Original Article

Available online at
www.bpasjournals.com

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.378-384
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00041.9

OPTIMALITY CONDITIONS FOR FUZZY NON-LINEAR UNCONSTRAINED MINIMIZATION PROBLEMS

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Received on 29.01.2019 Revised on 28.05.2019 Accepted on 03.06.2019

Abstract:
In this paper, optimality conditions for fuzzy non-linear unconstrained minimization problems are discussed. Here the cost coefficients are represented by triangular fuzzy numbers. Finally, these conditions are verified by some numerical examples.

Keywords: Fuzzy non-linear unconstrained minimization problem, triangular fuzzy number, triangular fuzzy matrix.

2010 Mathematics Subject Classification- 03E72, 90C30, 90C70.

1. INTRODUCTION

Many practical problems cannot be represented by linear programming model. Therefore, attempts were made to develop more general mathematical programming methods and many significant advances have been made in the area of nonlinear programming. Fuzzy set theory has been applied to many areas such as mathematical modeling, operations research, management sciences and many industrial applications. Many authors considered various types of the fuzzy non-linear programming problems and proposed several approaches by solving these problem[5, 12, 13, 14, 16, 17]. R. E. Bellman and L. A. Zadeh [5] have introduced the decision making in a fuzzy environment. M. Lalitha and C. Loganathan [12] have discussed an objective fuzzy nonlinear programming problem with symmetric trapezoidal fuzzy numbers. M. Lalitha and C. Loganathan [13] have presented solving nonlinear programming problem in fuzzy environment. C. Loganathan and M. Kiruthiga[14] have introduced solution of fuzzy nonlinear programming problem using ranking function. A. Nagoorgani and C. Arunkumar [16] have introduced the principal pivoting method for solving Fuzzy Quadratic Programming Problem. R. Saranya and Palanivel Kaliyaperumal [17] have presented fuzzy nonlinear programming problem for inequality constraints with alpha optimal solution in terms of trapezoidal membership functions. In this paper, we consider the objective function, which is of minimization and there are no constraints available. We have also discussed the optimality conditions for this problem. This scheme of the paper is as follows: section 2 provides some basic definitions such as triangular fuzzy number, triangular fuzzy matrix and some operations on these two. Section 3 provides the optimality conditions for fuzzy non-linear unconstrained minimization problems. Finally, section 4 provides some numerical examples based on these optimality conditions.

2. PRELIMINARIES

2.1. Fuzzy non-linear programming problem:

It refers to an optimization problem in which the variables are continuous variables and the problem is of the following general form:

Minimize $\theta(\tilde{x})$

Subject to h_i $(\tilde{x})$= 0, i = 1,2,..,m.

g_p $(\tilde{x})$ ≥0, p = 1,2,..,t.

where$\theta(\tilde{x})$, h_i $(\tilde{x})$, g_p $(\tilde{x})$ are all real valued continuous functions of $\tilde{x} = (\tilde{x}_1,..,\tilde{x}_n) \in \boldsymbol{R}^n$.

2.2. Fuzzy non-linear unconstrained minimization problem:

It is that minimization problem in which there are no constraints on the variables in the fuzzy non-linear programming problem.

2.3. Types of solutions for a fuzzy non-linear programming problem:

Consider a fuzzy non-linear programming problem in which a function $\theta(\tilde{x})$ is required to be optimized subject to some constraints on the variables $\tilde{x} = (\tilde{x}_1, \ldots\ldots, \tilde{x}_n)^T$. Let $\widetilde{\boldsymbol{K}}$ denote the set of fuzzy feasible solutions for this problem. For this problem a fuzzy feasible solution $\tilde{\bar{x}} \in$ K is said to be a

(a) **local minimum,** if there exists an $\in > 0$ such that $\theta(\tilde{x}) \geq \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}} \cap \{\tilde{x}: \|\tilde{x} - \tilde{\bar{x}}\| < \in\}$,

(b) **strong local minimum,** if there exists an $\in > 0$ such that $\theta(\tilde{x}) > \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}} \cap \{\tilde{x}: \|\tilde{x} - \tilde{\bar{x}}\| < \in\}$, $\tilde{x} \neq \tilde{\bar{x}}$,

(c) **weak local minimum,** if it is a local minimum, but not a strong one,

(d) **global minimum,** if $\theta(\tilde{x}) \geq \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}}$,

(e) **local maximum,** if there exists an $\in > 0$ such that $\theta(\tilde{x}) \leq \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}} \cap \{\tilde{x}: \|\tilde{x} - \tilde{\bar{x}}\| < \in\}$,

(f) **strong local maximum,** if there exists an $\in > 0$ such that $\theta(\tilde{x}) < \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}} \cap \{\tilde{x}: \|\tilde{x} - \tilde{\bar{x}}\| < \in\}$, $\tilde{x} \neq \tilde{\bar{x}}$,

(g) **weak local maximum,** if it is a local maximum, but not a strong one,

(h) **global maximum,** if $\theta(\tilde{x}) \leq \theta(\tilde{\bar{x}})$ for all $\tilde{x} \in \widetilde{\boldsymbol{K}}$,

(i) **stationary point,** if some necessary optimality conditions for the problem are satisfied at the point $\tilde{\bar{x}}$.

2.4. Fuzzy set

A fuzzy set $\tilde{A}$ is defined by $\tilde{A}$ = $\{(x, \mu_A(x))$: $x \in A$, $\mu_A(x) \in [0,1]\}$. In the pair (x, $\mu_A(x)$), the first element x belong to the classical set A, the second element$\mu_A(x)$ belong to the interval [0,1], called the membership function.

2.5. Fuzzy number

The notion of fuzzy numbers was introduced by D. Dubois D. and H. Prade. A fuzzy subset $\tilde{A}$ of the real line R with membership function $\mu_{\tilde{A}}$: $R \to [0,1]$ is called a fuzzy number if

i) A fuzzy set $\tilde{A}$ is normal.

ii) $\tilde{A}$ is fuzzy convex.
(i.e.) $\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2)$, $x_1, x_2 \in R$, $\forall \lambda \in [0,1]$.

iii) $\mu_{\tilde{A}}$ is upper continuous, and

iv) supp $\tilde{A}$ is bounded, where supp $\tilde{A}$= $\{x \in$ R:$\mu_{\tilde{A}}$ $(x)>0\}$.

2.6. Triangular Fuzzy Number

It is a fuzzy number represented with three points as follows: $\tilde{A}$ = (a_1, a_2, a_3). This representation in interpreted as membership functions

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & \text{for} a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{for} a_2 \leq x \leq a_3 \\ 0 & \text{for } \quad x > a_3 \end{cases}$$

2.7. Operations of Triangular Fuzzy Number using Function Principle

Let $\tilde{A}$ = (a_1, a_2, a_3) and $\tilde{B}$ = (b_1, b_2, b_3) be two triangular fuzzy numbers. Then

i. The addition of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A}+\tilde{B} = (a_1+b_1, a_2+b_2, a_3+b_3)$ where $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers.
ii. The product of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A}$ x $\tilde{B} = (c_1, c_2, c_3)$, where $T = \{a_1b_1, a_2b_2, a_3b_3\}$ where $c_1 = \min\{T\}$, $c_2 = a_2b_2$, $c_3 = \max\{T\}$. If $a_1, a_2, a_3, b_1, b_2, b_3$ are all non- zero positive real numbers, then $\tilde{A}$ x$\tilde{B} = (a_1b_1, a_2b_2, a_3b_3)$
iii. $-\tilde{B} = (-b_3, -b_2, -b_1)$ then the subtraction of $\tilde{B}$ from $\tilde{A}$ is $\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$ where $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers.
iv. The division of $\tilde{A}$ and $\tilde{B}$ is $\frac{\tilde{A}}{\tilde{B}} = (c_1, c_2, c_3)$, where $T = (\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1})$, where $c_1=\min\{T\}$, $c_2=\frac{a_2}{b_2}$, $c_3=\max\{T\}$.

If $a_1, a_2, a_3, b_1, b_2, b_3$ are all non-zero positive real numbers then $\frac{\tilde{A}}{\tilde{B}} = (\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1})$.

2.8. Triangular Fuzzy Matrix

A triangular fuzzy matrix of order mxn is defined as $A=(\tilde{a}_{ij})_{mxn}$, where $a_{ij}=(a_{ij1}, a_{ij2}, a_{ij3})$ is the ij^{th} element of A.

2.9. Operations on Triangular Fuzzy Matrices

Let $A=(\tilde{a}_{ij})$ and $B=(\tilde{b}_{ij})$ be two triangular fuzzy matrices of same order. Then we have the following:
i. $A+B=(\tilde{a}_{ij} + \tilde{b}_{ij})$
ii. $A-B=(\tilde{a}_{ij} - \tilde{b}_{ij})$
iii. For $A=(\tilde{a}_{ij})_{mxn}$ and $B=(\tilde{b}_{ij})_{nxk}$ then $AB=(\tilde{c}_{ij})_{mxk}$ where $\tilde{c}_{ij} = \sum_{p=1}^{n} \tilde{a}_{ip}.\tilde{b}_{pj}$, i=1,2,…,m and j=1,2,,k.
iv. $A^T=(\tilde{a}_{ji})$
v. $KA=(K\tilde{a}_{ij})$ where K is scalar.

2.10. Positive semidefinite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order n, whether it is symmetric or not, is said to be a positive semidefinite (PSD) fuzzy matrix if $\tilde{x}^T\tilde{A}\tilde{x} \geq 0$ for all $\tilde{x} \in \boldsymbol{R}^n$.

2.11. Positive definite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order n, whether it is symmetric or not, is said to be a positive definite (PD) fuzzy matrix if $\tilde{x}^T\tilde{A}\tilde{x} > 0$ for all $\tilde{x} \neq 0$.

3. MAIN RESULTS

3.1 Optimality conditions for fuzzy non-linear unconstrained minimization problems

Let $\widetilde{\boldsymbol{K}}$ denote the set of fuzzy feasible solutions for an optimization problem in which the objective function $\theta(\tilde{x})$ is to be minimized. Let $\bar{\tilde{x}} \in \widetilde{\boldsymbol{K}}$ be a fuzzy feasible solution. A fuzzy feasible direction at $\bar{\tilde{x}}$ for $\widetilde{\boldsymbol{K}}$ is a direction $\tilde{y}$ satisfying the property that beginning at $\bar{\tilde{x}}$, we can move a positive length along a straight line in the direction $\tilde{y}$, without leaving $\widetilde{\boldsymbol{K}}$. Necessary optimality conditions for this optimization problem are derived, based on two very simple principles. These are the following:
1. If $\bar{\tilde{x}}$ is a local minimum for this optimization problem, then, as we move from $\bar{\tilde{x}}$ straight along any fuzzy feasible direction at $\bar{\tilde{x}}$ for $\widetilde{\boldsymbol{K}}$, in a small neighbourhood of $\bar{\tilde{x}}$, the objective value cannot decrease.
2. Take a one dimensional, nonlinear, differentiable curve in the fuzzy feasible region $\widetilde{\boldsymbol{K}}$, passing through $\bar{\tilde{x}}$. If $\bar{\tilde{x}}$ is a local minimum for this fuzzy optimization problem, then, as we move from $\bar{\tilde{x}}$ along this curve, in a small neighbourhood of $\bar{\tilde{x}}$, the objective value cannot decrease (in effect this says that if $\bar{\tilde{x}}$ is a local minimum for $\theta(\tilde{x})$ in $\widetilde{\boldsymbol{K}}$, then $\bar{\tilde{x}}$ must be a local minimum for the one dimensional fuzzy optimization problem of minimizing $\theta(\tilde{x})$ on the curve).

Of course 1 is a special case of 2, since a straight line is a differentiable curve. These principles make it possible for us to derive necessary conditions for local minimality in higher dimensional fuzzy feasible regions using well known necessary conditions for local minimality in one-dimensional fuzzy optimization problems.
All the necessary optimality conditions are derived using the above principles. Even though the principles are the same, their application leads to optimality conditions which depend on the structure of the problem.

We will now derive optimality conditions for fuzzy nonlinear unconstrained minimization problems.
First consider the unconstrained minimization problem

$$\begin{aligned} &\text{minimize } \theta(\tilde{x}) \\ &\text{over } \tilde{x} \in \boldsymbol{R}^n. \end{aligned} \tag{3.1}$$

Given $\tilde{\bar{x}} \in \boldsymbol{R}^n$, $\tilde{y} \in \boldsymbol{R}^n$, $\tilde{y} \neq 0$, by differentiability of $\theta(\tilde{x})$, we know that limit of $(\theta(\tilde{\bar{x}} + \alpha\tilde{y}) - \theta(\tilde{\bar{x}}) - \alpha(\nabla\theta(\tilde{\bar{x}}))\tilde{y})/\alpha$ as α tends to zero is zero. So, if $(\nabla\theta(\tilde{\bar{x}}))\tilde{y} < 0$ by choosing α positive and sufficiently small, we will have $\theta(\tilde{\bar{x}} + \alpha\tilde{y}) < \theta(\tilde{\bar{x}})$. Similarly, if $(\nabla\theta(\tilde{\bar{x}}))\tilde{y} > 0$, by choosing α negative with sufficiently small absolute value we will have again $\theta(\tilde{\bar{x}} + \alpha\tilde{y}) < \theta(\tilde{\bar{x}})$. So if $\tilde{\bar{x}}$ is a local minimum for (3.1), we must have $(\nabla\theta(\tilde{\bar{x}}))\tilde{y} = 0$ for all $\tilde{y} \in \boldsymbol{R}^n$, that is

$$\nabla\theta(\tilde{\bar{x}}) = 0 \qquad (3.2)$$

(3.2) is the first order necessary condition for $\tilde{\bar{x}}$ to be a local minimum for (3.1).

If $\theta(\tilde{x})$ is twice continuously differentible at $\tilde{\bar{x}}$, we know that the limit of $(\theta(\tilde{\bar{x}} + \alpha\tilde{y}) - \theta(\tilde{\bar{x}}) - \alpha(\nabla\theta(\tilde{\bar{x}}))\tilde{y}) - (\alpha^2/2)\tilde{y}^T\widetilde{H}(\theta(\tilde{\bar{x}}))\tilde{y})/\alpha^2$ as α tends to zero is zero, where $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is the Hessian matrix (the matrix of second order partial derivatives) of $\theta(\tilde{x})$ at $\tilde{\bar{x}}$. So if $\tilde{\bar{x}}$ is such that (3.2) is satisfied, and $\tilde{y}$is such that $\tilde{y}^T\widetilde{H}(\theta(\tilde{\bar{x}}))\tilde{y} < 0$ then for $\alpha \neq 0$ and sufficiently small, we will have $\theta(\tilde{\bar{x}} + \alpha\tilde{y}) < \theta(\tilde{\bar{x}})$. So, if $\tilde{\bar{x}}$is a local minimum for (3.1) we must have $\tilde{y}^T\widetilde{H}(\theta(\tilde{\bar{x}}))\tilde{y} \geq 0$ for all $\tilde{y} \in \boldsymbol{R}^n$ when $\tilde{\bar{x}}$satisfies(3.2), that is

$\widetilde{H}(\theta(\tilde{\bar{x}}))$ must be PSD. (3.3)

(3.2) and (3.3) together are the second order necessary conditions for $\tilde{\bar{x}}$ to be a local minimum to (3.1).

Theorem 3.1.1(Gradient support inequality): Let $\theta(\tilde{x})$ be a real valued convex function defined on an open convex fuzzy set $\boldsymbol{\Gamma} \subset \boldsymbol{R}^n$ If $\theta(\tilde{x})$ is differentiable at $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$,

$$\theta(\tilde{x}) - \theta(\tilde{\bar{x}}) \geq (\nabla\theta(\tilde{\bar{x}}))(\tilde{x} - \tilde{\bar{x}}) \text{ for all } \tilde{x} \in \boldsymbol{\Gamma}. \qquad (3.4)$$

Conversely, if $\theta(\tilde{x})$ is a real valued differentiable function defined on $\boldsymbol{\Gamma}$ and (3.4) holds for all$\tilde{x}, \tilde{\bar{x}} \in \boldsymbol{\Gamma}$, $\theta(\tilde{x})$ is convex.

Proof: Suppose $\theta(\tilde{x})$ is convex. Let $\tilde{x} \in \boldsymbol{\Gamma}$. By convexity of $\boldsymbol{\Gamma}$, $\alpha\tilde{x} + (1-\alpha)\tilde{\bar{x}} = \tilde{\bar{x}} + \alpha(\tilde{x} - \tilde{\bar{x}}) \in \boldsymbol{\Gamma}$ for all $0 \leq \alpha \leq 1$. Since $\theta(\tilde{x})$ is convex we have $\theta(\tilde{\bar{x}} + \alpha(\tilde{x} - \tilde{\bar{x}})) \leq \alpha\theta(\tilde{x}) + (1-\alpha)\theta(\tilde{\bar{x}})$. So for $0 < \alpha \leq 1$, we have

$$\theta(\tilde{x}) - \theta(\tilde{\bar{x}}) \geq (\theta(\tilde{\bar{x}} + \alpha(\tilde{x} - \tilde{\bar{x}})) - \theta(\tilde{\bar{x}}))/\alpha. \qquad (3.5)$$

By definition of differentiability, the right hand side of (3.5) tends to $\nabla\theta(\tilde{\bar{x}})(\tilde{x} - \tilde{\bar{x}})$ as α tends to zero through positive values. Since (3.5) holds for all $0 < \alpha \leq 1$, this implies (3.4) as α tends to zero through positive values in (3.5).

Conversely, suppose $\theta(\tilde{x})$ is a real valued differentiable function defined on $\boldsymbol{\Gamma}$and suppose (3.4) holds for all $\tilde{x}$, $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$. Given $\tilde{x}^1, \tilde{x}^2 \in \boldsymbol{\Gamma}$, from (3.4) we have, for $0 < \alpha < 1$,

$$\theta(\tilde{x}^1) - \theta((1-\alpha)\tilde{x}^1 + \alpha\tilde{x}^2) \geq \alpha(\nabla\theta(1-\alpha)\tilde{x}^1 + \alpha\tilde{x}^2)\,(\tilde{x}^1 - \tilde{x}^2)$$

$$\theta(\tilde{x}^2) - \theta((1-\alpha)\tilde{x}^1 + \alpha\tilde{x}^2) \geq -(1-\alpha)(\nabla\theta((1-\alpha)\tilde{x}^1 + \alpha\tilde{x}^2))\,(\tilde{x}^1 - \tilde{x}^2).$$

Multiply the first inequality by $(1-\alpha)$ and the second by α and add. This leads to

$$(1-\alpha)\theta(\tilde{x}^1) + \alpha\theta(\tilde{x}^2) - \theta((1-\alpha)\tilde{x}^1 + \alpha\tilde{x}^2) \geq 0. \qquad (3.6)$$

Since (3.6) holds for all$\tilde{x}^1, \tilde{x}^2 \in \boldsymbol{\Gamma}$and $0 < \alpha < 1$, $\theta(\tilde{x})$ is convex.

Theorem 3.1.2: Let $\theta(\tilde{x})$ be a real valued fuzzy convex function defined on an open convex fuzzy subset $\boldsymbol{\Gamma} \subset \boldsymbol{R}^n$. If $\theta(\tilde{x})$ is twice differentiable at $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$, $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PSD. Conversely, if $\theta(\tilde{x})$ is a twice differentiable real valued function defined on $\boldsymbol{\Gamma}$ and $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PSD for all $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$, $\theta(\tilde{x})$ is convex.

Proof: Let $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$and $\tilde{y} \in \boldsymbol{R}^n$.Suppose $\theta(\tilde{x})$ is convex.For $\alpha > 0$ and sufficiently small, by theorem 3.1.1 we have

$$(\theta(\tilde{\bar{x}} + \alpha\tilde{y}) - \theta(\tilde{\bar{x}}) - \alpha(\nabla\theta(\tilde{\bar{x}}))\tilde{y})/\alpha \geq 0. \qquad (3.7)$$

Taking the limit as α tends to zero through positive values, from (3.7) we have $\tilde{y}^T\widetilde{H}(\theta(\tilde{\bar{x}}))\tilde{y} \geq 0$, and since this holds for all $\tilde{y} \in \boldsymbol{R}^n$, $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PSD.

Suppose $\theta(\tilde{\bar{x}})$ is twice differentiable on $\boldsymbol{\Gamma}$and $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PSD for all $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$. By Taylor's theorem of calculus we have, for $\tilde{x}^1, \tilde{x}^2 \in \boldsymbol{\Gamma}$, $\theta(\tilde{x}^2) - \theta(\tilde{x}^1) - (\nabla\theta(\tilde{x}^1))(\tilde{x}^2 - \tilde{x}^1)$
$= (\tilde{x}^2 - \tilde{x}^1)^T\widetilde{H}(\theta(\tilde{x}^1 + \alpha(\tilde{x}^2 - \tilde{x}^1)))(\tilde{x}^2 - \tilde{x}^1)\,/\,2$ for some $0 < \alpha < 1$. But the latter expression is ≥ 0 since $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PSD for all $\tilde{\bar{x}} \in \boldsymbol{\Gamma}$. So $\theta(\tilde{x}^2) - \theta(\tilde{x}^1) - (\nabla\theta(\tilde{x}^1))(\tilde{x}^2 - \tilde{x}^1) \geq 0$ for all $\tilde{x}^1, \tilde{x}^2 \in \boldsymbol{\Gamma}$. By theorem 3.1.1, this implies that $\theta(\tilde{x})$ is convex.

Theorem 3.1.3: A square symmetric fuzzy matrix is PD iff all its principal subdeterminants are strictly positive.
Proof:

$$\text{Let the matrix } \widetilde{H} = \begin{pmatrix} \tilde{d}_{11} & \cdots & \tilde{d}_{1n} & \tilde{d}_{1,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{n1} & \cdots & \tilde{d}_{nn} & \tilde{d}_{n,n+1} \\ \tilde{d}_{n+1,1} & \cdots & \tilde{d}_{n+1,n} & \tilde{d}_{n+1,n+1} \end{pmatrix}$$

If $\widetilde{H}$ is PD, all its principal subdeterminants are strictly positive.
On the other hand, if all the principal subdeterminants of $\widetilde{H}$ are strictly positive, the $n+1$ principal subdeterminants of $\widetilde{H}$ are strictly positive, and this implies that $\widetilde{H}$ is PD.
The following theorem states a sufficient optimality condition for (3.1).

Theorem 3.1.4: Suppose $\theta(\tilde{x})$ is twice continuously differentiable, and $\tilde{\bar{x}}$ is a point satisfying $\nabla\theta(\tilde{\bar{x}}) = 0$,
and $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PD (3.8)
then $\tilde{\bar{x}}$ is a local minimum for (3.1).

Proof: Since $\widetilde{H}(\theta(\tilde{\bar{x}}))$ is PD, all its principal subdeterminants are > 0. Since $\theta(\tilde{x})$ is twice continuously differentiable, all principal subdeterminants of the Hessian matrix $\widetilde{H}(\theta(\tilde{x}))$ are continuous functions.These facts imply that there exists an $\in > 0$, such that if $\boldsymbol{\Gamma} = \{\tilde{x} : \|\tilde{x} - \tilde{\bar{x}}\| < \in\}$, all principal subdeterminants of $\widetilde{H}(\theta(\tilde{x}))$ are > 0 for all $\tilde{x} \in \boldsymbol{\Gamma}$. Being a Hessian matrix $\widetilde{H}\,(\theta(\tilde{x}))$ is also symmetric, by theorem 3.1.3, these facts imply that $\widetilde{H}(\theta(\tilde{x}))$ is PSD for all $\tilde{x} \in \boldsymbol{\Gamma}$. By theorem 3.1.2, this implies that $\theta(\tilde{x})$ is convex over $\tilde{x} \in \boldsymbol{\Gamma}$. So by theorem 3.1.1(the gradient support inequality)

$$\theta(\tilde{x}) - \theta(\tilde{\bar{x}}) \geq ((\nabla\theta(\tilde{\bar{x}}))(\tilde{x} - \tilde{\bar{x}}) \text{ for all } \tilde{x} \in \boldsymbol{\Gamma}$$
$$\geq 0, \text{ since } \nabla\theta(\tilde{\bar{x}}) = 0 \text{ by (3.8).}$$

This proves that $\tilde{\bar{x}}$ is a local minimum for $\theta(\tilde{x})$.This is a sufficient condition for $\tilde{\bar{x}}$ to be a local minimum for (3.1) is (3.8).

4. NUMERICAL EXAMPLE

Example 4.1:
Consider the problem
minimize $\theta(\tilde{x}) = (1.75, 2, 2.25)\tilde{x}_1{}^2 + (0.75, 1, 1.25)\,\tilde{x}_2{}^2 + (0.75, 1, 1.25)\,\tilde{x}_3{}^2 + (0.75, 1, 1.25)\,\tilde{x}_1\tilde{x}_2 + (0.75, 1, 1.25)\,\tilde{x}_2\tilde{x}_3 + (0.75, 1, 1.25)\,\tilde{x}_3\tilde{x}_1 + (-9.25, -9, -8.75)\,\tilde{x}_1 + (-9.25, -9, -8.75)\,\tilde{x}_2 + (-8.25, -8, -7.75)\,\tilde{x}_3$ over $\tilde{x} \in \boldsymbol{R}^3$.
Solution:

From the necessary optimality conditions, every local minimum for this problem must satisfy

$$\frac{\partial\theta(\tilde{x})}{\partial\tilde{x}_1} = 0$$
$$\Rightarrow (3.5, 4, 5)\,\tilde{x}_1 + (0.75, 1, 1.25)\,\tilde{x}_2 + (0.75, 1, 1.25)\,\tilde{x}_3 + (-9.25, -9, -8.75) = 0.$$

$$\frac{\partial\theta(\tilde{x})}{\partial\tilde{x}_2} = 0$$
$$\Rightarrow (0.75, 1, 1.25)\,\tilde{x}_1 + (1.5, 2, 2.5)\,\tilde{x}_2 + (0.75, 1, 1.25)\,\tilde{x}_3 + (-9.25, -9, -8.75) = 0.$$

$$\frac{\partial\theta(\tilde{x})}{\partial\tilde{x}_3} = 0$$
$$\Rightarrow (0.75, 1, 1.25)\,\tilde{x}_1 + (0.75, 1, 1.25)\,\tilde{x}_2 + (1.5, 2, 2.5)\,\tilde{x}_3 + (-8.25, -8, -7.75) = 0.$$

Since by using defuzzification and solving the above system of equations, the unique solution is

$$\tilde{\bar{x}} = \begin{bmatrix} (-0.96, 1, 2.36) \\ (1.88, 3, 5.95) \\ (-0.59, 2, 39.64) \end{bmatrix}.$$

The Hessian matrix is

$$\widetilde{H}(\theta(\tilde{\tilde{x}})) = \begin{bmatrix} (3.5, 4, 5) & (0.75, 1, 1.25) & (0.75, 1, 1.25) \\ (075, 1, 1.25) & (1.5, 2, 2.5) & (0.75, 1, 1.25) \\ (0.75, 1, 1.25) & (0.75, 1, 1.25\ ) & (1.5, 2, 2.5) \end{bmatrix}$$

This matrix is PD(since $\tilde{\tilde{x}}^T \widetilde{H}\tilde{\tilde{x}}> 0$). So $\tilde{\tilde{x}}$ satisfies the sufficient conditions for a local minimum. Clearly, here, $\theta(\tilde{x})$ is convex and hence $\tilde{\tilde{x}}$ is a global minimum for $\theta(\tilde{x})$.

Example 4.2:

Let $\theta(\tilde{x}) = (-2.25,\ -2, -1.75)\ \tilde{x}_1{}^2 + (-1.25,\ -1, -0.75)\ \tilde{x}_2{}^2 + (0.75,\ 1,\ 1.25)\tilde{x}_1\tilde{x}_2 + (-10.25, -10, -9.75)\ \tilde{x}_1 + (5.75, 6, 6.25)\ \tilde{x}_2$ and consider the problem of minimizing $\theta(\tilde{x})$ over $\tilde{x} \in \boldsymbol{R}^2$.

Solution:

The first order necessary conditions for a local minimum are

$$\frac{\partial\theta(\tilde{x})}{\partial\tilde{x}_1} = 0$$

$\Rightarrow (-4.5, -4, -3.5)\ \tilde{x}_1 + (0.75,\ 1,\ 1.25)\ \tilde{x}_2 + (-10.25,\ -10, -9.75) = 0.$

$$\frac{\partial\theta(\tilde{x})}{\partial\tilde{x}_2} = 0$$

$\Rightarrow (0.75,\ 1,\ 1.25)\ \tilde{x}_1 + (-2.5, -2, -1.5)\ \tilde{x}_2 + (5.75,\ 6,\ 6.25) = 0.$

Since by using defuzzification and solving the above system of equations, the unique solution is

$$\tilde{\tilde{x}} = \begin{bmatrix} (-2.30, -2, -1.79) \\ (1.4, 2, 3.02) \end{bmatrix}.$$

The Hessian matrix is

$$\widetilde{H}(\theta(\tilde{\tilde{x}})) = \begin{bmatrix} (-4.5, -4, -3.5) & (0.75, 1, 1.25) \\ (0.75, 1, 1.25) & (-2.5, -2, -1.5) \end{bmatrix}.$$

Since H($\theta(\tilde{\tilde{x}})$) is not PSD (since $\tilde{\tilde{x}}^T \widetilde{H}\tilde{\tilde{x}}< 0$), $\tilde{\tilde{x}}$ violates the second order necessary conditions for being a local minimum of $\theta(\tilde{x})$. So $\theta(\tilde{x})$ has no local minimum. In fact, it can be verified that the Hessian matrix is negative definite (ND), so $\tilde{\tilde{x}}$ satisfies the sufficient condition for being a local maximum for $\theta(\tilde{x})$ (a local maximum for $\theta(\tilde{x})$ is a local minimum for $-\theta(\tilde{x})$). Actually, $\theta(\tilde{x})$ here is concave and $\tilde{\tilde{x}}$ is a global maximum point for $\theta(\tilde{x})$. It can be verified that $\theta(\tilde{x})$ is unbounded below on $\boldsymbol{R}^2$.

5. CONCLUSION

In this paper, the fuzzy nonlinear unconstrained minimization problem is defined and the optimality conditions for this problem are stated. Some examples are discussed based on these optimality conditions.

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