Original Article
Bulletin of Pure and Applied Sciences.
Mathematics & Statistics Vol.45 E, No.1.
January-June 2026: P.22-35

Print version ISSN 0970 6577
Online version ISSN 2320 3226
Available online at www.bpasjournals.com

A Geometric Approach to Fixed Point Theorems for Mappings Satisfying Implicit Relations in Quasi Multiplicative Metric Spaces

A. Kamal*

Author Affiliation:
Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah, 51452, Saudi Arabia

***Corresponding Author:** **A. Kamal**, Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah, 51452, Saudi Arabia
E-mail: ak.ahmed@qu.edu.sa

ABSTRACT
In this work, we introduce a generalized type of multiplicative metric space so-called quasi multiplicative metric space (abbreviated as q-multiplicative metric space). We state and prove fixed point theorems in this space. Also we investigate geometric properties of common fixed points in the context of the fixed-circle problem. To do this, we propose some new solutions to the fixed-circle problem on q-multiplicative metric space

Keyword: Multiplicative metric space; quasi multiplicative metric space; Fixed point.
AMS: 46E15, 47B33, 47B38, 54C35.

How to cite this article: Kamal A. A Geometric Approach to Fixed Point Theorems for Mappings Satisfying Implicit Relations in Quasi Multiplicative Metric Spaces. *Bulletin of Pure and Applied Sciences- Math & Stat.*, 2026;45E (1):22-35.

Received on 15.08.2025, Revised on 19.10.2025, Accepted on 12.11.2025

1. INTRODUCTION

In 1906, Frechet introduced the notion of metric space. We can see recent generalizations of metric space and concern topological structures in literature. (See, e.g.,[1-16]). Bashirov et al. [1] studied the concept of multiplicative calculus. After that Ö zavsar et al. [17] were the first researchers who discussed multiplicative metric mapping by giving some topological properties of the relevant multiplicative metric space. They observed that the set of positive real numbers R_+ is a complete multiplicative metric space with respect to the multiplicative absolute value function. Furthermore, they mentioned concept of multiplicative contraction mapping. For more details, (see, e.g., [17,18]).

Definition 1.1 *[17] Suppose that X is a non-empty set. A function $d : X \times X \to R$ is called a multiplicative metric if it satisfies following conditions:*

1. $d(\eta, \xi) \geq 1$; *for all* $\eta, \xi \in X$ *and* $d(\eta, \xi) = 1$ *if and only if* $\eta = \xi$;
2. $d(\eta, \xi) = d(\xi, \eta)$; *for all* $\eta, \xi \in X$;
3. $d(\eta, \xi) \leq d(\eta, z).d(z, \xi)$, *for all* $\eta, \xi, z \in X$.

Then the (X, d) is called multiplicative metric space (MMS).

Example 1.1 *[1] Let R_+^n be the set of all n- tuples of positive real numbers. Let $d^*: R_+^n \times R_+^n \to R^+$ be mapping defined as follows:*

$$d^*(x,y) = \left(\left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* . \left|\frac{x_3}{y_3}\right|^* \dots \left|\frac{x_n}{y_n}\right|^* \right)$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R_+^n$ and $|\,.\,|^*: R \to R^+$ is defined by

$$|a|^* = \begin{cases} a & if \; a \geq 1 \\ \dfrac{1}{a} & if \; 0 < a < 1 \end{cases}$$

Then it is easy to see that d is a multiplicative metric on R_+^n.

Example 1.2 *[1] Suppose that $w > 1$ is a fixed real number. Then $d_w: R^n \times R^n \to R$ defined by*

$$d_w(x,y) := |x-y|_w := \prod_{i=1}^{n} \left|\frac{x^{w_i}}{y^{w_i}}\right|^*$$

holds the multiplicative metric conditions. Note that d_w satisfies the following relation:

$$\prod_{i=1}^{n} \left|\frac{w^{x_i}}{w^{y_i}}\right|^* = w^{\sum_{i=1}^{n}|x_i - y_i|^*}$$

In this paper, we introduce the new notion of q-multiplicative metric space. We prove fixed point theorems for single mappings and a common fined point in q-multiplicative metric space.

2. FIXED POINT THEOREMS IN Q-MULTIPLICATIVE METRIC SPACES

First we introduce the concept of q-multiplicative metric space which generalizes the concept of multi-plicative metric space as following

Definition 2.1 *Let $X \neq \phi$. q-Multiplicative metric is a mapping $d_q : X \times X \to R^+$, for all $x, y, z \in X$ satisfying the following conditions,*

(i) $dq(x,y) \geq 1$, *and* $dq(x,y) = 1$ *if and only if* $x = y$;

(ii) $dq(x,y) \leq dq(x,z).dq(z,y)$.

where d_q is called q-multiplicative metric(q-MM) and (X, d_q) is called q-multiplicative metric spaces (q- MMS)

Example 2.1 *Let $X = R^+ \cup \{0\}$. Define $d : X \times X \to [1, \infty)$ as :*

$$d_q(\eta, \xi) = a^{|\eta - \xi|^2} \quad \forall\, \eta, \xi \in X, \quad a \geq 1.$$

Then (X, d) is q-multiplicative metric space.

Definition 2.2 *(Multiplicative continuity) Let (X, d_q1) and (Y, d_q2) be two q-multiplicative metric space and $f : X \to Y$ be a function. If f holds the requirement that, for every $\epsilon \geq 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.*

Definition 2.3 *Let (X, d_q) is q-multiplicative metric space, then we have*

(i) A point $x \in X$ is said to multiplicative limit point of $W \subset X$, if and only if $(B\epsilon(x)\{x\}) \cap W \neq \emptyset$ for every $\epsilon > 1$.

(ii) A set $W \subset X$ is multiplicative closed in (X, d_q) if W is contains all of its multiplicative limit point.

(iii) A set W is multiplicative bounded if there exist $x \in X$ and $L > 1$ such that $W \subseteq BL(x)$.

Definition 2.4 *A sequence $\{x_n\}$ in q-converges to x if*

$$\lim_{n\to\infty} dq(x_n, x) = \lim_{n\to\infty} dq(x, x_n) = 1$$

In this case x is called a q-limit of $\{x_n\}$.

Definition 2.5 *A sequence $\{x_n\}$ in q-multiplicative metric space (X, d_q) is called q−Cauchy if for all $\epsilon > 1$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$, we have $d_q(x_n, x_m) < \epsilon$, and $d_q(x_m, x_n) < \epsilon$.*
Definition 2.6 *A q-multiplicative metric space (X, d_q) is called complete if every q-Cauchy sequence in it is q-convergent.*
Definition 2.7 *(Multiplicative open set): Let (X, dq) be a q-multiplicative metric space and $W \subset X$. If every point of W is a multiplicative interior point of W, this mean, $W = int(W)$, then W is called a multiplicative open set.*

Now, we state the following lemma without proof.
Lemma 2.1 *(i) Let (X, dq) be an q-multiplicative metric space and $\{\eta_n\}_{n=1}^{\infty}$ is q-multiplicative Cauchy if and only if*

$$dq(\eta_n, \eta_m) \to_q 1, \quad as \quad n, m \to \infty.$$

(ii) An q-multiplicative metric space (X, dq) is complete if every multiplicative Cauchy sequence in X is convergent itself.

Lemma 2.2 *Suppose that (X, d_q) is q-multiplicative metric space. Then any subsequence of convergent sequence in X is convergent.*
Theorem 2.1 *Let (X, dq) be complete q-multiplicative metric space and $f : X \to X$ be a map such that*

$$d(fx, fy) \leq max\{dq(x, fx), dq(y, fy), dq(x, y), dq(x, fy), dq(fx, y)\}^{\lambda}. \tag{1}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$ and f is continuous. Then f has unique fixed point.
Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X by

$$x_n = fx_{n-1}, \quad for\ n = 1, 2, \ldots.$$

Case I: If $x_{n_0} = x_{n_0} + 1$ for some $n_0 = 1, 2, \ldots$, then it is clear that $\{x_n\}$ has a fixed point.
Case II: assume $x_n \neq x_{n+1} + 1$ for all n. Then from (1)

$$dq(x_n, x_{n+1}) = dq(fx_{n-1}, fx_n)$$

$$\leq max\{dq(x_{n-1}, fx_{n-1}), dq(x_n, fx_n), dq(x_{n-1}, x_n), dq(x_{n-1}, fx_n), dq(fx_{n-1}, x_n)\}^{\lambda}$$
$$\leq max\{dq(x_{n-1}, x_n), dq(x_n, x_{n+1}), dq(x_n, x_n), dq(x_{n-1}, x_{n+1}), dq(x_n, x_n)\}^{\lambda}$$
$$\leq max\{dq(x_{n-1}, x_n), dq(x_n, x_{n+1}), dq(x_{n-1}, x_n), dq(x_{n-1}, x_n). dq(x_n, x_{n+1}), dq(x_n, x_n)\}^{\lambda}$$
$$= dq(x_{n-1}, x_n)^{\lambda}. dq(x_n, x_{n+1})^{\lambda}$$
$$\leq dq(x_{n-1}, x_n)^{\{\lambda+\lambda 2\}}. dq(x_n, x_{n+1})^{\lambda}$$
$$\leq d_q(x_{n-1}, x_n)^{\sum_{k=1}^{n}\lambda^k}. (x_n, x_{n+1})^{\lambda^n}$$
$$\leq d_q(x_{n-1}, x_n)^{\sum_{k=1}^{n}\lambda^k}$$
$$dq(x_n, x_{n+1}) \leq dq(x_{n-1}, x_n)^{\frac{\lambda}{1-\lambda}}$$

for all $n \in$ N and take $h = \frac{\lambda}{1-\lambda}$
Thus

$$dq(x_n, x_{n+1}) \leq dq(x_{n-1}, x_n)^h \tag{2}$$

Therefore,

$$dq(fx_{n-1}, x_n) \leq dq(fx_{n-2}, fx_{n-1})^h (3)$$

Similarly

$$dq(fx_n, x_{n+1}) \leq dq(fx_{n-1}, fx_n)^h (4)$$

Therefore,

$$dq(fx_n\ ,x_{n+1}) \leq dq(fx_{n-1}, fx_n\ )^h$$

$$\leq dq(fx_{n-2}\ , fx_{n-1}\ )^{h^2}$$

$$\leq\ dq(fx_{n-3}, fx_{n-2})^{h^3}$$

.

.

$$\leq\ dq(fx_0, fx_1)^{h^n}$$

There exits $n_0 \in$ N and Consider $n, m \in$ N with $n > m$. From (*ii*) of Definition 2.1, we conclude inductively that

$$\Rightarrow\ dq(x_m, x_n) \leq \prod_{i=m}^{n-1} d_q(x_i, x_{i+1})$$

.

.

.

$$dq(x_m, x_n) \leq \prod_{i=m}^{n-1} d_q(x_i, x_{i+1})$$

$$dq(x_m, x_n) \leq \prod_{i=m}^{\infty}[\ d_q(x_0, x_1)]\text{^}h^i$$

We utilize the fact that the logarithmic is a continuous and increasing function. This leads to

$$\log dq(x_m, x_n)\ \leq \log\left(\prod_{i=m}^{\infty}[\ d_q(x_0, x_1)]\text{^}h^i\ \right)$$

$$= \quad (\sum_{i=m}^{\infty} h^i \log[\ d_q(x_0, x_1)])$$

$$= (\log\ [\ d_q(x_0, x_1)]) \quad \sum_{i=m}^{\infty} h^i$$

$$= (\log\ [\ d_q(x_0, x_1)])\ \frac{h^m}{1-h} \quad (h\ \in [0,1))$$

sum of a Geometric progression. As $m, n \to \infty$, we get

$$log\ [dq(x_m\ , x_n\ )] \to_q 0 \qquad (5)$$

We use the fact that the exponential function is continuous. We apply the exponential function on both sides of (5) and get

$$dq(x_m, x_n\ )\ \to_q\ 1\ as\ m, n\ \to\ \infty$$

This makes $\{fx_n\}$ a multiplicative Cauchy sequence by Lemma 2.1.
From assumption, we know that X is complete. This means that there is $z\ \in\ X$ such that $\lim_{n\to\infty} x_n\ = z$.
From the construction of the sequence $\{x_n\}$, we have $x_n\ = x_{n+1}$. Taking $n\ \to\ \infty$, we have

$$\lim_{n\to\infty} x_n\ = z = \lim_{n\to\infty} x_{n+1}.$$

As $z \in X$, there is $w \in X$ such that $fw = w$, we claim that w is a fixed point of f .
Let $x = x_n$ and $y = w$ in (1). Then we have

$$dq(fxn, fw) \leq max\{(xn, fxn), dq(w, fw), dq(xn, w), dq(xn, fw), dq(fxn, w)\}^h$$

$$dq(fxn, w) \leq max\{dq(xn, fxn), dq(w, w), dq(xn, w), dq(xn, w), dq(fxn, w)\}^h \quad (6)$$

Since f is continuous function. Hence, on taking $n \to \infty$ in (6) we have,

$$dq(fz, w) \leq max\{dq(z, fz), 1, dq(z, w), dq(z, w), dq(fz, w)\}^h \quad (7)$$

$$dq(z, w) \leq max\{1, 1, dq(z, w), dq(z, w), dq(z, w)\}^h \quad (8)$$

Suppose $u = d_q(z, w)$, $v = 1$ deduce that for some $0 \leq h < 1$, we have
$dq(z, w) < 1 \Rightarrow dq(z, w) = 1 (\because Definition 2.1) \Rightarrow z = w$
making z the unique fixed point of f.
Popa [18] utilized the idea of implicit function to unify the fixed point theorems.
Now, we define the following class of implicit functions as follows:
Let Θ be a family of all continuous real functions $\theta : [1, \infty)^6 \to [1, \infty)$ and the following conditions:
(θ_1) θ is non-increasing in the fifth variable and
$\Theta (u, v, v, u, v.u, 1) \leq 1$ for $u, v \geq 1$ implies that $v \leq u^h$ for some $h \in [0, 1)$.
(θ_2) $\theta (u, 1, 1, u, u, u) > 1$ for all $u > 1$.

Example 2.2 *The following functions $\theta \in \Theta$ satisfy the properties (θ_1) and (θ_2)*

(i) $\theta(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_4}{t_1^a}$ where $a \in [0,1)$

(ii) $\theta(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{(t_2 t_3)^b}$ where $t_1 \geq t_2$, $b \in \left[\mathbf{0}, \frac{1}{2}\right)$

(iii) $\theta(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_3\, t_4}{(t_5 t_6)^c}$ where $c \in \left[\mathbf{0}, \frac{1}{2}\right)$

Theorem 2.2 *Let (X, dq) be compete q-multiplicative metric space and $f : X \to X$ be a map. Assume that Θ is a family of all continuous real functions $\theta: [1, \infty)^6 \to [1, \infty)$ exists satisfying (θ_1) and (θ_2) such that*

$$\Theta (d_q(fx, fy), d_q(x, fx), d_q(y, fy), d_q(x, y), d_q(x, fy), d_q(fx, y)) \leq 1, \quad (9)$$

for all $x \in X$, and f is continuous. Then f has unique fixed point.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X by
$x_n = fx_{n-1}, \quad for\ n = 1, 2, \ldots.$
Firstly if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 1, 2, \ldots$, then it is clear that $\{x_n\}$ has a fixed point. Secondly, assume $x_n \neq x_{n+1}$ for all n. Then we have from (9)
$dq(x_n, x_{n+1}) = dq(fx_{n-1}, fx_n)$

$\theta(dq(fx, fy), dq(x, fx), dq(y, fy), dq(x, y), dq(x, fy), dq(fx, y)) \leq 1,$
$\theta (d_q(fx_{n-1}, fx_n), d_q(x_{n-1}, fx_{n-1}), d_q(x_n, fx_n), d_q(x_{n-1}, x_n), d_q(x_{n-1}, fx_n), d_q(fx_{n-1}, x_n)) \leq 1,$
$\theta (d_q(x_n, x_{n+1}), d_q(x_{n-1}, x_n), d_q(x_n, x_{n+1}), d_q(x_{n-1}, x_n), d_q(x_{n-1}, x_{n+1}), d_q(x_n, x_n)) \leq 1,$
$\theta (d_q(x_n, x_{n+1}), d_q(x_{n-1}, x_n), d_q(x_n, x_{n+1}), d_q(x_{n-1}, x_n), d_q(x_{n-1}, x_n).d_q(x_n, x_{n+1}), d_q(x_n, x_n)) \leq 1.$
From (*i*) of Definition 2.1, we note that $d_q(x_n, x_n) = 1$.
If we set $u = dq(x_n, x_{n+1})$, $v = dq(x_{n-1}, x_n)$, becomes

$$\theta (u, v, v, u, vu, 1)) \leq 1. \quad (10)$$

From (*ii*) of Definition 2.1, we note that

$$\begin{aligned} dq(x_{n-1}, x_{n+1}) &\leq dq(x_{n-1}, x_n). dq(x_n, x_{n+1}) \\ &\leq uv. \end{aligned}$$

From the assumption θ is non-increasing in fifth variable. Therefore, 10, implies that
$\Theta (u, v, v, u, vu, 1)) \leq 1$.
This implies that there is $h \in [0, 1)$ such that $u \leq v^h$

$$dq(x_{n-1}, x_n) \leq [dq(x_n, x_{n+1})]^h (11)$$

Extending (11) for large n, we get

$$dq(x_n, x_{n+1}) \leq [dq(x_{n-1}, x_n)]^h$$
$$\leq [dq(x_{n-2}, x_{n-1})]^{h^2}$$
$$\leq [dq(x_{n-3}, x_{n-2})]^{h^3}$$
$$\leq [dq(x_{n-4}, x_{n-3})]^{h^4}$$
.
.
.

$$\Rightarrow dq(x_n, x_{n+1}) \leq [dq(x_0, x_1)]^{h^n}$$

Consider $n, m \in \mathrm{N}$ with $n > m$. From (*ii*) of Definition 2.1, we conclude inductively that

$$dq(x_m, x_n) \leq \prod_{i=m}^{n-1} d_q(x_i, x_{i+1})$$
.
.
.

$$dq(x_m, x_n) \leq \prod_{i=m}^{n-1} d_q(x_i, x_{i+1})$$
$$dq(x_m, x_n) \leq \prod_{i=m}^{\infty} [\, d_q(x_0, x_1)]\verb|^|h^i$$

We utilize the fact that the logarithmic is a continuous and increasing function. This leads to

$$\log dq(x_m, x_n) \leq \log\left(\prod_{i=m}^{\infty} [\, d_q(x_0, x_1)]\verb|^|h^i\right)$$
$$= \log(\sum_{i=m}^{\infty} h^i \log[\, d_q(x_0, x_1)])$$
$$= (\log [\, d_q(x_0, x_1)]) \quad \textstyle\sum_{i=m}^{\infty} h^i$$
$$= (\log [\, d_q(x_0, x_1)]) \frac{h^m}{1-h} \quad (h \in [0,1))$$

sum of a Geometric progression. As $m, n \to \infty$, we get

$\log [d_q(x_m, x_n)] \to_q 0$ (12)

We use the fact that the exponential function is continuous. We apply the exponential function on both sides of (12) and get

$$d_q(x_m, x_n) \to_q 1 \; as \; m, n \to \infty$$

This makes $\{fx_n\}$ a multiplicative Cauchy sequence by Lemma 2.1.

From assumption, we know that X is complete. This means that there is $z \in X$ such that $\lim_{n\to\infty} x_n = z$. From the construction of the sequence $\{x_n\}$, we have $x_n = x_{n+1}$. Taking $n \to \infty$, we have

$$\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} x_{n+1}.$$

As $z \in X$, there is $w \in X$ such that $fw = w$, we claim that w is a fixed point of f.
Let $x = x_n$ and $y = w$ in (9). Then we have

Θ ($d_q(fx_n, fw)$, $d_q(x_n, fx_n)$, $d_q(w, fw)$, $d_q(x_n, w)$, $d_q(x_n, fw)$, $d_q(fx_n, w)$) ≤ 1

$$\Theta\,(d_q(fx_n, w), d_q(x_n, fx_n), d_q(w, w), d_q(x_n, w), d_q(x_n, w), d_q(fx_n, w)) \leq 1 \quad (13)$$

Since θ is continuous function. Hence, on taking $n \to \infty$ in (13) we have,

Θ ($d_q(fz, w)$, $d_q(z, fz)$, 1, $d_q(z, w)$, $d_q(z, w)$, $d_q(fz, w)$) ≤ 1 (14)

Θ ($d_q(z, w)$, 1, 1, $d_q(z, w)$, $d_q(z, w)$, $d_q(z, w)$) ≤ 1 (15)

By the property (θ_2) with $u = d_q(z, w), v = 1$ deduce that for some $0 \leq h < 1$, we have $u \leq v^h \Rightarrow d_q(z, w) < 1 \Rightarrow d_q(z, w) = 1(\because Definition 2.1) \Rightarrow z = w$

making z the unique fixed point of f.

From the above theorem, we obtain the following results as special cases.

Theorem 2.3 *Let* (X, d_q) *be compete* q *multiplicative metric space and* $f : X \to X$ *be a map satisfying the condition*

$$dq(fx, fy) \leq dq(x, y)^{h_1}$$

for all $x, y \in X$, $0 < h_1 < 1$. *Then f has a unique fixed point in X.*

Theorem 2.4 *Let* (X, d_q) *be compete* q *multiplicative metric space and* $f : X \to X$ *be a map satisfying the condition*

$$dq(fx, fy) \leq (dq(x, fy).dq(y, fx))^{h_2}$$

for all $x, y \in X$, $0 < h_2 < \frac{1}{2}$ *Then f has a unique fixed point in X.*

Theorem 2.5 *Let* (X, d_q) *be compete* q *multiplicative metric space and* $f : X \to X$ *be a map satisfying the condition*

$$dq(fx, fy) \leq (dq(x, fx).dq(y, fy))^{h_3}$$

for all $x, y \in X$, $0 < h_3 < \frac{1}{2}$ *Then f have a unique fixed point in X.*

Theorem 2.6 *Let* (X, d_q) *be compete q- multiplicative metric space and* $f : X \to X$ *be a map said to be a Zamfirescu satisfying the condition is true:*

1. $dq(fx, fy) \leq dq(x, y)^{h_1}$
2. $dq(fx, fy) \leq (dq(x, fy).dq(y, fx))^{h_2}$
3. $dq(fx, fy) \leq (dq(x, fx).dq(y, fy))^{h_3}$

$\forall\, x, y \in X, h_1 \in [0, 1)$ *and* $h_2, h_3 \in [0, \frac{1}{2})$. *Then f has a unique fixed point in X.*

Corollary 2.1 *Let* (X, d_q) *be a q- multiplicative metric space and* $f : X \to X$ *be a contraction map. If* (X, d_q) *is complete, then f has a unique fixed point.*

Theorem 2.7 *Let* (X, d_q) *be compete q-multiplicative metric space and T,* $f : X \to X$ *be a map. Assume that* Θ *is a family of all continuous real functions* $\theta: [1, \infty)^6 \to [1, \infty)$ *exists satisfying* (θ_1) *and* (θ_2) *such that*

$$\Theta\,(dq(Tx, Ty), dq(fx, fy), dq(fx, Tx), dq(fy, Ty), dq(fx, Ty), dq(fy, Tx)) \leq 1, \quad (16)$$

for all $x \in X$, *and f is continuous. Then f has unique fixed point.*

3. Common fixed point theorems in q-multiplicative metric spaces

Definition 3.1 *Let* Θ *be a family of all continuous real functions* $\theta: [1, \infty)^4 \to [1, \infty)$ *and the following conditions:*

a. (a_1) $u \leq \theta\,(v, u, v, \sqrt{u.v})$ *or* (a_2) $u \leq \theta(u, v, v, x)$.

b. (b_1) $u \leq \theta\,(v, \sqrt{u.v}, 1, u.v)$ *or* (b_2) $x \leq \theta(u, v, u, u)$.

then there exists a real number $0 < h < 1$ *such that* $u \leq v^h$.

Now we define the following conditions:

Condition (I) : Let X be a q- multiplicative metric space and S, T be two self-mappings of X such that for all $x, y \in X$ satisfying the condition:

$$d_q(Sx, Ty) \leq \theta(d_q(x, y), d_q(x, Sx), d_q(y, Ty), \sqrt{dq(x, Ty).dq(y, Sx)}) \quad (17)$$

Condition (II): Let X be a q- multiplicative metric space and S, T be two self-mappings of X such that for all $x, y \in X$ satisfying the condition:

$$d_q(Sx, Ty) \leq \theta(d_q(x, y), \sqrt{dq(x, Sx).dq(y, Ty)}, 1, d_q(x, Ty).d_q(y, Sx)) \quad (18)$$

Theorem 3.1 *Let X be a q- multiplicative metric space and S, T be two continuous self-mappings of X satisfying the condition (I). Then S and T have a unique common fixed point in X.*

Proof: For a given $x_0 \in X$ and $n \geq 1$, take $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define a sequence of elements of X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n} = Tx_{2n-1}$ for $n = 0, 1, 2, 3, \cdots$.
Now for all $u \in X$, by condition (*I*), we obtain

$d_q(x_{2n+1}, x_{2n}) \;=\; d_q(Sx_{2n}, Tx_{2n-1})$

$\leq \theta(dq(x2nx2n-1), dq(x2n, Sx2n), dq(x2n-1, Tx2n-1),$
$\sqrt{dq(x2n, Tx2n-1) + dq(x2n-1, Sx2n)})$

$= \theta(dq(x2n, x2n-1), dq(x2n, x2n+1), dq(x2n-1, x2n),$
$\sqrt{dq\ (x2n, x2n).dq\ (x2n-1, x2n+1)}\ )$

$= \theta(dq(x2n, x2n-1), dq(x2n, x2n+1), dq(x2n, x2n-1),$
$\sqrt{dq\ (x2n-1, x2n+1)})$

$= \theta(\ dq(x2n, x2n-1), dq(x2n, x2n+1), dq(x2n, x2n-1),$
$\sqrt{dq\ (x2n-1, x2n).dq\ (x2n, x2n+1)})$

Hence from definition 3.1 (a_1), we have

$$d_q(x_{2n+1}, x_{2n}) \leq [d_q(x_{2n}, x_{2n-1})]^h \quad (19)$$

where, $0 < h < 1$. Similarly, we have

$$d_q(x_{2n}, x_{2n-1}) \leq [d_q(x_{2n-1}, x_{2n-2})]^h. \quad (20)$$

Hence from (19) and (20), we have

$$dq(x_{2n+1}, x_{2n}) \;\leq\; [dq(x_{2n-1}, x_{2n-2})]^{h^2}$$

on continuing this process, we get

$$dq(x_{2n+1}, x_{2n}) \;\leq\; [dq(x_1, x_0)]^{h^{2n}}$$

For every $n > m$, we have

$d_q(x_n, x_m) \;\leq\; d_q(x_n, x_{n-1}) + d_q(x_{n-1}, x_{n-2}) + \cdots + d_q(x_{m+1}, x_m)$

$\leq\; [dq(x_1, x_0)]^{(h^{n-1}+h^{n-2}+\cdots+h^m)}$

$\leq\; [dq(x_1, x_0)]^{\frac{h^m}{1-h}}.$

Since $0 < h < 1$, by Definition 3.1 $\left(\frac{h^m}{1-h}\right) < 1$ as $m \to \infty$. Hence $d_q(x_n, x_m) \to_q 1$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of S and T that $Sz = Tz = z$. Thus z is a common fixed point of S and T.

Uniqueness Let w be another common fixed point of S and T, that is $Sw = Tw = w$. Then, we have

$d_q(z, w) \;=\; d_q(Sz, Tw)$

$\leq \theta(d_q(z, \mathrm{w}), d_q(z, \mathrm{Sz}), d_q(w, \mathrm{Tw}), \sqrt{dq(z, Tw).dq(w, Sz)})$

$\leq \theta(d_q(z, \mathrm{w}), 1, 1, d_q(z, \mathrm{w}),$ (*)

By Definition 3.1 (a_2) and the inequality (*), we get

$$d_q(z, w) \leq 1.$$

Hence $z = w$ and for all $u \in X$. Thus z is a unique common fixed point of S and T.

Corollary 3.1 *Let X be a q- multiplicative metric space and T be a self-mapping of X satisfying the condition*

$$dq(Tx, Ty) \;\leq\; \theta(dq(x, y), dq(x, Tx), dq(y, Ty), \sqrt{dq(x, Ty).dq(y, Tx)}) \quad (21)$$

for all x, y ∈X. Then T has a unique fixed point in X.

Proof: The proof of corollary has immediately follows from above Theorem 3.1 by taking $f = T$. This completes the proof.

We prove the following theorem using the condition (*II*).

Theorem 3.2 *Let X be a q- multiplicative metric space and f, T be two continuous self-mappings of X* *satisfying the condition (II). Then f and T have a unique common fixed point in X.*

Proof: For a given $x_0 \in X$ and $n \geq 1$, take $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define a sequence of elements of X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n} = Tx_{2n-1}$ for $n = 0, 1, 2, 3, \cdots$.
Now for all $u \in X$, by condition (II), we obtain

$$dq(x_{2n+1}, x_{2n}) = dq(Sx_{2n}, Tx_{2n-1})$$
$$\leq \theta(dq(x_{2n}, x_{2n-1}), \sqrt{(dq(x_{2n-1}), Tx_{2n-1}). dq(x_{2n}, Sx_{2n})},$$
$$dq(x_{2n}, Tx_{2n-1}), dq(x_{2n-1}, Sx_{2n}))$$
$$= \theta(dq(x_{2n}, x_{2n-1}), \sqrt{(dqx_{2n-1}, x_{2n}). dq(x_{2n}, x_{2n+1})},$$
$$dq(x_{2n}, x_{2n}), dq(x_{2n}, x_{2n+1}))$$

$$= \theta(dq(x_{2n}, x_{2n-1}), \sqrt{(dq(x_{2n-1}, x_{2n}). dq(x_{2n}, x_{2n+1})},$$
$$1, dq(x_{2n}, x_{2n+1}))$$

$$\leq \theta(dq(x_{2n}, x_{2n-1}), \sqrt{(dq(x_{2n-1}, x_{2n}). dq(x_{2n}, x_{2n+1})},$$
$$1, dq(x_{2n-1}, x_{2n}) + dq(x_{2n}, x_{2n+1}))$$

Hence from Definition 3.1 (b_1), we have
$d_q(x_{2n+1}, x_{2n}) \leq d_q(x_{2n}, x_{2n-1})^h$ *where* $0 < h < 1$. (22)

Similarly, we have

$$dq(x_{2n}, x_{2n-1}) \leq dq(x_{2n-1}, x_{2n-2})^h \quad (23)$$

Hence from (22) and (23), we have

$$dq(x_{2n+1}, x_{2n}) \leq dq(x_{2n-1}, x_{2n-2})^{h^2}$$

on continuing this process, we get

$$dq(x_{2n+1}, x_{2n}) \leq dq(x_1, x_0)^{h^{2n}}$$

For every $n > m$, we have

$$dq(x_n, x_m) \leq dq(x_n, x_{n-1}) + dq(x_{n-1}, x_{n-2}) + \cdots + dq(x_{m+1}, x_m)$$
$$\leq [dq(x_1, x_0)]^{(h^{n-1} + h^{n-2} + \cdots + h^m)}$$
$$\leq [dq(x_1, x_0)]^{\frac{h^m}{1-h}}.$$

Since $0 < h < 1$, by Definition 3.1, $\left(\frac{h^m}{1-h}\right) < 1$ as $m \to \infty$. Hence $dq(x_n, x_m)$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of S and T that $Sz = Tz = z$. Thus z is a common fixed point of S and T.

Uniqueness Let w be another common fixed point of S and T, that is $Sw = Tw = w$. Then, we have

$$d_q(z, w) = d_q(Sz, Tw) \quad (24)$$

$$\leq \theta(dq(z, w), \sqrt{dq(w, Tw). dq(z, Sz)}, dq(z, Tw), dq(w, Sz)) \quad (25)$$
$$\leq \theta(dq(z, w), 1, dq(z, w), dq(z, w)). \quad (26)$$

By Definition 3.1 (b_2) and the inequality (26), we get

$$d_q(z, w) \leq 0.$$

Hence $z = w$ and for all $u \in X$. Thus z is a unique common fixed point of S and T.

Corollary 3.2 *Let X be q- multiplicative metric space and T be a self-mapping of X satisfying the con- dition*

$$d_q(Tx, Ty) \leq \theta(d_q(x, y), \sqrt{dq(x,Tx).dq(y,Ty)}, d_q(x, Ty), d_q(x, Ty).d_q(y, Tx)) \quad (27)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof: The proof of corollary has immediately follows from above Theorem 3.2 by taking $f = T$. This completes the proof.

Theorem 3.3 *Let (X, d_q) be a q-multiplicative metric space and $T, f : X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that, for all $x, y \in X$, the following condition holds:*

$$d(Tx,Ty) \leq [d(fx,fy)]a[d(fx,Tx)]b[d(fy,Ty)]^c, \quad (28)$$

where, 0 $a + b + c < 1$. If $TX \subseteq f X$ and fX is a complete subspace of X, then T and f have a coincidence point. Moreover, if T and f are coincidentally commuting, then T and f have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Hypothesis, $TX \subseteq fX$. There is $x_1 \in X$ such that $fx_1 = Tx_0$, in general $fx_{n+1} = Tx_n$.

If we set $x = x_n$, $y = x_n$ in (28), we get

$$d(Tx_{n-1}, Tx_n) = d(fx_n, fx_{n+1})$$

$$\leq [d(fx_{n-1}, fx_n)] \ . [d(fx_{n-1}, Tx_{n-1})] \ .[d(fx_n, fx_{n+1})]$$
$$= [d(fx_{n-1}, fx_n)]^a.[d(fx_{n-1}, fx_n)]^b.[d(fx_n, fx_{n+1})]^c$$

$$d(fx_n, fx_{n+1}) \leq [d(fx_{n-1}, fx_n)]^{\frac{a+b}{1-c}} \quad (29)$$

then we have $0 \leq a + b + c < 1$. This implies $h = \frac{a+b}{1-c} < 1$. Hence, for all $n \geq 1$, we have

$$dq(fx_n, fx_{n+1}) \leq [dq(fx_{n-1}, fx_n)]^h$$
$$\leq [dq(fx_{n-2}, fx_{n-1})]^{h^2}$$
$$\leq [dq(fx_{n-3}, fx_{n-2})]^{h^3}$$
$$\leq [dq(fx_{n-4}, fx_{n-3})]^{h^4}$$
$$\leq [dq(fx_0, fx_1)]^{h^n}$$
$$\Rightarrow dq(fx_n, fx_{n+1}) \leq [dq(fx_0, fx_1)]^{h^n} \quad (30)$$

Consider $n, m \in$ N with $n > m$. From (*ii*) of Definition 2.1, we conclude inductively that

$$\Rightarrow dq(fx_m, fx_n) \leq \prod_{i=m}^{n-1} d_q(fx_i, fx_{i+1})$$
$$dq(fx_m, fx_n) \leq \prod_{i=m}^{n-1} [d_q(fx_0, fx_1)]^{h^i}$$
$$dq(fx_m, fx_n) \leq \prod_{i=m}^{\infty} [d_q(fx_0, fx_1)]^{h^i}$$

We utilize the fact that the logarithmic is a continuous and increasing function. This leads to

$$\log dq(fx_m, fx_n) \leq \log [\prod_{i=m}^{\infty} [d_q(fx_0, fx_1)]^{h^i}]$$
$$= \sum_{i=m}^{\infty} h^i \ \log \ [d_q(fx_0, fx_1)]$$

$$= \log \ [d_q(fx_0, fx_1)] \sum_{i=m}^{\infty} h^i$$

$$= \log \ [d_q(fx_0, fx_1)] \ \frac{h^m}{1-h}, \quad h \in [0,1)$$

sum of a Geometric progression. As $m, n \to \infty$, we get

$\log [d_q(fx_m, fx_n)] \ \to_q \ 0$ (31)

We use the fact that the exponential function is continuous. We apply the exponential function on both sides of (31) and get

$$d_q(fx_m, fx_n) \to_q 1 \ as \ m, n \to \infty$$

This makes $\{fx_n\}$ a multiplicative Cauchy sequence by Lemma 2.1.

From assumption, we know that X is complete. This means that there is $z \in X$ such that $\lim_{n\to\infty} x_n = z$. From the construction of the sequence $\{fx_n\}$, we have $fx_n = Tx_{n+1}$. Taking $n \to \infty$, we have

$$\lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} Tx_n.$$

As $z \in X$, there is $w \in X$ such that $fw = z$, we claim that w is a fixed point of f and T.

Let $x = x_n$ and $y = w$ in (9). Then we have

$$d(Tx_n, Tw) \leq [d(fx_n, fw)]^a[d(fx_n, Tx_n)]^b[d(fw, Tw)]^c$$
$$d(z, Tw) \leq [d(z, z)]^a[d(z, z)]^b[d(z, Tw)]^c$$

Then we have

$$\Rightarrow d(z, Tw) \leq 1$$
$$\Rightarrow d(z, Tw) = 1 \ ((i) \ of \ Definition 2.1)$$
$$\Rightarrow z = Tw.$$

Thus w is a coincidence point for f and T because $Tw = z = fw$.

If f and T are coincidentally commuting at point w, then we have

$$f \ Tw = Tfw \Rightarrow fz = Tz. \quad (32)$$

We claim that z is a common fixed point of f and T. Put $x = w, y = z$ in (28).

$d(Tw, Tz) \leq [d(fw, fz)]^a[d(fw, Tw)]^b[d(fz, Tz)]^c$,

$d(z, fz) \leq [d(z, fz)]^a[d(z, z)]^b[d(fz, fz)]^c$, $d(z, fz) \leq [d(z, fz)]^a$,

which is a contraction. Hence

$\Rightarrow d(z, fz) \leq 1$

$\Rightarrow d(z, fz) = 1 \ ((i) \ of \ Definition 2.1)$

$z = Tw$. by (i) Definition 2.1 and 32.

$\Rightarrow$ Therefore, z is a common fixed point of f and T.

Prove that, z is an unique common fixed point of f and T. Put $x = w, y = z$ in (28).

$d(Tz, Tz^*) \leq [d(fz, fz^*)]^a[d(fz, Tz)]^b[d(fz^*, Tz^*)]^c$,

$d(z, z^*) \leq [d(z, z^*)]^a[d(z, z)]^b[d(z^*, z^*)]^c$, $d(z, z^*) \leq [d(z, z^*)]^a$,

which is a contraction. Hence

$\Rightarrow d(z, z \ ) \leq 1$

$\Rightarrow d(z, z \ ) = 1 \ ((i) of \ Definition 2.1)^{*}_{*}$

$\Rightarrow z = z * Tw$. by (i) Definition 2.1 and (32).

Therefore, z is an unique common fixed point of f and T.

Corollary 3.3 *Let (X, d_q) be a q-multiplicative metric space and $T, f : X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that, for all $x, y \in X$, the following condition holds:*

$$d(Tx, Ty) \leq [d(fx, fy)]^a, \quad (33)$$

where, $0 \leq a < 1$. *If* $TX \subseteq fX$ *and* $f\ X$ *is a complete subspace of* X, *then* T *and* f *have a coincidence point. Moreover, if* T *and* f *are coincidentally commuting, then* T *and* f *have a unique common fixed point.*

Corollary 3.4 *Let* (X, d_q) *be a q-multiplicative metric space and* $T, f : X \to X$ *be self-mappings such that* $TX \subseteq fX$. *Assume that, for all* $x, y \in X$, *the following condition holds:*

$$d(Tx,Ty) \leq [d(fx,Tx)]^b [d(fy,Ty)]^c, \quad (34)$$

where, $0 \leq b + c < 1$. *If* $TX \subseteq fX$ *and* $f X$ *is a complete subspace of* X, *then* T *and* f *have a coincidence point. Moreover, if* T *and* f *are coincidentally commuting, then* T *and* f *have a unique common fixed point.*

4. AN APPLICATION TO THE FIXED-CIRCLE PROBLEM

In this section, we focus on the geometric properties of common fixed points when they are not unique. We prove a new common fixed-circle result on a q- multiplicative metric space. For more details, (see, e.g., [19-22]). To do this, we define the following notions: Now, we state the following definitions.

Definition 4.1 *(Multiplicative ball) Let* (X, d_q) *be a q multiplicative metric space,* $x \in X$ *and* $\epsilon > 1$. *Then*

$$B(x_0, \epsilon) = \{x : dq(x_0, x) < \epsilon\},$$

is called q-open ball of radius ϵ *with center x.*

Definition 4.2 (1) *Let* $d_q : X \to X$ *and* $\epsilon \in R$ *with* $\epsilon > 0$ *or* $\epsilon = \theta$. *Then the set*

$$C^2_{x_0,\epsilon} = C(x_0, r) = \{x : dq(x, x_0)\}$$

(2) *Let* $dq : X \to X$ *and* $\epsilon \in \mathrm{R}$ *with* $\epsilon > 0$ *or* $\epsilon = \theta$. *Then the set*

$$B[x_0, \epsilon] = B(x_0, \epsilon) \cup C^2_{x_0,\epsilon}$$

is called a closed ball centered at x_0 *with radius* ϵ.

(3) *The circle* $\cup\ C^2_{x_0,\epsilon x_0,\epsilon}$ *(or the closed ball* $B[x_0, \epsilon]$ *is called as the fixed circle (or fixed disc) of a self-mapping* T *if* $Tx = x$ *for all* $x \in C^2_{x_0,\epsilon}$ *(or* $x \in B[x_0, \epsilon]$, *respectively.*

We give the following fixed-circle (or fixed-disc) results:

Theorem 4.1 *Let X be a q-multiplicative metric space,* $T : X \to X$ *be a self-mapping,* $x_0 \in X$ *and*

$\epsilon = inf_{x\in X} \{ dq(Tx, x) : Tx \neq x\}$. . (35)

If T *satisfies the following conditions, then* $C^2_{x_0,\epsilon}$ *is a fixed circle of* T :

(1) *If* $Tx \neq x$, *then*

$$dq(Tx, x) \leq \theta\left(dq(x, x_0), dq(Tx, x), dq(x, Tx_0), \sqrt{dq(x, Tx_0).dq(Tx, x)}\right),$$

where $\theta \in \Phi$.

(2) $Tx_0 = x_0$.

Proof: Case 1: Let $\epsilon = \theta$. Then we have

$$dq(x, x_0) = 1 \quad \Rightarrow max\{dq(x, x_0)\} = 1$$

$$\Rightarrow dq(x, x_0) = 1 \quad for\ all \quad i = 1, 2, \ldots d$$

$$\Rightarrow x = x_0$$

$$\Rightarrow C^2_{x_0\epsilon} = \{x_0\}$$

Using the condition (2), we know $Tx_0 = x_0$ and so $C^2_{x_0\epsilon}$ is a fixed circle of T.

Case 2: Let $\epsilon > 0$ and $x \in C^2_{x_0\epsilon}\ C^2_{x_0\epsilon}$ with $Tx \neq x$. By the definition of ϵ, we have $\epsilon \leq dq(Tx, x)$. Using the conditions (1), (2) and the property of θ, we obtain

$$dq(Tx, x) \leq \theta\left(dq(x, x_0), dq(Tx, x), dq(x, Tx_0), \sqrt{dq(x, Tx_0).dq(Tx, x)}\right),$$

$$\leq \theta\left(\epsilon, dq(Tx, x), d\epsilon, \sqrt{\epsilon.dq(Tx, x)}\right)$$

From Definition 3.1 (a_1), we have

$$dq(Tx, x) \leq \epsilon^h, \quad h \in (0, 1),$$

which is a contradiction with the definition of ϵ. Therefore, it should be $Tx = x$. Consequently, $C^2_{x_0\epsilon}$ is a fixed circle of T.

Corollary 4.1 *Let X be a q-multiplicative metric space,* $T : X \to X$ *be a self-mapping,* $x_0 \in X$ *and r be defined as in (35). If T satisfies the following conditions, then T fixes the closed ball* $B[x_0, \rho]$ *with* $\rho \leq \epsilon$ *(or* $B[x_0, \epsilon]$ *is the fixed disc of T) :*

(1) *If* $T \neq x$ *then*

$$dq(Tx, x) \leq \theta\left(dq(x, x_0), dq(Tx, x), dq(x, Tx_0), \sqrt{dq(x, Tx_0).dq(Tx, x)}\right)$$ *where* $\theta \in \Theta$.

(2) $Tx_0 = x_0$.

Proof: The proof can be easily seen by the similar arguments used in the proof of Theorem 4.1.

Theorem 4.2 *Let X be a q-multiplicative metric space,* $T : X \to X$ *be a self-mapping,* $x_0 \in X$ *and r be defined as in (35). If T satisfies the following conditions, then* $C^2_{x_0\epsilon}$ *is a fixed circle of T :*

(1) *If* $T \neq x$ *then*

$$dq(Tx, x) \leq \theta\left(dq(x, x_0)\ , \sqrt{dq(x, Tx_0).dq(Tx, x)}\ , 1,\ dq(Tx, x).dq(x, Tx_0)\right)$$

where $\theta \in \Theta$.

(2) $Tx_0 = x_0$.

Proof: Case 1: Let $\epsilon = 0$. Then we have $C^2_{x_0\epsilon}$

$= \{x_0\}$. Using the condition (2), we know $Tx_0 = x_0$ and so $C^2_{x_0\epsilon}$ is a fixed circle of T.

Case 2: Let $\epsilon > 0$ and $x \in C^2_{x_0\epsilon}$ with $T \neq x$. By the definition of ϵ, we have $\epsilon \leq dq(Tx, x)$. Using the conditions (1), (2) and the property of θ, we obtain

$$dq(Tx, x) \leq \theta\left(dq(x, x_0)\ , \sqrt{dq(x, Tx_0).dq(Tx, x)}\ , 1,\ dq(Tx, x) + dq(x, Tx_0)\right)$$

$$\leq \theta(\epsilon\ , \sqrt{(dq(x, Tx\_0\ ).dq(Tx, x)\ )}\ , 1,\ dq(Tx, x) + \epsilon\ )$$

From Definition 3.1 (b_1), we have

$$dq(Tx, x) \leq \epsilon^{h}\ ,\ h \in (0, 1),$$

which is a contradiction with the definition of r. Therefore, it should be $Tx = x$. Consequently, $C^2_{x_0\epsilon}$ is a fixed circle of T.

Corollary 4.2 *Let X be a q-multiplicative metric space,* $T : X \to X$ *be a self-mapping,* $x_0 \in X$ *and* ϵ *be defined as in (35). If T satisfies the following conditions, then T fixes the closed ball* $B[x_0, \rho]$ *with* $\rho \leq \epsilon$ *(or* $B[x_0, \epsilon]$ *is the fixed disc of T*

(1) *If* $T \neq x$ *then*

$$dq(Tx, x) \leq \theta\left(dq(x, x_0)\ , \sqrt{dq(x, Tx_0).dq(Tx, x)}\ , 1,\ dq(Tx, x) + dq(x, Tx_0)\right)$$

where $\theta \in \theta$.

(2) $T\, x_0 = x_0$.

Proof: The proof can be easily seen by the similar arguments used in the proof of Theorem 4.2.

5. CONCLUSION

In this paper, we introduced the concept of q-multiplicative metric spaces. We studied some of the fixed point theorems in these spaces.

6. ACKNOWLEDGEMENTS

The author is grateful to the referees for their valuable proposals to improve this paper.

REFERENCES

1. Bashirov AE, Kurpınar EM, Özyapıcı A. Multiplicative calculus and its applications. *J Math Anal Appl*. 2008;337:36–48.
2. Ali MU, Kamran T, Kurdi A. Fixed point theorems in b-multiplicative metric spaces. *UPB Sci Bull Ser A*. 2017;79(3):107–116.
3. Ait Mansour M, Bahraoui MA, El Bekkali A. A global approximate contraction mapping principle in non-complete metric spaces. *J Nonlinear Var Anal*. 2020;4:153–157.
4. Czerwik S. Contraction mappings in b-metric spaces. *Acta Math Inform Univ Ostrav*. 1993;1:5–11.
5. Došenović T, Postolache M, Radenović S. On multiplicative metric spaces. *Fixed Point Theory Appl*. 2016;2016:92.
6. Heilpern S. Fuzzy mappings and fixed point theorems. *J Math Anal Appl*. 1981;83:566–569.
7. Gu F, Shatanawi W. Some new results on common coupled fixed points of two hybrid pairs of mappings in partial metric spaces. *J Nonlinear Funct Anal*. 2019;2019:Article ID 13.
8. Simkhah M, Türkoğlu D, Sedghi S, Shobe N. Suzuki type fixed point results in p-metric spaces. *Commun Optim Theory*. 2019;2019:Article ID 13.
9. Subrahmanyam PV. A common fixed point theorem in fuzzy metric spaces. *Inf Sci*. 1995;83:109–112.
10. Gregori V. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst*. 1988;27:385–389.
11. Kamal A, Abd-Elal AM. On fixed point findings for diverse contractions in b-dislocated multiplicative metric spaces. *J Math*. 2021;2021:4762281.
12. Kang SM, Kumar P, Kumar S, Nagpal P, Garg SK. Common fixed points for compatible mappings and its variants in multiplicative metric spaces. *Int J Pure Appl Math*. 2015;102(2):383–406.
13. Vasuki R. A common fixed point theorem in a fuzzy metric space. *Fuzzy Sets Syst*. 1998;97:395–397.
14. Weiss MD. Fixed points and induced fuzzy topologies for fuzzy sets. *J Math Anal Appl*. 1975;50:142–150.
15. Butnariu D. Fixed point for fuzzy mapping. *Fuzzy Sets Syst*. 1982;7:191–207.
16. Özavsar M, Çevikel AC. Fixed point of multiplicative contractive metric space. *arXiv* preprint arXiv:1205.5131; 2012.
17. Popa V. Fixed point theorems for implicit contractive mappings. *Stud Cerc St Ser Mat Univ Bacău*. 1997;7:127–133.
18. Özgür NY, Taş N. Some fixed-circle theorems on metric spaces. *Bull Malays Math Sci Soc*. 2017. doi:10.1007/s40840-017-0555-z.
19. Özgür NY, Taş N. Fixed-circle problem on S-metric spaces with a geometric viewpoint. *arXiv* preprint arXiv:1704.08838.
20. Özgür NY, Taş N. Some fixed-circle theorems and discontinuity at fixed circle. In: *AIP Conf Proc*. 2018;1926:020048.
21. Mlaiki N, Çelik U, Taş N, Özgür NY, Mukheimer A. Wardowski type contractions and the fixed-circle problem on S-metric spaces. *J Math*. 2018;2018:9127486.
22. Mlaiki N, Taş N, Özgür NY. On the fixed-circle problem and Khan type contractions. *Axioms*. 2018;7:80.