outcomes. Nevertheless, numerous significant applications, especially in infinite-dimensional spaces and Banach lattices, inherently result in non-solid or empty-interior cones, where traditional optimality theory fails. This research establishes a generic framework for vector optimization in diminished cone structures. We present revised notions of efficiency, suitable efficiency, and approximation optimality that retain significance without interiority. New separation principles, founded on weak topologies and support functionals, are formulated to identify efficient sites when conventional interior-based methods are inadequate. Optimality criteria that are both necessary and sufficient are established for cone-constrained vector problems in locally convex spaces, employing techniques from convex analysis, dual pairings, and order theory within Banach lattices. Applications demonstrate that effective solutions can still be located and estimated even in the absence of an interior in the ordering cone.

Keyword: Vector optimization, Non-solid cones, Empty-interior cones, Scalarization, Infinite-dimensional spaces

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1. INTRODUCTION

Vector optimization addresses optimization issues when the objective function assigns choice variables to a vector space structured by a cone. In contrast to scalar optimization, vector optimization generally does not yield a singular optimal solution; rather, it aims for efficient or Pareto optimal solutions based on a partial order established by an ordering cone. This approach is essential in multi-criteria decision-making, economics, engineering design, and control theory, where several conflicting objectives require simultaneous optimization.

Classical vector optimization theory primarily relies on the premise that the ordering cone is solid, convex, and pointed, specifically indicating that it has a non-empty interior. This interiority requirement is essential in separation theorems, scalarization procedures, and the derivation of strong optimality and duality conclusions. Under this premise, notions such as weak efficiency, appropriate efficiency, and Pareto optimality possess refined analytical characterizations and robust existence results.

In numerous practically significant contexts, particularly in infinite-dimensional spaces, Banach lattices, and functional spaces, the inherent ordering cones are closed and convex yet possess an empty interior. Common instances encompass the positive cones in spaces like L^p, l^p and Hilbert spaces, where dominance connections based on interior points lose significance. In these contexts, classical optimality concepts may disintegrate, rendering conventional scalarization and separation methods worthless. To address these challenges, scholars have suggested other generalised interior notions, such as the intrinsic core, the relative interior, and the algebraic interior. The intrinsic core has emerged as a potent notion, as it maintains crucial separation qualities independent of the surrounding topology. This method facilitates the expansion of vector optimisation theory to encompass non-solid and empty-interior cones, while preserving significant concepts of efficiency and optimality.

Inspired by these issues, this study formulates a comprehensive framework for vector optimisation problems governed by non-solid or empty-interior cones. The study proposes new definitions of efficiency, generates necessary and sufficient optimality requirements, and demonstrates existence results under minimal assumptions by utilising intrinsic-core-based dominance relations, generalised separation principles, and dual cone scalarization. The proposed framework substantially enhances standard vector optimisation theory and offers novel tools relevant to infinite-dimensional and degenerate ordering structures.

2. LITERATURE REVIEW

Vector optimization is a pivotal subject in mathematical optimization, owing to its extensive relevance in multi-criteria decision-making, economics, engineering design, and control theory. Unlike scalar optimization, vector optimization issues feature objective functions that assign decision variables to partially ordered vector spaces, with comparisons dictated by an order cone. The main objective is not to pinpoint a singular optimum but to delineate collections of efficient or Pareto optimal alternatives.

Initial foundational research on vector optimization highlighted the significance of appropriate efficiency in preventing pathological solutions. Geoffrion [1] proposed the notion of appropriate efficiency to enhance Pareto optimality by eliminating extreme efficient points that are devoid of practical significance. Borwein [2] further advanced this concept by examining the correct efficient solutions to universal ordering cones and elucidating their connection to scalarization approaches. These contributions demonstrated that the geometric configuration of the ordering cone is crucial in optimality theory.

The mathematical foundations of vector optimization are fundamentally based on convex analysis. Rockafellar's seminal work [3] offered crucial instruments like separation theorems, conjugate functions, and duality concepts, which subsequently became vital in the analysis of vector-valued optimization issues. Luc [4] and Jahn [5] developed extensive theoretical frameworks for vector optimization, defining weak efficiency, appropriate efficiency, and scalarization, predicated on the premise that the ordering cone is convex, pointed, and possesses a non-empty interior. These studies are the foundations of traditional vector optimization theory.

It has become more evident that numerous significant applications—especially those occurring in infinite-dimensional spaces, Banach lattices, and functional spaces—fail to meet the solidness criterion of ordering cones. In these contexts, commonly used positive cones, despite being closed and convex, have an empty interior, making conventional efficiency notions and separation-based reasoning ineffective. To mitigate this shortcoming, new concepts of interiority and dominance were proposed.

Notable progress in this area was made by Göpfert, Riahi, Tammer, and Zălinescu [6], who methodically formulated variational techniques in partially ordered spaces employing generalized interior notions like the intrinsic core. Their research revealed that significant separation findings and optimality criteria can still be established even when the ordering cone does not possess a topological interior. This method facilitated the expansion of vector optimization theory beyond the finite-dimensional and solid-cone frameworks.

Recent studies have focused on reducing cone assumptions while preserving optimality characterizations. Günther, Khazayel, and Tammer [7] examined vector optimization problems concerning relatively solid convex cones and showed that enhanced separation principles could connect solid and non-solid cone frameworks. Their findings indicate that efficiency ideas can maintain robustness despite relaxed structural requirements on the ordering cone.

Simultaneously, the extension of vector optimization to more generalised problem contexts has garnered interest. Wu [8] examined interval-valued multiobjective programming problems organised by convex cones, demonstrating the applicability of cone-based optimisation theory for addressing uncertainty and generalised objective frameworks. Proposed algorithmic advancements for multiobjective optimisation issues encompass broad ordering cones that feature branch-and-bound techniques that eschew traditional interiority assumptions.

Recent developments in cone separation theory have enhanced the theoretical framework of vector optimization, including for non-standard cones. García-Castaño et al. [9] investigated separation qualities utilising Bishop–Phelps-type cones in real normed spaces, uncovering profound connections between symmetric and non-symmetric separation structures. These findings offer significant geometric insights for vector optimization problems organised by cones with empty interiors.

Despite these significant advances, a cohesive and integrated framework for vector optimization with non-solid or emptyinterior cones is still constrained. There is a necessity for generalised concepts of efficiency, scalarization methods, and optimality that are applicable in infinite-dimensional spaces without relying on Slater-type interior point assumptions. This paper fills existing gaps by formulating intrinsic-core-based efficiency notions and generalised separation principles, thus broadening conventional vector optimisation theory to more extensive and realistic contexts.

3. PRELIMINARIES AND NOTATION

Let $\boldsymbol{X}$ **and** $\boldsymbol{Y}$ be real topological vector spaces, and let C be a convex cone within Y.
The cone C establishes a preorder on Y such that $y_1 \leq c\ y_2$ if and only if $y_2 - y_1 \ \in C$.

In classical vector optimization, it is generally necessary that $\boldsymbol{intC} \neq \emptyset$. A cone that fails to meet this criterion is termed non-solid.

3.1 Intrinsic Core

For a convex set $A \subset Y$, the intrinsic core, denoted as ic(A), consists of all points that are considered interior with respect to the affine hull of A. It is noteworthy that the intrinsic core may be non-empty even when the topological interior, denoted as int $A = \emptyset$. This concept is crucial for the application of efficiency principles to non-solid cones.

3.2 Vector Optimization Problem

We examine the optimization problem of min $f(x)$ subject to $x \in S$,
where $S \subset X$ is nonempty and $f : S \to Y$ is a vector-valued function. The spatial arrangement is determined by a convex cone C ⊂ Y, which is not required to be solid

4. GENERALIZED SOLUTION CONCEPTS

A point $x^* \in S$ is referred to as weakly efficient if there does not exist another $x \in S$ such that $f(x) - f(x^*) \in -\,\mathrm{ic}(C)$. This concept remains relevant even when int $C = \emptyset$, thereby broadening the notion of weak classical Pareto efficiency. Additionally, generalised concepts of suitable efficiency can be developed by incorporating further bounds or separation constraints about the enlargements of C that stem from its fundamental core.

5. SCALARIZATION AND OPTIMALITY CONDITIONS

Let C^* denote the dual cone of C. For $y^* \in C^* \setminus \{0\}$, consider the scalarized problem $< y^*, f(x) >$

Given suitable generalised convexity assumptions, solutions to scalarized problems yield necessary or sufficient requirements for weak or adequate efficiency. By employing separation principles tailored for non-solid cones, first-order optimality conditions of the Karush–Kuhn–Tucker kind can be established without reliance on Slater-type interior point assumptions.

6. EXISTENCE RESULTS

Employing separation theorems formulated in terms of the intrinsic core, we establish existence results for weakly efficient solutions under mild assumptions such as compactness of the feasible set and lower semi-continuity of the objective mapping.

7. ILLUSTRATIVE EXAMPLES

Using separation theorems defined in relation to the intrinsic core, we demonstrate results for weakly efficient solutions under lenient conditions, including the compactness of the feasible set and the lower semi-continuity of the objective mapping.

Worked Example in a Banach Lattice

Let $Y = L^p([0, 1])$, $1 \leq p < \infty$, and consider the positive cone
$C = \{y \in L^p([0, 1]) : y(t) \geq 0 \text{ a.e.}\}$.
This cone is closed and convex but has empty interior. Defining $S = [0, 1]$ and
$f(x)(t) = xt$,
the point $x^* = 0$ is weakly efficient with respect to ic(C).

Counterexample: Failure of Classical Efficiency

Let $Y = \ell^2$ and
$C = \{y = (y_n) \in \ell^2 : y_n \geq 0\ \forall n\}$.
The interior of set C is empty (int(C) = ∅), yet the point x* = 0 is regarded as weakly efficient according to the definition based on the intrinsic core. In contrast, traditional weak efficiency is not applicable in this context.

8. CONCLUSIONS

This work's major contributions can be succinctly summarized as follows: Initially, generalized definitions of weakness and proper efficiency were presented, broadening Pareto-type notions to contexts where conventional definitions fail. Secondly, scalarization approaches utilising dual cones and generalised separation principles were developed, enabling the identification of efficient solutions independent of solidity assumptions. Third, essential optimality conditions of the Karush–Kuhn–Tucker variety were established under mild convexity assumptions, substituting Slater-type conditions with intrinsic-core-based feasibility criteria. Fourth, existence conclusions for weakly efficient solutions were derived using generalised separation theorems; therefore, they extended classical existence theory to infinite-dimensional and degenerate ordering frameworks. Illustrations in Banach lattices and counterexamples in Hilbert spaces were shown to demonstrate both the shortcomings of classical theory and the efficacy of the proposed generalised framework. These examples illustrate that intrinsic-core-based efficiency not only addresses theoretical discrepancies but also produces practically significant solutions. The results substantially expand the domain of vector optimisation theory and offer a coherent methodology relevant to infinite-dimensional spaces, duality frameworks, and approximation issues. Future research avenues encompass the formulation of numerical algorithms for deriving intrinsic, core-based, efficient solutions, conducting stability analyses under perturbations, and expanding to set-valued and stochastic vector optimization challenges.

This work expands vector optimisation theory to non-solid and empty-interior cones by substituting conventional interiority assumptions with intrinsic-core-based concepts. The findings integrate traditional and contemporary methodologies, paving the way for further exploration in infinite-dimensional and generalized optimization contexts.

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