

A new type of $\alpha - F -$ contraction on common fixed point theorems in metric spaces and its application *

Garima Gadkari¹, M.S. Rathore² and Naval Singh³

1. Department of Mathematics, Mahakal Institute of Technology,

Ujjain-456010, Madhya Pradesh, India.

2. Department of Mathematics, C.S.A. Govt. P.G. College, Sehore, Madhya Pradesh, India.

3. Govt. Science and Commerce P.G. College, Benazeer, Bhopal, Madhya Pradesh, India.

1. E-mail: garima27gadkari@gmail.com

Abstract In this paper we introduce the notion of $F -$ contraction via $\alpha -$ admissible pair of mappings. We also provide many common fixed point results regarding rational expressions in the setting of metric spaces. Moreover, we also present some illustrated examples as an application of this concept and we also establish an existence theorem for integral equations.

Key words Common fixed point, Metric space, $\alpha -$ admissible mappings, $F -$ contraction.

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1 Introduction and preliminaries

Banach fixed point theorem is one of the most remarkable results in the theory of metric spaces. In 1922, the Polish mathematician Banach [1] established a very important result regarding a contraction mapping, known as the Banach contraction principle. One of the biggest common applications of fixed points of contractive mappings noted for particular types of spaces is the acceptance of the existence and uniqueness of solutions of nonlinear functional equations. One of such tools was newly concluded by Wardowski [2], where the author introduced a new family of mappings, the so called F or $\mathfrak{F}$ family. He notified a new concept of contraction condition called $F -$ contraction and proved a fixed point theorem which generalizes the Banach contraction principle. Secelean [3] proved fixed point theorems by iterated function systems consisting of $F -$ contractions. In 2013, Sgroi and Vetro [4] discussed the multivalued $F -$ contractions and the solution of certain functional and integral equations. Afterwards Piri and Kumam [5] proved fixed point result for $F -$ Suzuki contractions which generalizes the result of Wardowski [2]. $F -$ contractive mappings of Hardy-Rogers type and Ćirić type were investigated by Cosentino and Vetro [6] and Minak et al. [7]. Later on, Ahmad et al. [8] recalled the concept of new fixed point theorems for generalized $F -$ contraction in complete metric spaces. Klim and Wardowski [9] investigated the fixed points dynamic processes of set valued $F -$ contractions. Further Khan et al. [10] studied two new types of fixed point theorems for $F -$ contraction. Some fixed point theorems for

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Corresponding author Garima Gadkari, E-mail: garima27gadkari@gmail.com

generalized $F -$ contractions involving new rational expressions were also established by Paesano and Vetro [11], Arshad et al. [12], Ali and Kamran [13] and Budhia et al. [14]. Very recently, Tomar and Sharma [15] established coincidence and common fixed point theorems for a discontinuous non-compatible pair of self maps in noncomplete metric space.

In 2012, Samet et al. [16] introduced the class of $\alpha - \psi$ contractive type mappings and established many fixed point theorems for such mappings in the set up of complete metric space which generalize and extend the Banach contraction principle. Many authors appreciate these conditions which can be seen in [17–33]. In recent paper, Al-Rawashdeh et al. [33] introduced the concept of modified $F -$ contractions via $\alpha -$ admissible pair of mappings. In the present work, we extend the notion $\alpha - \psi$ contractions appraised by Al-Rawashdeh et al. [33].

Wardowski [2] introduced the concept of $F -$ contraction as follows:

Definition 1.1. [2] Let $\mathfrak{F}$ be the family of all functions $F : (0, +\infty) \rightarrow \mathbb{R}$ such that

(F₁): F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F₂): For each sequence $\{\alpha_n\}$ of positive number, the following holds:

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \text{ iff } \lim_{n \rightarrow \infty} F(\alpha_n) = -\infty;$$

(F₃): There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow \infty} (\alpha^k F(\alpha)) = 0$.

Let (X, d) be a metric space. A map $T : X \rightarrow X$ is said to be an $F -$ contraction on (X, d) if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Some examples of the functions belonging to F are:

$$F(\alpha) = \ln \alpha ; \quad (1.1)$$

$$F(\alpha) = \ln \alpha + \alpha, \alpha > 0 ; \quad (1.2)$$

$$F(\alpha) = \frac{-1}{\sqrt{\alpha}}, \alpha > 0 ; \quad (1.3)$$

$$F(\alpha) = \ln(\alpha^2 + \alpha), \alpha > 0. \quad (1.4)$$

The author [2] has proved fixed point theorem which generalizes Banach contraction principle by using the notion of $F -$ contraction.

Theorem 1.2. [2] Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an $F -$ contraction. Then T has a unique fixed point $x^\bullet \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is convergent to $x^\bullet$.

Remark 1.3. [2] Let T be an $F -$ contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous map.

In 2012, Samet et al. [16] introduced the class of $\alpha -$ admissible mappings.

Definition 1.4. Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be given mappings. We say that T is an $\alpha -$ admissible mapping if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Many authors have considered this result for generalized contraction using $\alpha -$ admissible mappings, see [17], [18], [19], [22], [28] and [31]. Recently, Aydi [20] generalized the Definition 1.4 and introduced the following:

Definition 1.5. [20] Let $A, B : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be the given mappings. We say that (A, B) is a generalized $\alpha -$ admissible pair if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Ax, By) \geq 1 \text{ and } \alpha(By, Ax) \geq 1.$$

More recently, Al-Rawashdeh et al. [33] modified the notion of F -contraction via α -admissible pair of mappings.

Definition 1.6. Let $A, B : X \rightarrow X$ be self mappings and let (X, d) be a metric space. The pair (A, B) is α - F -contractive if there exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x, y) \geq 1$

$$d(Ax, By) > 0 \Rightarrow \tau + F(d(Ax, By)) \leq F(M(x, y)) ,$$

$$\text{where } F \in \mathfrak{F} \text{ and } M(x, y) = \max \left\{ d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{2} \right\} .$$

Starting from the work of Wardowski [2], the object of this paper is to develop fixed point theory in this direction: we study the notion of F -contraction via α -admissible pair of mappings and to prove some common fixed point theorems for this type of contractions, which are more general than the F -contraction introduced by Wardowski [2]. We obtain the appropriate results and verify with the help of illustrative examples associated with the Volterra type integral equation.

2 Main results

We introduced the concept of an α - F -contraction defined in the following manner:

Definition 2.1. Let $M, N : X \rightarrow X$ be self mappings and let (X, d) be a metric space. The pair (M, N) is α - F -contractive if there exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x, y) \geq 1$

$$d(Mx, Ny) > 0 \Rightarrow \tau + F(d(Mx, Ny)) \leq F(G(x, y)) , \quad (2.1)$$

where $F \in \mathfrak{F}$ and

$$G(x, y) = \max \left\{ d(x, y), d(x, Mx), d(y, Ny), \left(\frac{d(x, Ny) + d(y, Mx)}{d(x, Mx) + d(y, Ny) + 1} \right) d(x, y) \right\} \quad (2.2)$$

In the case where $F(t) = \ln(t)$ for $t > 0$, (2.1) becomes

$$d(Mx, Ny) \leq e^{-\tau} G(x, y) = kG(x, y) \quad (2.3)$$

for all $\alpha(x, y) \geq 1$, $Mx \neq Ny$ and $k = e^{-\tau} < 1$. Note that (2.3) is also satisfied for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $Mx = Ny$.

Now, let us prove the following main theorem:

Theorem 2.2. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be such that (M, N) is α - F -contractive. Suppose that

- (i) (M, N) is a generalized α -admissible pair;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Mx_0) \geq 1$ and $\alpha(Mx_0, x_0) \geq 1$;
- (iii) M and N are continuous.

Then M and N have a common fixed point.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Mx_0) \geq 1$ and $\alpha(Mx_0, x_0) \geq 1$. Take $x_1 = Mx_0$ and $x_2 = Nx_1$. By induction, we construct a sequence $\{x_n\}$ such that

$$x_{2n} = Nx_{2n-1} \text{ and } x_{2n+1} = Mx_{2n} \quad \forall n = 1, 2, \dots \quad (2.4)$$

Let $s_n = d(x_n, x_{n+1})$ for $n \geq 0$.

We split the proof of our result into several steps:

Step 1: $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \geq 0$.

We have $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_0) \geq 1$. (M, N) is a generalized α -admissible pair of mappings, so $\alpha(x_1, x_2) = \alpha(Mx_0, Nx_1) \geq 1$ and $\alpha(x_2, x_1) = \alpha(Nx_1, Mx_0) \geq 1$.

We also have $\alpha(x_3, x_2) = \alpha(Mx_2, Nx_1) \geq 1$ and $\alpha(x_2, x_3) = \alpha(Nx_1, Mx_2) \geq 1$.

Now we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{n+1}, x_n) \geq 1 \quad (2.5)$$

for all $n = 0, 1, 2, \dots$.

Step 2: We shall prove that

$$\lim_{n \rightarrow \infty} s_n = 0. \quad (2.6)$$

If $d(x_{2n}, x_{2n+1}) = 0$ for some n , then we prove that $d(x_{2n+1}, x_{2n+2}) = 0$. On the contrary suppose that, $d(x_{2n+1}, x_{2n+2}) = d(Mx_{2n}, Nx_{2n+1}) > 0$. From (2.1) and (2.5) (i.e., $\alpha(x_{2n}, x_{2n+1}) \geq 1$), by triangular inequality, we have

$$\tau + F(d(x_{2n+1}, x_{2n+2})) \leq \tau + F(d(Mx_{2n}, Nx_{2n+1})) \leq F(G(x_{2n}, x_{2n+1})),$$

$$\begin{aligned} \text{where } G(x_{2n}, x_{2n+1}) &= \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, Mx_{2n}), d(x_{2n+1}, Nx_{2n+1}), \right. \\ &\quad \left. \left(\frac{d(x_{2n}, Nx_{2n+1}) + d(x_{2n+1}, Mx_{2n})}{d(x_{2n}, Mx_{2n}) + d((x_{2n+1}, Nx_{2n+1}) + 1)} \right) d(x_{2n}, x_{2n+1}) \right\} \\ &= \max \{0, 0, d(x_{2n+1}, x_{2n+2}), 0\} = d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Then, $\tau + F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n+1}, x_{2n+2}))$. This implies that

$$F(d(x_{2n+1}, x_{2n+2})) < F(d(x_{2n+1}, x_{2n+2})).$$

From (F_1) ,

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction.

Clearly, we have $x_{2n} = x_{2n+1} = x_{2n+2}$. Then $x_{2n} = x_{2n+m}$, for all $m = 0, 1, 2, \dots$, we have $x_{2n} = x_{2n+1} = Mx_{2n}$ and $x_{2n} = x_{2n+2} = Nx_{2n+1} = Nx_{2n}$. Hence x_{2n} is a common fixed point of M and N . Similarly, if $d(x_{2n+1}, x_{2n+2}) = 0$ for some n , we find that x_{2n+1} is a common fixed point of M and N and this completes the proof.

Now suppose that $d(x_{2n}, x_{2n+1}) > 0$ for all $n \geq 0$. Since $d(x_{2n}, x_{2n+1}) = d(Mx_{2n}, Nx_{2n-1}) > 0$, by (2.1) and (2.5) (i.e., $\alpha(x_{2n}, x_{2n-1}) \geq 1$), we have

$$\tau + F(d(x_{2n+1}, x_{2n})) = \tau + F(d(Mx_{2n}, Nx_{2n-1})) \leq F(G(x_{2n}, x_{2n-1})),$$

where

$$\begin{aligned} G(x_{2n}, x_{2n-1}) &= \max \left\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \right. \\ &\quad \left. \left(\frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d((x_{2n-1}, x_{2n}) + 1)} \right) d(x_{2n}, x_{2n-1}) \right\} \\ &= \max \left\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \left(\frac{d(x_{2n-1}, x_{2n+1})}{d((x_{2n+1}, x_{2n-1}) + 1)} \right) d(x_{2n}, x_{2n-1}) \right\} \\ &= \max \{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} \end{aligned}$$

Therefore, by (F_1) ,

$$d(x_{2n+1}, x_{2n}) < \max \{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}.$$

If $\max \{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$, then $0 < d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n+1})$, which is a contradiction. Thus, for all $n \geq 0$,

$$F(d(x_{2n+1}, x_{2n})) \leq F(d(x_{2n}, x_{2n-1})) - \tau. \quad (2.7)$$

Again, we have $d(x_{2n+1}, x_{2n+2}) = d(Mx_{2n}, Nx_{2n+1}) > 0$. Then, by (2.1) and (2.5) (i.e., $\alpha(x_{2n}, x_{2n+1}) \geq 1$), we get

$$\tau + F(d(x_{2n+1}, x_{2n+2})) = \tau + F(d(Mx_{2n}, Nx_{2n+1})) \leq F(G(x_{2n}, x_{2n+1})),$$

where,

$$G(x_{2n}, x_{2n+1}) = \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \right. \\ \left. \left(\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d((x_{2n+1}, x_{2n+2}) + 1)} \right) d(x_{2n}, x_{2n+1}) \right\}$$

$$\begin{aligned}
&= \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \left(\frac{d(x_{2n}, x_{2n+2})}{d((x_{2n}, x_{2n+2}) + 1)} \right) d(x_{2n}, x_{2n+1}) \right\} \\
&= \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}.
\end{aligned}$$

Then, by $(\mathbf{F}_1)$,

$$d(x_{2n+1}, x_{2n+2}) < \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}.$$

If $\max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} = d(x_{2n+1}, x_{2n+2})$, then

$$0 < d(x_{2n+1}, x_{2n+2}) < \psi(d(x_{2n+1}, x_{2n+2})) \leq d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction. Thus

$$F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})) - \tau \quad (2.8)$$

for all $n \geq 0$. Combining (2.7) and (2.8), we get

$$F(s_n) \leq F(s_{n-1}) - \tau \quad (2.9)$$

for all $n \geq 1$. We have

$$F(s_n) \leq F(s_{n-1}) - \tau \leq F(s_{n-2}) - 2\tau \leq \dots \leq F(s_0) - n\tau \quad (2.10)$$

for all $n \geq 1$. From (2.10), we obtain $\lim_{n \rightarrow \infty} F(s_n) = -\infty$. Applying $(\mathbf{F}_2)$, we get

$$\lim_{n \rightarrow \infty} s_n = 0. \quad (2.11)$$

Step 3: We shall prove that $\{x_n\}$ is a Cauchy sequence. From (2.11) and $(\mathbf{F}_3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} s_n^k F(s_n) = 0. \quad (2.12)$$

By (2.10), we have for all $n = 1, 2, \dots$

$$s_n^k F(s_n) - s_n^k F(s_0) \leq s_n^k (F(s_0) - n\tau) - s_n^k F(s_0) = -n\tau s_n^k \leq 0. \quad (2.13)$$

Letting $n \rightarrow \infty$ in (2.13), by (2.11) and (2.12), we obtain

$$\lim_{n \rightarrow \infty} n s_n^k = 0. \quad (2.14)$$

This implies that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$s_n \leq \frac{1}{n^{\frac{1}{k}}}. \quad (2.15)$$

Then for all $n \geq n_0$ and $p \in \mathbb{N}$

$$d(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}) = \sum_{i=n}^{n+p-1} s_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} < \infty$, $\lim_{n \rightarrow \infty} d(x_n, x_{n+p}) = 0$. Thus $\{x_n\}$ is a Cauchy sequence.

As (X, d) is a complete metric space, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, u) = 0$.

Step 4: We shall prove that u is a common fixed point of M and N . Having $\lim_{n \rightarrow \infty} d(x_n, u) = 0$, then $\lim_{n \rightarrow \infty} d(x_{2n}, u) = \lim_{n \rightarrow \infty} d(x_{2n+1}, u) = 0$. By continuity of M and N , we obtain that $\lim_{n \rightarrow \infty} d(x_{2n+1}, Mu) = \lim_{n \rightarrow \infty} d(Mx_{2n}, Mu) = 0$ and $\lim_{n \rightarrow \infty} d(x_{2n+2}, Nu) = \lim_{n \rightarrow \infty} d(Nx_{2n+1}, Nu) = 0$. Hence $Mu = u = Nu$, i.e., u is a common fixed point of M and N . This completes the proof. $\square$

Now, let ψ be the family of continuous functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following condition: $\psi(t) < t$ for all $t > 0$. As in Definition 2.1, we introduce the concept of an $\alpha - \psi - F$ -contraction as follows:

Definition 2.3. Let (X, d) be a metric space and $M, N : X \rightarrow X$ be self mappings. The pair (M, N) is $\alpha - \psi - F$ -contractive if there exists $\tau > 0$ such that for all $x, y \in X$ with

$$\alpha(x, y) \geq 1$$

$$d(Mx, Ny) > 0 \Rightarrow \tau + F(d(Mx, Ny)) \leq F(\psi(G(x, y))) , \quad (2.16)$$

where $F \in \mathfrak{F}$, $\psi \in \Psi$ and $G(x, y)$ is defined by (2.2).

In our next result, we replace the continuity hypothesis by the following property:

(H): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists a sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and $\alpha(x, x_{n(k)}) \geq 1$ for all k .

So we have, the ensuing theorem:

Theorem 2.4. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be self mappings such that the pair (M, N) is $\alpha - \psi - F$ -contractive. Suppose that

- (i) (M, N) is a generalized α -admissible pair;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Mx_0) \geq 1$ and $\alpha(Mx_0, x_0) \geq 1$;
- (iii) **(H)** holds.

Then M and N have a common fixed point.

Proof. Following the proof of Theorem 2.2, it is obvious that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d) and converges to some $u \in X$. Now we shall prove that $Mu = u = Nu$. Suppose on the contrary that, $Mu \neq u$, or, $Nu \neq u$. If $x_n = Mu$ and $x_n = Nu$ for arbitrary large n , so, necessarily $Mu = u = Nu$. Therefore, we assume that $x_n \neq Mu$ or $x_n \neq Nu$ for infinitely many n . Let us suppose that $x_n \neq Mu$ for all $n \in \mathbb{N}$.

Since for all $k \in \mathbb{N}$ we have $d(Mu, x_{2n(k)}) = d(Mu, Nx_{2n(k)-1}) > 0$. Then, by assumption (iii) (i.e., $\alpha(u, x_{2n(k)-1}) \geq 1$) and (2.16), we have the following

$$\tau + F(d(Mu, Nx_{2n(k)-1})) = \tau + F(d(Mu, Nx_{2n(k)-1})) \leq F(\psi(G(u, x_{2n(k)-1}))) , \quad (2.17)$$

where

$$\begin{aligned} G(u, x_{2n(k)-1}) &= \max \left\{ \frac{d(u, x_{2n(k)-1}), d(u, Mu), d(x_{2n(k)-1}, Nx_{2n(k)-1}),}{\left(\frac{d(u, Nx_{2n(k)-1}) + d(x_{2n(k)-1}, Mu)}{d(u, Mu) + d((x_{2n(k)-1}, Nx_{2n(k)-1}) + 1)} \right)} d(u, x_{2n(k)-1}) \right\} \\ &= \max \left\{ \frac{d(u, x_{2n(k)-1}), d(u, Mu), d(x_{2n(k)-1}, x_{2n(k)}),}{\left(\frac{d(u, x_{2n(k)}) + d(x_{2n(k)-1}, Mu)}{d(u, Mu) + d((x_{2n(k)-1}, x_{2n(k)}) + 1)} \right)} d(u, x_{2n(k)-1}) \right\} . \end{aligned}$$

We know that $\lim_{n \rightarrow \infty} d(u, x_{2n(k)-1}) = \lim_{n \rightarrow \infty} d(x_{2n(k)-1}, x_{2n(k)}) = \lim_{n \rightarrow \infty} d(u, x_{2n(k)}) = 0$,

and $\lim_{n \rightarrow \infty} d(x_{2n(k)-1}, Mu) = d(u, Mu)$.

On the other hand, by (2.17) and **(F₁)**, we have for all $k \in \mathbb{N}$

$$d(Mu, x_{2n(k)}) = d(Mu, Nx_{2n(k)-1}) < \psi(G(u, x_{2n(k)-1})) , \quad (2.18)$$

for all $k \in \mathbb{N}$.

Mention to above limits and using the continuity of ψ in (2.18), as $k \rightarrow \infty$, we get

$$d(Mu, u) \leq \psi(d(u, Mu)).$$

Remembering that $\psi(t) < t$ for all $t > 0$, so the above inequality becomes

$$0 < d(Mu, u) \leq \psi(d(u, Mu)) < d(Mu, u) ,$$

This is a contradiction. Hence, we find that u is a fixed point of M . Similarly, we can show that u is a fixed point of N . Thus, u is a common fixed point of M and N . $\square$

We furnish the following example in support of our hypothesis.

Example 2.5. Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the mapping $M, N : X \rightarrow X$ by

$$Mx = \begin{cases} \frac{x}{4} & ; \text{ if } x \in [0, 1] \\ 3x - 1 & ; \text{ if } x > 0 \end{cases} \quad \text{and} \quad Nx = \begin{cases} 0 & ; \text{ if } x \in [0, 1] \\ x & ; \text{ if } x > 0. \end{cases}$$

Define the mapping $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 + e^{(x^2+y)} & ; \text{ if } x, y \in [0, 1] \\ 0 & ; \text{ otherwise.} \end{cases}$$

Let $\psi(t) = \frac{3}{5}t$, $F(t) = \ln(t^2 + t)$ for all $t > 0$ and $\tau = \ln \frac{3}{2}$. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α , this implies that $x, y \in [0, 1]$. Thus

$$\alpha(Mx, Ny) = \alpha\left(\frac{x}{4}, 0\right) = 1 + e^{(\frac{x^2}{16} + 0)} \geq 1 \quad \text{and} \quad \alpha(Ny, Mx) = \alpha\left(0, \frac{x}{4}\right) = 1 + e^{(0 + \frac{x}{4})} \geq 1.$$

Then, (M, N) is a generalized α -admissible pair.

Pointing out that M and N are noncontinuous mappings, now, we show that **(H)** is verified.

Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$, for all n and $x_n \rightarrow u \in X$.

Then $\{x_n\} \subset [0, 1]$. Consequently, $u \in [0, 1]$. Thus, $\alpha(x_n, u) = 1 + e^{(x_n^2 + u)} \geq 1$ and $\alpha(u, x_n) = 1 + e^{(u^2 + x_n)} \geq 1$ for all n . Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Mx_0) \geq 1$ and $\alpha(Mx_0, x_0) \geq 1$.

In fact, for $x_0 = 1$, we have $\alpha(1, M1) = \alpha(1, \frac{1}{4}) = 1 + e^{\frac{5}{4}} \geq 1$ and $\alpha(M1, 1) = \alpha(\frac{1}{4}, 1) = 1 + e^{\frac{17}{16}} \geq 1$.

Now, we prove that (M, N) is $\alpha - \psi - F$ -contractive. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$.

So, $x, y \in [0, 1]$. In the case, we have $d(Mx, Ny) = |Mx - Ny| = \frac{x}{4}$ and we also have

$$\begin{aligned} G(x, y) &= \max \left\{ |x - y|, \frac{3x}{4}, y, \left(\frac{x + |y - \frac{x}{4}|}{\frac{3x}{4} + y + 1} \right) |x - y| \right\} \\ &= \begin{cases} x - y & ; 0 \leq y \leq \frac{x}{4} \\ \frac{x}{3} & ; \frac{x}{4} < y \leq \frac{3x}{4} \\ y & ; \frac{3x}{4} < y \leq 1. \end{cases} \end{aligned}$$

that is,

$$d(Mx, Ny) \leq \frac{3}{4} \psi(G(x, y)),$$

for all $x, y \in X$ such that $\alpha(x, y) \geq 1$. Therefore,

$$\begin{aligned} d(Mx, Ny)(1 + d(Mx, Ny)) &\leq \frac{3}{4} \psi(G(x, y))(1 + \frac{3}{4} \psi(G(x, y))) \\ &\leq \frac{3}{4} \psi(G(x, y))(1 + \psi(G(x, y))). \end{aligned}$$

So, for all $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $d(Mx, Ny) > 0$, we have

$$\tau + F(d(Mx, Ny)) \leq F(\psi(G(x, y))).$$

Hence, all hypotheses of Theorem 2.4 are verified. Indeed, $\{0, \frac{5}{2}\}$ is the set of common fixed points of M and N .

Note that Theorem 2.4 assures only of the existence of a fixed point but not the uniqueness. In this example 0 and $\frac{5}{2}$ are two fixed points of M and N . So for the uniqueness, we need the following condition.

(U): For all $x, y \in CF(M, N)$, we have $\alpha(x, y) \geq 1$, where $CF(M, N)$ denotes the set of common fixed points of M and N .

Theorem 2.6. Adding condition **(U)** to the hypotheses of Theorem 2.2 (resp., Theorem 2.4), we obtain that u is the unique common fixed point of M and N .

Proof. We claim by contradiction, that is, there exist $u, v \in X$ such that $u = Mu = Nu$ and $v = Mv = Nv$ with $u \neq v$. By assumption **(U)** we have $\alpha(u, v) \geq 1$. First, assume that hypotheses of Theorem 2.2 hold. Since $d(u, v) = d(Mu, Nv) > 0$, by (2.1), we have

$$\tau + F(d(u, v)) = \tau + F(d(Mu, Nv)) \leq F(G(u, v)) = F(d(u, v)).$$

Then, by **(F₁)**

$$0 < d(u, v) < d(u, v),$$

which is a contradiction. Hence $u = v$. Secondly, suppose that the hypotheses of Theorem 2.4 hold.

Similarly, using (2.16), we get

$$\tau + F(d(u, v)) \leq F(\psi(G(u, v))) = F(\psi(d(u, v))).$$

Again, by **(F₁)**

$$0 < d(u, v) < \psi(d(u, v)) \leq d(u, v),$$

which is a contradiction, so, $u = v$. □

In the following, we present some illustrated consequences and corollaries of our obtained results given by Theorem 2.2 and Theorem 2.4.

Corollary 2.7. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be given continuous mappings. Suppose there exists $\tau > 0$ such that

$$d(Mx, Ny) > 0 \Rightarrow \tau + F(d(Mx, Ny)) \leq F(G(x, y)) \quad (2.19)$$

for all $x, y \in X$, where $F \in \mathfrak{F}$ and $M(x, y)$ is defined by (2.2). Then M and N are unique common fixed points.

Proof. To prove the above corollary it suffices to take $\alpha(x, y) = 1$ in Theorem 2.2 and to apply Theorem 2.6. □

Corollary 2.8. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be given continuous mappings. Suppose there exists $k \in (0, 1)$ such that

$$d(Mx, Ny) \leq k G(x, y), \quad (2.20)$$

for all $x, y \in X$, where $\psi \in \Psi$ and $M(x, y)$ is defined by (2.2). Then M and N are unique common fixed points.

Proof. To prove the above corollary it suffices to take $F(t) = \ln(t)$ and $\tau = -\ln(k)$ in Corollary 2.7 for $Mx \neq Nx$. Note that (2.20) is also satisfied for all $x, y \in X$ with $Mx = Nx$. □

Corollary 2.9. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be given continuous mappings. Suppose there exists $k \in (0, 1)$ such that

$$d(Mx, Ny) \leq k d(x, y), \quad (2.21)$$

for all $x, y \in X$. Then M and N are unique common fixed points.

Corollary 2.10. ([2], Theorem 2.1) Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point $x^\bullet \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to $x^\bullet$.

Proof. Taking $M = N = T$ in Theorem 2.2, then by **(F₁)**, if $d(Tx, Ty) > 0$ and $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, we have $\tau + F(d(Tx, Ty)) \leq F(G(x, y))$. The proof is then concluded by Theorem 2.2. □

Corollary 2.11. Let (X, d) be a complete metric space and $M, N : X \rightarrow X$ be given continuous mappings. Suppose there exists $\tau > 0$ such that

$$d(Mx, Ny) > 0 \Rightarrow \tau + F(d(Mx, Ny)) \leq F(\psi(G(x, y))), \quad (2.22)$$

for all $x, y \in X$, where $F \in \mathfrak{F}$, $\psi \in \Psi$ and $G(x, y)$ is defined by (2.2). Then M and N are unique common fixed points.

Proof. To prove the above corollary it suffices to take $\alpha(x, y) = 1$ in Theorem 2.4 and to apply Theorem 2.6. $\square$

Now, we give an example in which we ensure the uniqueness of the common fixed point.

Example 2.12. Let $X = \mathbb{R}$ be endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the mapping $M, N : X \rightarrow X$ by

$$Mx = \begin{cases} \frac{x}{2} & ; \text{ if } x \in [0, 1) \\ x - 1 & ; \text{ if } x > 0 \end{cases} \quad \text{and} \quad Nx = \begin{cases} 0 & ; \text{ if } x \in [0, 1) \\ x^2 & ; \text{ if } x > 0. \end{cases}$$

Define the mapping $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 + [\cos x + \cos y] & ; \text{ if } x, y \in [0, 1] \\ 0 & ; \text{ otherwise.} \end{cases}$$

Let $\psi(t) = \frac{3}{5}t$, $F(t) = \ln(t^2 + t)$ for all $t > 0$ and $\tau = \ln \frac{3}{2}$. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α , this implies that $x, y \in [0, 1]$. Thus

$$\alpha(Mx, Ny) = \alpha\left(\frac{x}{2}, 0\right) \geq 1 \quad \text{and} \quad \alpha(Ny, Mx) = \alpha\left(0, \frac{x}{2}\right) \geq 1.$$

Then, (M, N) is a generalized α -admissible pair.

3 Application to integral equations

In this section, we discuss the application of fixed point theorem proved by us in the previous section to the following Volterra type integral equation.

$$q(t) = \int_0^t L(t, s, w(s))ds + g(t) \quad (3.1)$$

for $t \in [0, a]$, where $a > 0$. We find the solution of (3.1). Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $q \in C([0, a], \mathbb{R})$, define the supremum norm as: $\|q\|_\tau = \sup_{t \in [0, a]} \{q(t)e^{-\tau(t)}\}$, where $\tau > 0$ is taken arbitrary. Let $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ be endowed with the metric

$$d_\tau(q, p) = \sup_{t \in [0, a]} \| |q(t) - p(t)| e^{-\tau t} \|_\tau \quad (3.2)$$

for all $q, p \in C([0, a], \mathbb{R})$. With these settings $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes a Banach space.

Now we prove the following theorem to ensure the existence of the solution of the integral equation (3.1).

Theorem 3.1. Assume that the following conditions are satisfied:

- (i) $L : [0, a] \times [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : [0, a] \rightarrow \mathbb{R}$ are continuous;
- (ii) Define

$$Tq(t) = \int_0^t L(t, s, q(s))ds + g(t).$$

Suppose there exist $\tau > 1$, such that

$$|L(t, s, q) - L(t, s, p)| \leq \tau e^{-\tau} [G(q, p)].$$

For all $t, s \in [0, a]$ and $q, p \in C([0, a], \mathbb{R})$, where

$$G(q, p) = \max\{|q(t) - p(t)|, |q(t) - Tq(t)|, |p(t) - Tp(t)|, \left(\frac{|q(t) - Tp(t)| + |p(t) - Tq(t)|}{|q(t) - Tq(t)| + |p(t) - Tp(t)| + 1}\right) |q(t) - p(t)|\},$$

then integral equation given in (3.1) has a solution.

Proof. By assumption (ii)

$$\begin{aligned} |Tq(t) - Tp(t)| &= \int_0^t |L(t, s, q(s) - L(t, s, p(s)))| ds \\ &\leq \int_0^t \tau e^{-\tau} ([G(q, p)] e^{-\tau s}) e^{\tau s} ds \leq \int_0^t \tau e^{-\tau} \|G(q, p)\|_{\tau} e^{\tau s} ds \\ &\leq \tau e^{-\tau} \|G(q, p)\|_{\tau} \int_0^t e^{\tau s} ds \leq \tau e^{-\tau} \|G(q, p)\|_{\tau} \frac{1}{\tau} e^{\tau t} \\ &\leq e^{-\tau} \|G(q, p)\|_{\tau} e^{\tau t}. \end{aligned}$$

This implies that

$$|Tq(t) - Tp(t)| e^{-\tau t} \leq e^{-\tau} \|G(q, p)\|_{\tau},$$

i.e.,

$$\|Tq(t) - Tp(t)\|_{\tau} \leq e^{-\tau} \|G(q, p)\|_{\tau},$$

which further implies that

$$\tau + \ln \|Tq(t) - Tp(t)\|_{\tau} \leq \ln \|G(q, p)\|_{\tau}.$$

Thus, all the conditions of Theorem 2.2 are satisfied. Hence integral equation given in (3.1) has a unique solution. $\square$

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