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A study on strong arcs in fuzzy graphs and union of fuzzy graphs *

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Abstract In the present work, we discuss the union operation and strong arc domination on fuzzy graphs. We determine the bounds for the strong arc domination number of fuzzy graphs. Further, this new domination parameter is discussed in the union of fuzzy graphs. Some basic theorems and results are obtained in the union of fuzzy graphs.

Key words Fuzzy graph, Strong arc, Domination number, Effective edge domination, Strong arc domination number.

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1 Introduction

Zadeh introduced the concept of fuzzy relations [1]. Rosenfeld [5] introduced the notion of a fuzzy graph and several fuzzy analogs of graph-theoretic concepts such as path, cycles, connectedness and etc. The work on fuzzy graphs was also done by Monderson and Nair [3], Debnath [10], and Yeh and Bang [11]. The concept of domination in graphs was introduced by Ore in 1962 [2] and further studied by Cockayne and Hedetniemi [12]. Bhutani and Rosenfeld [8] and Bhutani et al. [9] introduced the concept of strong arcs in the fuzzy graph.
Nagoorgani and Chandrasekaran [7] discussed domination in fuzzy graphs utilizing strong arcs. In this paper, we determine the upper bound of strong arc domination number of fuzzy graphs and we discuss the strong arcs in the union of fuzzy graphs. We recall some basic definitions in fuzzy graphs and introduce some new definition and notation. For graph-theoretic terminology, we refer to Harary [6].

2 Preliminaries

Definition 2.1. Fuzzy graph $G(\sigma, \mu)$ is a pair of function $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Definition 2.2. The fuzzy graph $H(\tau, \rho)$ is called a fuzzy subgraph of $G(\sigma, \mu)$ if $\tau(u) \le \sigma(u)$ for all u in V and $\rho(u, v) \le \mu(u, v)$ for all u, v in V.

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Definition 2.3. A fuzzy subgraph $H(\tau,\rho)$ is said to be a spanning subgraph of $G(\sigma,\mu)$ if $\tau(u)=\sigma(u)$ for all u in V. In this case, the two graphs have the same fuzzy node set, but they differ only in the arc weights.

Definition 2.4. Let $G(\sigma,\mu)$ be a fuzzy graph and τ be fuzzy subset of σ, that is, $\tau(u)\le\sigma(u)$ for all u in V. Then the fuzzy subgraph of $G(\sigma,\mu)$ induced by τ is the maximal fuzzy subgraph of $G(\sigma,\mu)$ that has fuzzy node set τ. Evidently, this is just the fuzzy graph $H(\tau,\rho)$ where $\rho(u,v)=\tau(u)\wedge\tau(v)\wedge\mu(u,v)$ for all u,v in V.

Definition 2.5. The underlying crisp graph of a fuzzy graph $G(\sigma,\mu)$ is denoted by $G^*=(\sigma^*,\mu^*)$, where $\sigma^*=\{u\in V\mid\sigma(u)>0\}$ and $\mu^*=\{(u,v)\in V\times V\mid\mu(u,v)>0\}$.

Definition 2.6. A fuzzy graph $G(\sigma,\mu)$ is a strong effective fuzzy graph if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all $(u,v)\in\mu^*$ and is a complete fuzzy graph if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all u,v in σ^*. Two nodes u and v are said to be neighbors if $\mu(u,v)>0$.

Definition 2.7. A fuzzy graph $G(\sigma,\mu)$ is said to be bipartite if the node set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1,v_2)=0$ if $v_1,v_2\in V_1$ or $v_1,v_2\in V_2$. Further if $\mu(v_1,v_2)>0$ for all $v_1\in V_1$ and $v_2\in V_2$ then G is called a complete bipartite graph and it is denoted by K_{σ_1,σ_2} where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

Definition 2.8. The complement of a fuzzy graph $G(\sigma,\mu)$ is a subgraph $\overline{G}(\overline{\sigma},\overline{\mu})$ where $\overline{\sigma}=\sigma$ and $\overline{\mu}(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v)$ for all u,v in V. A fuzzy graph is self-complementary if $G=\overline{G}$.

Definition 2.9. The order p and size q of a fuzzy graph $G(\sigma,\mu)$ is defined as $p=\sum_{u\in V}\sigma(u)$ and $q=\sum_{(u,v)\in E}\mu(u,v)$.

Definition 2.10. The degree of a vertex u is defined as the number of arcs incident at u and is denoted by $d(u)$.

Definition 2.11. An arc (u,v) of the fuzzy graph $G(\sigma,\mu)$ is called an effective edge if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ and effective edge neighborhood of $u\in V$ is $N_e(u)=\{v\in V:\text{edge}\,(u,v)\text{ is effective}\}$. $N_e[u]=N_e(u)\cup\{u\}$ is the closed neighborhood of u. The minimum cardinality of effective neighborhood $\delta_e(G)=\min\{|N_e(u)|:\ u\in V(G)\}$. The maximum cardinality of effective neighborhood is

$$\triangle_e(G)=\max\{|N_e(u)|:\ u\in V(G)\}.$$

Definition 2.12. A path ρ in a fuzzy graph $G(\sigma,\mu)$ is a sequence of distinct nodes $v_0,v_1,\ldots,v_n$ such that $\mu(v_{i-1},v_i)>0$ where $1\le i\le n$ and n is called the length of ρ. The path ρ is called v_0-v_n path. A path is called a cycle if $v_0=v_n$ and $n\ge 3$. Two vertices x,y in a fuzzy graph $G(\sigma,\mu)$ are said to be connected if there exists an $x-y$ path in $G(\sigma,\mu)$. The strength of the path ρ is defined to be $\bigwedge_{i=i}^{n}\mu(v_{i-1},\ v_i)$. A single vertex is considered as a path of length zero.

Definition 2.13. A fuzzy graph $G(\sigma,\mu)$ is said to be a fuzzy cycle if $G(\sigma,\mu)$ is itself a cycle. A fuzzy cycle $G(\sigma,\mu)$ with n vertices is denoted by $C_n(\sigma,\mu)$. A fuzzy graph $G(\sigma,\mu)$ is said to be a fuzzy path if $G(\sigma,\mu)$ is itself a path. A fuzzy path $G(\sigma,\mu)$ with n vertices is denoted by $P_n(\sigma,\mu)$.

Definition 2.14. Two nodes that are joined by a path are said to be connected. The relation of connectedness is reflexive, symmetric and transitive. If u and v are connected by means of length k, then $\mu^k(u,v)=\sup\{\mu(u,v_1)\wedge\mu(v_1,v_2)\wedge\ldots\wedge\mu(v_{k-1},v)\mid u,v_1,v_2,\ldots,v\text{ in such path }\rho\}$

Definition 2.15. The strongest path joining any two nodes u,v is a path corresponding to the maximum strength between u and v. The strength of the strongest path is denoted by $\mu^\infty(u,v)$.

$$\mu^\infty(u,v)=\sup\left\{\mu^k(u,v)\,|k=1,2,\ldots,\infty\right\}.$$

Example 2.16. In the fuzzy graph of Fig. 1, w,v,x is a $w-x$ path of length 2 and its strength is 0.3. Another $w-x$ path is w,u,v,x of length 3 and strength 0.4. The strength of the strongest path joining w and x is $\mu^\infty(w,x)=\sup\{0.3,0.4\}=0.4$.

Definition 2.17. A node is a fuzzy cut node of $G(\sigma,\mu)$ if removal of it reduces the strength of the connectedness between some other pair of nodes. That is, w is a fuzzy cut node of $G(\sigma,\mu)$ iff there exist u,v such that w is on every strongest path from u to v.

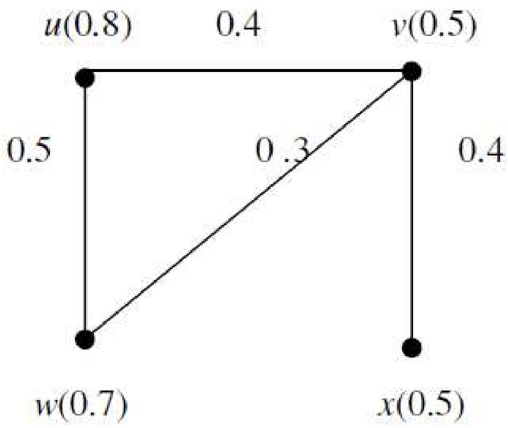

Fig. 1: Graph of Example 2.16.

3 Domination in fuzzy graph using strong arcs

Definition 3.1. Let $G(\sigma, \mu)$ be a fuzzy graph. Let u, v be two nodes of $G(\sigma, \mu)$. We say that u dominates v if the edge (u, v) is an effective edge. A subset D of V is called a dominating set of $G(\sigma, \mu)$ if for every $v \in V - D$, there exists $u \in D$ such that u dominates v . A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The minimum fuzzy cardinality taken over all minimal dominating sets of a graph G is called the effective edge domination number and is denoted by $\gamma_E(G)$ and the corresponding dominating set is called minimum effective edge dominating set. The minimum number of elements in the minimal effective edge dominating set is denoted by $N[\gamma_E(G)]$.

Definition 3.2. An arc (u, v) of a fuzzy graph $G(\sigma, \mu)$ is called a strong arc if $\mu^\infty(u, v) = \mu(u, v)$, otherwise it is called a non-strong arc. A strong neighborhood of $u \in V$ is $N(u) = \{v \in V : arc(u, v) \text{ is strong}\}$. $N[u] = N(u) \cup \{u\}$ is the closed neighborhood of u.

Definition 3.3. Let $G(\sigma, \mu)$ be a fuzzy graph. Let u, v be two nodes of $G(\sigma, \mu)$. We say that u dominates v if the arc (u, v) is strong. A subset D of V is called a strong arc dominating set of $G(\sigma, \mu)$ if for every $v \in V - D$, there exists $u \in D$ such that u dominates v. A strong arc dominating set D is called a minimal strong arc dominating if no proper subset of D is a strong arc dominating set. The minimum cardinality taken over all minimal dominating sets is called the strong arc domination number, denoted by $\gamma_s(G)$ and the corresponding dominating set is called the minimum strong arc dominating set.

Definition 3.4. A fuzzy graph $G(\sigma, \mu)$ is said to be a strong fuzzy graph if all of its edges are strong arcs.

Definition 3.5. Let v be a vertex in a fuzzy graph $G(\sigma, \mu)$. We define $\sigma^s(v) = \max\{\mu(v, x)/x \in V\}$ that is $\sigma^s(v)$ is the maximum of the weights of the edges incident at v.

Theorem 3.6. *Let $G(\sigma, \mu)$ be a fuzzy graph and if arc (x, y) in Gis such that $\sigma^s(x) = \mu(x, y)$ then (x, y) is a strong arc.*

Proof. Suppose (x, y) is not a strong arc then there exist a path $\rho : x = x_0, x_1, \ldots, x_m = y$ such that $s_\rho > \mu(x, y)$, which implies that $\mu(x_i, x_{i+1}) > \mu(x, y)$ for all $i = 0, 1, 2, \ldots, m - 1$. In particular $\mu(x_0, x_1) = \mu(x, x_1) > \mu(x, y)$.This implies that $\sigma^s(x) > \mu(x, y)$. This is a contradiction, therefore, the proof.

□

Theorem 3.7. *If (x, y) be an arc in the fuzzy graph $G(\sigma, \mu)$ such that x is an end vertex, then it is a strong arc.*

Proof . Let (x, y) be an arc and x is an end vertex. Therefore $\sigma^s(x) = \mu[(x,\ y)]$. By Theorem 3.6, (x, y) is a strong arc.

□

Theorem 3.8. *If an edge is a strong arc in a fuzzy graph $G(\sigma, \mu)$ then it is not the weakest arc of any cycle.*

Proof. Let C_n be any cycle of $G(\sigma, \mu)$ and let $x_1, x_2, \ldots, x_n$ be the edges of C_n. Let $x_i = (u, v)$ be an edge of C_n such that $\mu(x_i) < \mu(x_j)$, $j = 1, 2, \ \ldots, n$. Now $C_n - \{x_i\}$ is a $u - v$ path in $G(\sigma, \mu)$ and it has a strength $> \mu(x_i)$. This implies that x_i cannot be a strong arc.

□

Remark 3.9.

1. If a fuzzy graph $G(\sigma, \mu)$ has n end vertices then it has at least n strong arcs.
2. Let $G(\sigma, \mu)$ be a fuzzy graph with n vertices namely $u_1, u_2, \ldots, u_n$. Then every dominating set of $G(\sigma, \mu)$ contains u_i or $N_s(u_i)$, $i = 1, 2, \ldots, n$.

Theorem 3.10. *Let $G(\sigma, \mu)$ be a fuzzy graph with $|V(G(\sigma, \mu))| = 2n + 1, n \in \mathbb{Z}$. If $G(\sigma, \mu)$ has no isolated vertices then there exist a vertex u in $G(\sigma, \mu)$ such that $|N_s(u)| \geq 2$.*

Proof. Suppose each vertex u of $G(\sigma, \mu)$ has $|N_s(u)| = 1$, then $\sum_{u \in V} N_s[u] = 2n$. therefore $|(G(\sigma, \mu))| = \sum_{u \in V} N_s[u] = 2n$. This is a contradiction.

□

Theorem 3.11. *Let $G(\sigma, \mu)$ be a fuzzy graph. If $N_s(u) = \emptyset$ for some $u \in V[\mathrm{G}(\sigma, \mu)]$, then $d(u) = 0$.*

Proof. Suppose $d(u) > 0$ then we can find an edge in $G(\sigma, \mu)$ such that $\sigma^s(u) = \mu(ux)$. This implies that (u, x) is a strong arc and hence $N_s(u) \neq \emptyset$. □

Remark 3.12. By the above Theorems 3.10 and 3.6, we conclude that each non-isolated vertex u of a fuzzy graph $G(\sigma, \mu)$ strongly dominates at least one vertex of $G(\sigma, \mu)$.

Theorem 3.13. *Let $G(\sigma, \mu)$ be a fuzzy graph with p vertices. If $G(\sigma, \mu)$ has no isolated vertices then $\gamma_s(G) \leq \frac{p}{2}$.*

Proof. By the Theorem 3.10 and Remark 3.9 the result follows. □

Definition 3.14. The crown graph $C_n.K_1$ is the graph obtained from a cycle C_n by attaching the pendant edges at each vertex of the cycle.

Theorem 3.15. *Let G be a crown graph $C_n.K_1$. Then for the fuzzy graph $G(\sigma, \mu)$, $\gamma_s(G) = n$.*

Proof. By the definition of $G(\sigma, \mu)$, it has exactly n end vertices namely $u_1, u_2, \ldots, u_n$. By the Theorem 3.7 each end vertex u_i strongly dominates a unique vertex of C_n. Therefore, $D = \{u_1, u_2, \ldots, u_n\}$ is a minimal strong arc dominating set. Therefore $\gamma_s[G(\sigma, \mu)] = n$.

□

Definition 3.16. The star graph S_n of order n is a tree on n nodes with one node having a vertex of degree $n - 1$ and the other $n - 1$ nodes having vertex degree 1. That is the star graph S_n is the complete bipartite graph $K_{1,n}$.

Theorem 3.17. *Let G be a star graph S_n. Then for the fuzzy graph $G(\sigma, \mu)$, $\gamma_s[G] = 1$.*

Proof. By the definition of $G(\sigma, \mu)$, there exist a vertex u in G such that $d(u) = n - 1$ and also G has $n - 1$ end vertices. By the Theorem 3.7, $G(\sigma, \mu)$ has exactly $n - 1$ strong arcs. But $G(\sigma, \mu)$ gas exactly $n - 1$edges. Therefore $D = \{u\}$ is a strong arc dominating set.

□

4 Strong arcs in union of fuzzy graphs

Definition 4.1. Consider the union $G = G_1 \cup G_2$, of two graphs $G_1 = (V_1,\ X_1)$ and $G_2 = (V_2,\ X_2)$. Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of $X_i,\ i = 1, 2$. Define the fuzzy subsets $\sigma_1 \cup \sigma_2$ of $X_1 \cup X_2$ as follows
$(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u)$ if $u \in V_1 - V_2$,
$(\sigma_1 \cup \sigma_2)(u) = \sigma_2(u)$ if $u \in V_2 - V_1$,
$(\sigma_1 \cup \sigma_2)(u) = \max\{\sigma_1(u), \sigma_2(u)\}$ if $u \in V_1 \cap V_2$,
$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v)$ if $(u, v) \in X_1 - X_2$,
$(\mu_1 \cup \mu_2)(u, v) = \mu_2(u, v)$ if $(u, v) \in X_2 - X_1$
$(\mu_1 \cup \mu_2)(u, v) = \max\ \{\mu_1(u, v),\ \mu_2(u, v)\}$ if $(u, v) \in X_1 \cap X_2$.
The fuzzy graph $G = (\sigma_1 \cup \sigma_2,\ \mu_1 \cup \mu_2)$ is the union of the fuzzy graphs $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$.

Remark 4.2.
1. We denote the union of the fuzzy graphs $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ simply by $G_1 \cup G_2$.
2. If $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are two fuzzy graphs then $G = G_1 \cup G_2$ is also a fuzzy graph.

Theorem 4.3. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with the strong arc dominating sets D_1 and D_2 respectively. If $V_1 \cap V_2 = \emptyset$ then $D_1 \cup D_2$ is a strong arc dominating set of $G_1 \cup G_2$.*

Proof. Since $V_1 \cap V_2 = \emptyset$, then $G_1 \cup G_2$ is disconnected and so any strong arc in $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are the strong arcs of $G_1 \cup G_2$. Hence if D_1 and D_2 are the strong arc dominating sets of $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ then $D_1 \cup D_2$ is a strong arc dominating set of $G_1 \cup G_2$. □

Corollary 4.4. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. If $V_1 \cap V_2 = \emptyset$, then, $\gamma_s(G_1 \cup G_2) = \gamma_s(G_1) + \gamma_s(G_2)$.*

Remark 4.5.
1. Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. A strong arc in $G_1(\sigma_1, \mu_1)$ need not be a strong arc in $G_1 \cup G_2$. Likewise, a strong arc in $G_2(\sigma_2, \mu_2)$ need not be a strong arc in $G_1 \cup G_2$. Hence in general, $\gamma_s(G_1 \cup G_2) \neq \gamma_s(G_1) + \gamma_s(G_2)$.
2. Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs having strong arc dominating sets D_1 and D_2 respectively. $D_1 \cup D_2$ is a strong arc dominating set of $G_1 \cup G_2$ iff every strong arc of $G_1(\sigma_1, \mu_1)$ is a strong arc of $G_1 \cup G_2$ and every strong arc of $G_2(\sigma_2, \mu_2)$ is a strong arc of $G_1 \cup G_2$.

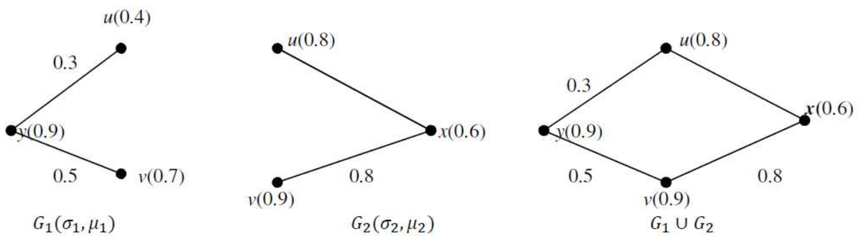

Fig. 2: Graph of Example 4.6.

Example 4.6. In the above figure (Fig. 2) the arc (x, u) is a strong arc in $G_1(\sigma_1, \mu_1)$, but it is not a strong arc in $G_1 \cup G_2$.

Theorem 4.7. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. If $x = (u, v)$ is a non-strong arc in $G_1(\sigma_1, \mu_1)$, then it is a non-strong arc in $G_1 \cup G_2$.*

Proof. Let $x = (u, v)$ be a non- strong arc in $G_1(\sigma_1, \mu_1)$. Then there exist a $u - v$ path ρ, $u = u_1, u_2, \ldots, u_n = v$ in $G_1(\sigma_1, \mu_1)$ such that whose strength $s_\rho > \mu(u, v)$. Let this path ρ be also in $G_1 \cup G_2$. Also if any edge (u_i, u_{i+1}) of ρ belongs to $X_1 \cap X_2$ then $(\mu_1 \cup \mu_2)(u_i, u_{i+1}) \geq \mu_1(u_i, u_{i+1})$. This implies that s_ρ in $G_1(\sigma_1, \mu_1) \leq s_\rho$ in $G_1 \cup G_2$. Therefore $\mu(u, v) < s_\rho \leq s_\rho$ in $G_1 \cup G_2$. Hence $x = (u, v)$ is not a strong arc in $G_1 \cup G_2$.

□

Corollary 4.8. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. If $x = (u, v)$ is a non-strong arc in $G_2(\sigma_2, \mu_2)$ then it is a non-strong arc in $G_1 \cup G_2$.*

Theorem 4.9. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with vertex sets V_1 and V_2 respectively. If $V_1 \cap V_2 = \{u\}$ then any strong arc in $G_1(\sigma_1, \mu_1)$ is a strong arc in $G_1 \cup G_2$.*

Proof. Let $x = (u, v)$ be a strong arc in $G_1(\sigma_1, \mu_1)$. Then for any $u-v$ path ρ, $u = u_1, u_2, \ldots, u_n = v$, $s_\rho \leq \mu(u, v)$. Since $V_1 \cap V_2 = \{u\}$, by definition of $G_1 \cup G_2$, there is no $u - v$ path in $G_1 \cup G_2$ other than that in $G_1(\sigma_1, \mu_1)$. Hence the strength of the $u - v$ path in $G_1(\sigma_1, \mu_1)$ is equal to the strength of the $u - v$ path in $G_1 \cup G_2$. Therefore for any $u - v$ path ρ in $G_1 \cup G_2$, $s_\rho \leq \mu(u, v)$. Hence $x = (u, v)$is a strong arc in $G_1 \cup G_2$.

□

Corollary 4.10. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with vertex sets V_1 and V_2 respectively. If $V_1 \cap V_2 = \{u\}$ then any strong arc in $G_2(\sigma_2, \mu_2)$ is a strong arc in $G_1 \cup G_2$.*

Corollary 4.11. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with the strong arc dominating sets D_1 and D_2 respectively. If $V_1 \cap V_2 = \{u\}$ then $D_1 \cup D_2$ is a strong arc dominating set of $G_1 \cup G_2$.*

Corollary 4.12. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with vertex sets V_1 and V_2 respectively. If $V_1 \cap V_2 = \{u\}$ then $\gamma_s(G_1 \cup G_2) = \gamma_s(G_1) + \gamma_s(G_2)$.*

Theorem 4.13. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs with exactly one edge (x, y) in common. Let (u, v) be an arc in $G_1(\sigma_1, \mu_1)$ such that $\mu(x, y) \leq \mu(u, v)$. If (u, v) be a strong arc in $G_1(\sigma_1, \mu_1)$ then it is a strong arc in $G_1 \cup G_2$.*

Proof. Let (x, y) be an arc common to both $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$. Let (u, v) be a strong arc in $G_1(\sigma_1, \mu_1)$. Therefore, for any path ρ: $u = u_1, u_2, \ldots, u_n = v$ in $G_1(\sigma_1, \mu_1)$, $s_\rho \leq \mu(u, v)$. Since $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ share exactly one edge (x, y), we have a $u - v$ path ρ_1 of $G_1 \cup G_2$ containing an edge (x, y), or, all the edges of ρ_1 are in $G_1(\sigma_1, \mu_1)$. Since, $\mu(x, y) \leq \mu(u, v)$, in both the cases $s_{\rho_1} \leq \mu(u, v)$.

□

Definition 4.14. Let (x, y) be an $M-$ arc in $G_2(\sigma_2, \mu_2)$. If $\mu(x, y) \leq \mu(u, v)$ for all (u, v) in $G_1(\sigma_1, \mu_1)$ then we say that the fuzzy graph $G_1(\sigma_1, \mu_1)$ strongly dominates the fuzzy graph $G_2(\sigma_2, \mu_2)$ and it is denoted by $G_2\mathcal{S}G_1$.

Theorem 4.15. *Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. If $G_2\mathcal{S}G_1$ then any strong arc in $G_1(\sigma_1, \mu_1)$ is a strong arc in $G_1 \cup G_2$.*

Proof. Let (x, y) be a strong arc in $G_1(\sigma_1, \mu_1)$. Suppose it is not a strong arc in $G_1 \cup G_2$ then there exists an $x - y$ path ρ_1: $x = x_0, x_1, , \ldots, x_n = y$ in $G_1 \cup G_2$ such that $s_{\rho_1} > \mu(x, y)$. This implies that $(\mu_1 \cup \mu_2)(x_i, x_{i+1}) > \mu_1(x, y)$ for all $i = 0, 1, \ldots, n - 1$. If all the edges of ρ_1 are in $G_1(\sigma_1, \mu_1)$ then it is a (x, y) path in $G_1(\sigma_1, \mu_1)$. Since (x, y) is a strong arc in $G_1(\sigma_1, \mu_1)$, there exists an edge (x_i, x_{i+1}) such that $\mu_1 \cup \mu_2[(x_i, x_{i+1})] = \mu_1[(x_i, x_{i+1})] \leq \mu_1(x, y)$. This is a contradiction. Suppose not all the edges of ρ_1 are in $G_1(\sigma_1, \mu_1)$, then there exists an edge (x_k, x_{k+1}) of ρ_1 which belongs to $X_2 - X_1$. Now $s_{\rho_1} > \mu(x, y)$ implies that $\mu_1 \cup \mu_2[(x_k, x_{k+1})] > \mu_1(x, y)$.That is, $\mu_1 \cup \mu_2[(x_k, x_{k+1})]$ $= \mu_2[(x_k, x_{k+1})] > \mu_1(x, y)$. This contradicts the fact that $G_2\mathcal{S}G_1$.

□

Remark 4.16. Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs. If $G_2\mathcal{S}G_1$ then any strong arc in $G_2(\sigma_2, \mu_2)$ need not be a strong arc in $G_1 \cup G_2$ and hence $\gamma_s(G_1 \cup G_2) \neq \gamma_s(G_1) + \gamma_s(G_2)$.

Example 4.17. In the figure below (Fig. 3) $G_2\mathcal{S}G_1$, the arc (d, x) is a strong arc in $G_2(\sigma_2, \mu_2)$ but it is not a strong arc in $G_1 \cup G_2$.

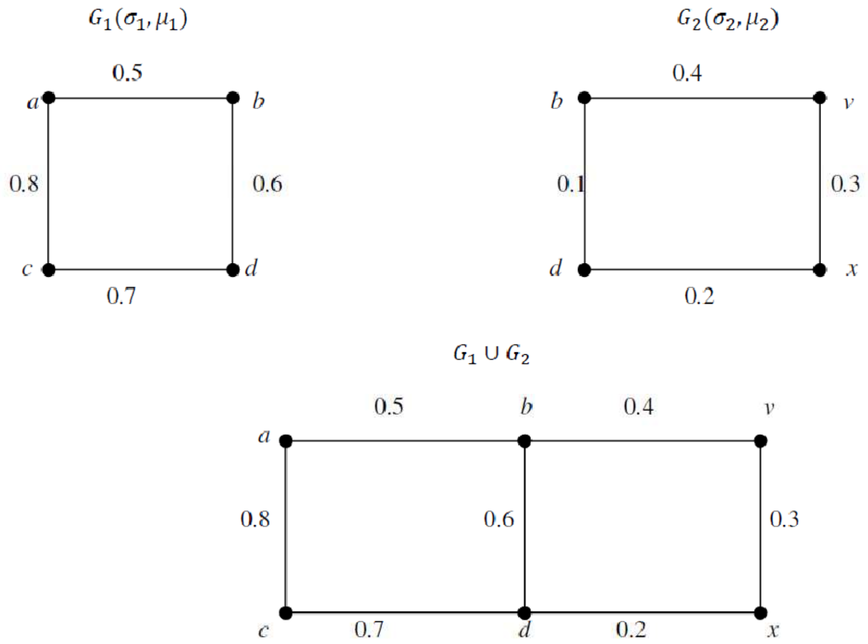

Fig. 3: Graph of Example 4.17.

5 Conclusion

In this paper, the strong arc domination number of some fuzzy graphs is obtained. Also, we have determined the upper bound for the strong arc domination number of fuzzy graphs. We have done a complete analysis of strong arcs in the union of fuzzy graphs. We shall work on this strong arc domination parameter in the Cartesian product of fuzzy graphs in our forthcoming papers.

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References

[1] Zadeh, L.A. (1965). Fuzzy sets, *Information and Control*, 8, 338 – 353.

[2] Ore, O. (1962). Theory of Graphs, Amer. Math. Soc. Colloq., Publication 38, Providence, U.S.A.

[3] Monderson, N. John and Nair, S. Premchand (2000). Fuzzy Graphs and Fuzzy Hypergraphs, Physica Verlag, New York.

[4] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. (1998). Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York.

[5] Rosenfeld, A. (1975). Fuzzy graphs, in Zadeh, L.A., Fu, K.S. and Shimura,M. (Editors), Fuzzy Sets and Their Applications, Academic Press, New York.

[6] Harary, F. (1969). Graph Theory, Addison Wesley, Reading, MA.

[7] Nagoorgani, A. and Chandrasekaran, V.T (2006). Domination in fuzzy graph, *Advances In Fuzzy Sets and Systems*, 1(1), 17–26.

[8] Bhutani, K.R. and Rosenfeld, A. (2003). Storng arcs in fuzzy graph, *Information Sciences*, 152, 319–322.

[9] Bhutani, K.R., Mordeson, J. and Rosenfeld, A. (2004). On degrees of end node and cut node in fuzzy graphs, *Iranian Journal of Fuzzy Systems*, 1(1), 57–64.

[10] Debnath, Pradip (2013). Domination in interval valued fuzzy graphs, *Ann.Fuzzy Math. Inform.*, 6(2) , 363–370.

[11] Yeh, R.T., Bang, S.Y. (1975). Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in Zadeh, L.A., Fu, K.S. and Shimura, M. (Editors), Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, New York, 125–149.

[12] Cockayne, E.J. and Hedetnieme, S.T. (1977). Towards a theory of domination in graphs, *Networks*, 7, 247–261.