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A new decomposition method - the "Upadhyaya decomposition method" - for the solution of the linear Volterra integro-differential equations of the second kind *

Sudhanshu Aggarwal[1,†] and Lalit Mohan Upadhyaya[2]

1. Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur, Uttar Pradesh-273402, India.
2. Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand-248179, India.

1. E-mail: sudhanshu30187@gmail.com
2. E-mail: lmupadhyaya@rediffmail.com , hetchres@gmail.com

Abstract This study introduces a new decomposition method called the "Upadhyaya decomposition method" for solving the problem of linear Volterra integro-differential equations of the second kind. This method is the combination of the Upadhyaya transform and the decomposition method. Three numerical problems provide a detailed description and illustration of the procedure. According to the results the current approach is quite effective and it provides the answers without requiring laborious computational efforts.

Key words Integral Transform, Upadhyaya Transform, Inverse Upadhyaya Transform, Convolution Theorem, Decomposition Method.

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1 Introduction

The integral and the integro-differential equations are widely used in the development of mathematical models for the solutions of various problems, including those involving electrical circuits, mechanical vibration, heat transfer, compartment problems, and bacterial growth [1,2]. In order to find the answers to many of the problems in physics, chemical sciences, mathematics, mechanics and medical sciences etc. researchers now-a-days are applying a variety of integral transformations [3–22]. Utilizing the Kamal, Mahgoub, Sadik, Aboodh, Mohand, Elzaki, Laplace-Carson, Laplace, Sawi, Sumudu, Shehu and the other transforms, researchers [23–35] were able to fully solve the first and second kinds of Volterra integro-differential equation problems. Comparative analyses of the Mohand and other transformations were conducted by Aggarwal and other researchers [36–41]. Duality relations of various integral transforms were also studied by many researchers [42–49]. In 2019, Upadhyaya [50] introduced a new integral transform the "Upadhyaya integral transform". Upadhyaya et al. [51] further gave an update on this transform. In the subject of mathematical analysis, there is a strong emphasis on investigating various strategies and approaches for solving the second-kind linear Volterra integro-differential equations.

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†Corresponding author Sudhanshu Aggarwal, E-mail: sudhanshu30187@gmail.com

These equations, which have both differential and integral components, present a unique difficulty that necessitates specialized strategies for effective resolution. Researchers and practitioners in this field work hard to investigate and create ways that can accurately and efficiently solve these types of equations. They intend to analyze and understand complicated systems and phenomena represented by such equations by utilizing advanced mathematical tools and sophisticated algorithms. This motivates us to discuss an effective new method in this paper which we call the **Upadhyaya decomposition method** to find the solution of the linear Volterra integro-differential equation of the second kind.
Elucidating the scheme of the paper we mention that in section 2 we give the necessary basic definitions and results which we employ in the sequel for establishing our results of this paper. The Upadhyaya decomposition technique id discussed bu us in section 3 and the same is applied by us to solve some interesting problems in section 4. The brief conclusion of the study is summed up finally in section 5.

2 Preliminaries

In this section we detail some preliminary results including the definition and properties of the Upadhyaya transform which shall be utilized by us in the succeeding sections of this paper for developing the technique of the Upadhyaya decomposition method and for its applications. For more details of these results we refer the reader to Upadhyaya [50] and Upadhyaya et al. [51].

Definition 2.1. Upadhyaya transform [50]: The Upadhyaya transform of a function $\theta(x) \in \mathcal{C}, x \geq 0$, where $\mathcal{C}$ is the class of all sequentially continuous exponential order functions, is given by [50]

$$\mathcal{U}\{\theta(x)\} = \alpha \int_0^\infty \theta(\gamma x) e^{-\beta x} dx = \mathcal{T}(\alpha,\beta,\gamma), \quad \alpha,\beta,\gamma > 0. \tag{2.1}$$

We remark here that in the original formulation of the Upadhyaya transform, the parameters α,β and γ are complex numbers and they are specially defined by Upadhyaya [50] and further generalized by Upadhyaya et. al. in [51] so that the Upadhyaya transform encompasses all the presently known and the upcoming integral transforms of the Laplace class, which make it the most generalized, powerful and robust transform of the Laplace class existing till date in the mathematics research literature. For the purposes of the present study, we have restricted ourselves to taking all the three parameters $\alpha,\beta,\gamma \in \mathbb{R}^+$, i.e., the set of the positive real numbers.

Definition 2.2. The inverse Upadhyaya transform [50]: The inverse Upadhyaya transform of $\mathcal{T}(\alpha,\beta,\gamma)$, designated by $\mathcal{U}^{-1}\{\mathcal{T}(\alpha,\beta,\gamma)\}$, is another function $\theta(x)$ having the property that $\mathcal{U}\{\theta(x)\} = \mathcal{T}(\alpha,\beta,\gamma)$. For a rigorous formulation of the inverse Upadhyaya transform we refer the reader to Upadhyaya [50].

For a ready reference we summarize the Upadhyaya transforms and the inverse Upadhyaya transforms of some of the most commonly encountered elementary function in Tables 1 and 2 respectively:

Definition 2.3. The linearity property of the Upadhyaya transform [50]: If $\theta_i(x) \in \mathcal{C}$ and $\mathcal{U}\{\theta_i(x)\} = \mathcal{T}_i(\alpha,\beta,\gamma)$ then $\mathcal{U}\left\{\sum_{i=1}^n a_i\theta_i(x)\right\} = \sum_{i=1}^n a_i\mathcal{U}\{\theta_i(x)\} = \sum_{i=1}^n a_i\,\mathcal{T}_i(\alpha,\beta,\gamma)$, where a_i are arbitrary constants.

Definition 2.4. The translation property of the Upadhyaya transform [50]: If $\theta(x) \in \mathcal{C}$ and $\mathcal{U}\{\theta(x)\} = \mathcal{T}(\alpha,\beta,\gamma)$ then $\mathcal{U}\{e^{ax}\theta(x)\} = \mathcal{T}(\alpha,\beta - a\gamma,\gamma)$, where a is an arbitrary constant.

Definition 2.5. The change of scale property of the Upadhyaya transform [50]: If $\theta(x) \in \mathcal{C}$ and $\mathcal{U}\{\theta(x)\} = \mathcal{T}(\alpha,\beta,\gamma)$ then $\mathcal{U}\{\theta(ax)\} = \mathcal{T}\left(\frac{\alpha}{a},\frac{\beta}{a},\gamma\right)$, where a is arbitrary constant.

Theorem 2.6. The convolution (Faltung) the Upadhyaya transform [50]: *If* $\theta_i(x) \in \mathcal{C},\ i = 1,\ 2$ *and* $\mathcal{U}\{\theta_i(x)\} = \mathcal{T}_i(\alpha,\beta,\gamma), i = 1,\ 2$ *then*

$$\mathcal{U}\{\theta_1(x) * \theta_2(x)\} = \left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{\theta_1(x)\}\mathcal{U}\{\theta_2(x)\} = \left(\frac{\gamma}{\alpha}\right)\ \mathcal{T}_1(\alpha,\beta,\gamma)\ \mathcal{T}_2(\alpha,\beta,\gamma).$$

Theorem 2.7. The Upadhyaya transform of the derivatives of a function [50]: *If* $\theta(x) \in \mathcal{C}$ *and* $\mathcal{U}\{\theta(x)\} = \mathcal{T}(\alpha,\beta,\gamma)$ *then*

Table 1: The Upadhyaya transform of some elementary functions [50].

S.N.	$\theta(x) \in \mathcal{C}, x \geq 0$	$\mathcal{U}\{\theta(x)\} = \mathcal{T}(\alpha, \beta, \gamma)$
1	1	$\left(\frac{\alpha}{\beta}\right)$
2	e^{ax}	$\left(\frac{\alpha}{\beta - a\gamma}\right)$
3	$x^a, a \in \mathbb{N}$	$a!\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right)$
4	$x^a,\ a > -1, a \in \mathbb{R}$	$\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right)\Gamma(a+1)$
5	$\sin(ax)$	$\frac{a\alpha\gamma}{(\beta^2+a^2\gamma^2)}$
6	$\cos(ax)$	$\frac{\alpha\beta}{(\beta^2+a^2\gamma^2)}$
7	$\sinh(ax)$	$\frac{a\alpha\gamma}{(\beta^2-a^2\gamma^2)}$
8	$\cosh(ax)$	$\frac{\alpha\beta}{(\beta^2-a^2\gamma^2)}$

Table 2: The inverse Upadhyaya transform of some elementary functions [50].

S.N.	$\mathcal{T}(\alpha, \beta, \gamma)$	$\theta(x) = \mathcal{U}^{-1}\{\mathcal{T}(\alpha, \beta, \gamma)\}$
1	$\left(\frac{\alpha}{\beta}\right)$	1
2	$\left(\frac{\alpha}{\beta - a\gamma}\right)$	e^{ax}
3	$\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right), a \in \mathbb{N}$	$\frac{x^a}{a!}$
4	$\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right)$, $a > -1, a \in \mathbb{R}$	$\frac{x^a}{\Gamma(a+1)}$
5	$\frac{\alpha\gamma}{(\beta^2+a^2\gamma^2)}$	$\frac{\sin(ax)}{a}$
6	$\frac{\alpha\beta}{(\beta^2+a^2\gamma^2)}$	$\cos(ax)$
7	$\frac{\alpha\gamma}{(\beta^2-a^2\gamma^2)}$	$\frac{\sinh(ax)}{a}$
8	$\frac{\alpha\beta}{(\beta^2-a^2\gamma^2)}$	$\cosh(ax)$

2.7.1 $\mathcal{U}\left\{\theta'(x)\right\} = \left(\frac{\beta}{\gamma}\right)\mathcal{T}(\alpha,\beta,\gamma) - \left(\frac{\alpha}{\gamma}\right)\theta(0)$

2.7.2 $\mathcal{U}\left\{\theta''(x)\right\} = \left(\frac{\beta}{\gamma}\right)^2\mathcal{T}(\alpha,\beta,\gamma) - \left(\frac{\alpha\beta}{\gamma^2}\right)\theta(0) - \left(\frac{\alpha}{\gamma}\right)\theta'(0)$

2.7.3 $\mathcal{U}\left\{\theta'''(x)\right\} = \left(\frac{\beta}{\gamma}\right)^3\mathcal{T}(\alpha,\beta,\gamma) - \left(\frac{\alpha\beta^2}{\gamma^3}\right)\theta(0) - \left(\frac{\alpha\beta}{\gamma^2}\right)\theta'(0) - \left(\frac{\alpha}{\gamma}\right)\theta''(0)$

2.7.4 $\mathcal{U}\left\{\theta^n(x)\right\} = \left(\frac{\beta}{\gamma}\right)^n\mathcal{T}(\alpha,\beta,\gamma) - \left(\frac{\alpha\beta^{n-1}}{\gamma^n}\right)\theta(0) - \left(\frac{\alpha\beta^{n-2}}{\gamma^{n-1}}\right)\theta'(0) - \left(\frac{\alpha\beta^{n-3}}{\gamma^{n-2}}\right)\theta''(0) - \ldots - \left(\frac{\alpha}{\gamma}\right)\theta^{(n-1)}(0).$

3 The Upadhyaya decomposition method for the solution of the linear Volterra integro-differential equation of the second kind

The linear Volterra integro-differential equation of the second kind is given by [1]

$$\theta^n(x) = h(x) + \lambda\int_0^x K(x,t)\theta(t)\,dt \tag{3.1}$$

where

$$\left.\begin{array}{c} \theta(x) = unknown\ function \\ h(x) = known\ function \\ \lambda = non-zero\ real\ number \\ K(x,t) = kernel \\ \theta^n(x) = n^{th}\ derivative\ of\ \theta(x)\ with\ respect\ to\ x \end{array}\right\}$$

For determining the particular solution of equation (3.1), it is necessary to define the initial conditions $\theta(0), \theta'(0), \theta''(0), \ldots, \theta^{(n-1)}(0)$.

In this work, we will assume that the kernel $K(x,t)$ of equation (3.1) is a difference kernel that can be expressed as $K(x,t) = K(x-t)$. Putting this in equation (3.1), it becomes

$$\theta^n(x) = h(x) + \lambda\int_0^x K(x-t)\theta(t)\,dt \tag{3.2}$$

Consider the initial condition as

$$\left.\begin{array}{c} \theta(0) = k_0 \\ \theta'(0) = k_1 \\ \theta''(0) = k_2 \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \theta^{(n-1)}(0) = k_{n-1} \end{array}\right\} \tag{3.3}$$

Applying the Upadhyaya transform on both sides of equation (3.2), we get

$$\mathcal{U}\left\{\theta^n(x)\right\} = \mathcal{U}\left\{h(x)\right\} + \lambda\mathcal{U}\left\{\int_0^x K(x-t)\theta(t)\,dt\right\}$$

$$\begin{aligned} \Rightarrow \mathcal{U}\left\{\theta(x)\right\} = {} & \left(\frac{\alpha}{\beta}\right)\theta(0) + \left(\frac{\alpha\gamma}{\beta^2}\right)\theta'(0) + \left(\frac{\alpha\gamma^2}{\beta^3}\right)\theta''(0) + \ldots + \left(\frac{\alpha\gamma^{n-1}}{\beta^n}\right)\theta^{(n-1)}(0) \\ & + \left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\left\{h(x)\right\} + \lambda\left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\left\{\int_0^x K(x-t)\,\theta(t)\,dt\right\} \end{aligned} \tag{3.4}$$

Using (3.3) in (3.4), we have

$$\begin{aligned} \mathcal{U}\left\{\theta(x)\right\} = {} & \left(\frac{\alpha}{\beta}\right)k_0 + \left(\frac{\alpha\gamma}{\beta^2}\right)k_1 + \left(\frac{\alpha\gamma^2}{\beta^3}\right)k_2 + \ldots + \left(\frac{\alpha\gamma^{n-1}}{\beta^n}\right)k_{n-1} + \left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\left\{h(x)\right\} \\ & + \lambda\left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\left\{\int_0^x K(x-t)\theta(t)\,dt\right\} \end{aligned} \tag{3.5}$$

Using the convolution theorem of the Upadhyaya transform in (3.5), we get

$$\begin{aligned} \mathcal{U}\left\{\theta(x)\right\} = {} & \left(\frac{\alpha}{\beta}\right)k_0 + \left(\frac{\alpha\gamma}{\beta^2}\right)k_1 + \left(\frac{\alpha\gamma^2}{\beta^3}\right)k_2 + \ldots + \left(\frac{\alpha\gamma^{n-1}}{\beta^n}\right)k_{n-1} + \left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\left\{h(x)\right\} \\ & + \lambda\left(\frac{\gamma^n}{\beta^n}\right)\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\left\{K(x)\right\}\mathcal{U}\left\{\theta(x)\right\} \end{aligned} \tag{3.6}$$

Operating by the inverse Upadhyaya transform operator on both sides of (3.6), we get

$$\theta(x) = k_0 + k_1 x + k_2 \left(\frac{x^2}{2}\right) + \ldots + k_{n-1}\left(\frac{x^{n-1}}{(n-1)!}\right) + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\{h(x)\}\right\} + \lambda \mathcal{U}^{-1}\left\{\left(\frac{\gamma^{n+1}}{\alpha\beta^n}\right)\mathcal{U}\{K(x)\}\mathcal{U}\{\theta(x)\}\right\} \tag{3.7}$$

The Upadhyaya decomposition method assumes the solution into infinite series as

$$\theta(x) = \sum_{i=0}^{\infty} \theta_i(x) \tag{3.8}$$

Using equation (3.8) into equation (3.7), we have

$$\sum_{i=0}^{\infty} \theta_i(x) = \left\{k_0 + k_1 x + k_2\left(\frac{x^2}{2}\right) + \ldots + k_{n-1}\left(\frac{x^{n-1}}{(n-1)!}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\{h(x)\}\right\} + \lambda\mathcal{U}^{-1}\left\{\left(\frac{\gamma^{n+1}}{\alpha\beta^n}\right)\mathcal{U}\{K(x)\}\mathcal{U}\left\{\sum_{i=0}^{\infty}\theta_i(x)\right\}\right\}$$

In general, the recursive relation for the required solution is given by

$$\left.\begin{array}{r}\theta_{i+1}(x) = \lambda\mathcal{U}^{-1}\left\{\left(\frac{\gamma^{n+1}}{\alpha\beta^n}\right)\mathcal{U}\{K(x)\}\mathcal{U}\{\theta_i(x)\}\right\},\ i \geqslant 0 \\ \text{with }\ \theta_0(x) = \left\{k_0 + k_1 x + k_2\left(\frac{x^2}{2}\right) + \ldots + k_{n-1}\left(\frac{x^{n-1}}{(n-1)!}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^n}{\beta^n}\right)\mathcal{U}\{h(x)\}\right\}\end{array}\right\}$$

4 Numerical Applications

In this section we solve some numerical problems by using the Upadhyaya decomposition method discussed above in section 3.

Problem 4.1. Consider the following second kind linear Volterra integro-differential equation

$$\theta'(x) = 2 + x - \frac{x^3}{3!} + \int_0^x (x-t)\theta(t)\,dt \tag{4.1}$$

with

$$\theta(0) = 1. \tag{4.2}$$

Solution: Taking the Upadhyaya transform of both sides of (4.1), we get

$$\mathcal{U}\left\{\theta'(x)\right\} = 2\mathcal{U}\{1\} + \mathcal{U}\{x\} - \frac{1}{3!}\mathcal{U}\{x^3\} + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\}$$

$$\Rightarrow \left(\frac{\beta}{\gamma}\right)\mathcal{U}\{\theta(x)\} - \left(\frac{\alpha}{\gamma}\right)\theta(0) = 2\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) - \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\} \tag{4.3}$$

Using (4.2) in (4.3), we have

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left(\frac{\alpha}{\beta}\right) + 2\left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) - \left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\gamma}{\beta}\right)\mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\} \tag{4.4}$$

Using the convolution theorem of the Upadhyaya transform in (4.4), we get

$$\mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + 2\left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) - \left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\gamma}{\beta}\right)\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{x\}\mathcal{U}\{\theta(x)\}\right]$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + 2\left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) - \left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\gamma}{\beta}\right)\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha\gamma}{\beta^2}\right)\mathcal{U}\{\theta(x)\}\right]$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + 2\left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) - \left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\{\theta(x)\}\right] \tag{4.5}$$

Operating by the inverse Upadhyaya transform on both sides of (4.5), we get

$$\theta(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\{\theta(x)\}\right\} \tag{4.6}$$

The Upadhyaya decomposition method assumes the solution into the form of an infinite series as

$$\theta(x) = \sum_{i=0}^{\infty}\theta_i(x) \tag{4.7}$$

Using (4.7) into (4.6), we obtain

$$\sum_{i=0}^{\infty}\theta_i(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\left\{\sum_{i=0}^{\infty}\theta_i(x)\right\}\right\}$$

From the above equation, the recursive relation for the required solution is given by

$$\left.\begin{array}{c}\theta_{i+1}(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\{\theta_i(x)\}\right\},\ i \geq 0 \\ \text{with } \theta_0(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!}\end{array}\right\}$$

Using the above recursive relation, the first few components of $\theta_i(x)$ are given as

$$\begin{aligned}
\theta_1(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\{\theta_0(x)\}\right\} \\
\Rightarrow \theta_1(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\left\{1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!}\right\}\right\} \\
\Rightarrow \theta_1(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\left[\mathcal{U}\{1\} + 2\mathcal{U}\{x\} + \frac{1}{2}\mathcal{U}\{x^2\} - \frac{1}{4!}\mathcal{U}\{x^4\}\right]\right\} \\
\Rightarrow \theta_1(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\left[\left(\frac{\alpha}{\beta}\right) + 2\left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) - \left(\frac{\alpha\gamma^4}{\beta^5}\right)\right]\right\} \\
\Rightarrow \theta_1(x) &= \left[\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right\} + 2\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^4}{\beta^5}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^5}{\beta^6}\right)\right\} - \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^7}{\beta^8}\right)\right\}\right] \\
\Rightarrow \theta_1(x) &= \left(\frac{x^3}{3!}\right) + 2\left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) - \left(\frac{x^7}{7!}\right) \qquad (4.8) \\
\theta_2(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\{\theta_1(x)\}\right\} \\
\Rightarrow \theta_2(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\left\{\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) - \left(\frac{x^7}{7!}\right)\right\}\right\} \\
\Rightarrow \theta_2(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\left[\left(\frac{1}{3!}\right)\mathcal{U}\{x^3\} + 2\left(\frac{1}{4!}\right)\mathcal{U}\{x^4\} + \left(\frac{1}{5!}\right)\mathcal{U}\{x^5\} - \frac{1}{7!}\mathcal{U}\{x^7\}\right]\right\} \\
\Rightarrow \theta_2(x) &= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^3}{\beta^3}\right)\left[\left(\frac{\alpha\gamma^3}{\beta^4}\right) + 2\left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\alpha\gamma^5}{\beta^6}\right) - \left(\frac{\alpha\gamma^7}{\beta^8}\right)\right]\right\} \\
\Rightarrow \theta_2(x) &= \left[\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^6}{\beta^7}\right)\right\} + 2\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^7}{\beta^8}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^8}{\beta^9}\right)\right\} - \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{10}}{\beta^{11}}\right)\right\}\right] \\
\Rightarrow \theta_2(x) &= \left(\frac{x^6}{6!}\right) + 2\left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) - \left(\frac{x^{10}}{10!}\right) \qquad (4.9)
\end{aligned}$$

Using (4.7), the required solution of (4.1) with (4.2) is given by

$$\theta(x) = x + \left[1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) + \ldots\right]$$

that converges to the exact solution $\theta(x) = x + e^x$.

Problem 4.2. Consider the following second kind linear Volterra integro-differential equation

$$\theta^{''}(x) = 1 + x + \int_0^x (x-t)\theta(t)\,dt \tag{4.10}$$

with

$$\theta(0) = 1, \theta^{'}(0) = 1. \tag{4.11}$$

Solution: Applying the Upadhyaya transform on both the sides of (4.10), we get

$$\mathcal{U}\left\{\theta^{''}(x)\right\} = \mathcal{U}\{1\} + \mathcal{U}\{x\} + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\}$$

$$\Rightarrow \left(\frac{\beta}{\gamma}\right)^2 \mathcal{U}\{\theta(x)\} - \left(\frac{\alpha\beta}{\gamma^2}\right)\theta(0) - \left(\frac{\alpha}{\gamma}\right)\theta^{'}(0) = \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\} \tag{4.12}$$

Using (4.11) in (4.12), we have

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\gamma^2}{\beta^2}\right)\mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\} \tag{4.13}$$

Invoking the convolution theorem of the Upadhyaya transform in (4.13), we get

$$\mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\gamma^2}{\beta^2}\right)\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{x\}\mathcal{U}\{\theta(x)\}\right]$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\gamma^2}{\beta^2}\right)\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha\gamma}{\beta^2}\right)\mathcal{U}\{\theta(x)\}\right]$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = \left[\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\{\theta(x)\}\right] \tag{4.14}$$

Operating now by the inverse Upadhyaya transform on both sides of (4.14), we get

$$\theta(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\{\theta(x)\}\right\} \tag{4.15}$$

The Upadhyaya decomposition method assumes the solution into the form of an infinite series as

$$\theta(x) = \sum_{i=0}^{\infty}\theta_i(x) \tag{4.16}$$

Substituting (4.16) into (4.15), we have

$$\sum_{i=0}^{\infty}\theta_i(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\left\{\sum_{i=0}^{\infty}\theta_i(x)\right\}\right\}$$

From the above equation, the recursive relation for the required solution is given by

$$\left.\begin{array}{c}\theta_{i+1}(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\{\theta_i(x)\}\right\},\ i \geq 0 \\ \text{with } \theta_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\end{array}\right\}.$$

Using the above recursive relations, the first few components of $\theta_i(x)$ are given as

$$\theta_1(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\{\theta_0(x)\}\right\} = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right\}\right\}$$

$$\Rightarrow \theta_1(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\left[\mathcal{U}\{1\} + \mathcal{U}\{x\} + \frac{1}{2!}\mathcal{U}\{x^2\} + \frac{1}{3!}\mathcal{U}\{x^3\}\right]\right\}$$

$$\Rightarrow \theta_1(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\left[\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right)\right]\right\}$$

$$\Rightarrow \theta_1(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^4}{\beta^5}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^5}{\beta^6}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^6}{\beta^7}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^7}{\beta^8}\right)\right\}\right]$$

$$\Rightarrow \theta_1(x) = \left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) \tag{4.17}$$

$$\theta_2(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\{\theta_1(x)\}\right\}$$

$$= \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\mathcal{U}\left\{\left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right)\right\}\right\}$$

$$\Rightarrow \theta_2(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\left[\left(\frac{1}{4!}\right)\mathcal{U}\{x^4\} + \left(\frac{1}{5!}\right)\mathcal{U}\{x^5\} + \left(\frac{1}{6!}\right)\mathcal{U}\{x^6\} + \frac{1}{7!}\mathcal{U}\{x^7\}\right]\right\}\right]$$

$$\Rightarrow \theta_2(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^4}{\beta^4}\right)\left[\left(\frac{\alpha\gamma^4}{\beta^5}\right) + \left(\frac{\alpha\gamma^5}{\beta^6}\right) + \left(\frac{\alpha\gamma^6}{\beta^7}\right) + \left(\frac{\alpha\gamma^7}{\beta^8}\right)\right]\right\}$$

$$\Rightarrow \theta_2(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^8}{\beta^9}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^9}{\beta^{10}}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{10}}{\beta^{11}}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{11}}{\beta^{12}}\right)\right\}\right]$$

$$\Rightarrow \theta_2(x) = \left(\frac{x^8}{8!}\right) + \left(\frac{x^9}{9!}\right) + \left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{11}}{11!}\right) \tag{4.18}$$

Using (4.16), the required solution of (4.10) with (4.11) is given by

$$\theta(x) = 1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^9}{9!}\right) + \left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{11}}{11!}\right) + \dots$$

that converges to the exact solution $\theta(x) = e^x$.

Problem 4.3. Consider the following second kind linear Volterra integro-differential equation

$$\theta'''(x) = 1 + x - 2x^2 + \int_0^x (x-t)\theta(t)\,dt \tag{4.19}$$

with

$$\theta(0) = 5, \theta'(0) = 1, \theta''(0) = 1 \tag{4.20}$$

Solution: Applying Upadhyaya transform on both sides of equation (4.19), we get

$$\mathcal{U}\left\{\theta'''(x)\right\} = \mathcal{U}\{1\} + \mathcal{U}\{x\} - 2\mathcal{U}\{x^2\} + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\}$$

$$\begin{aligned}\Rightarrow \left(\frac{\beta}{\gamma}\right)^3 \mathcal{U}\{\theta(x)\} - \left(\frac{\alpha\beta^2}{\gamma^3}\right)\theta(0) - \left(\frac{\alpha\beta}{\gamma^2}\right)\theta'(0) - \left(\frac{\alpha}{\gamma}\right)\theta''(0) \\ = \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) - 2\left(\frac{\alpha\gamma^2}{\beta^3}\right) + \mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\}\end{aligned} \tag{4.21}$$

Using (4.20) in (4.21), we have

$$\begin{aligned}\Rightarrow \mathcal{U}\{\theta(x)\} = 5\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\alpha\gamma^4}{\beta^5}\right) - 2\left(\frac{\alpha\gamma^5}{\beta^6}\right) \\ + \left(\frac{\gamma^3}{\beta^3}\right)\mathcal{U}\left\{\int_0^x (x-t)\theta(t)\,dt\right\}\end{aligned} \tag{4.22}$$

Using the convolution theorem of Upadhyaya transform in (4.22), we get

$$\begin{aligned}\mathcal{U}\{\theta(x)\} = 5\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\alpha\gamma^4}{\beta^5}\right) - 2\left(\frac{\alpha\gamma^5}{\beta^6}\right) \\ + \left(\frac{\gamma^3}{\beta^3}\right)\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{x\}\mathcal{U}\{\theta(x)\}\end{aligned}$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = 5\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\alpha\gamma^4}{\beta^5}\right) - 2\left(\frac{\alpha\gamma^5}{\beta^6}\right) + \left(\frac{\gamma^3}{\beta^3}\right)\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha\gamma}{\beta^2}\right)\mathcal{U}\{\theta(x)\}$$

$$\Rightarrow \mathcal{U}\{\theta(x)\} = 5\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\alpha\gamma^4}{\beta^5}\right) - 2\left(\frac{\alpha\gamma^5}{\beta^6}\right) + \left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\{\theta(x)\} \tag{4.23}$$

Applying the inverse Upadhyaya transform operator on both sides of (4.23), we get

$$\theta(x) = 5 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\{\theta(x)\}\right\} \tag{4.24}$$

The Upadhyaya decomposition method assumes the solution to be in the form of an infinite series as

$$\theta(x) = \sum_{i=0}^{\infty}\theta_i(x) \tag{4.25}$$

Using (4.25) into (4.24), we obtain

$$\sum_{i=0}^{\infty}\theta_i(x) = \left[5 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\left\{\sum_{i=0}^{\infty}\theta_i(x)\right\}\right\}\right].$$

From the above equation, the recursive relation for the required solution is given by

$$\left.\begin{array}{l}\theta_{i+1}(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\{\theta_i(x)\}\right\},\ i \geq 0 \\ \text{with } \theta_0(x) = 5 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!}\end{array}\right\}.$$

Using the above recursive relation, the first few components of $\theta_i(x)$ are given by

$$\theta_1(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\{\theta_0(x)\}\right\}$$

$$\Rightarrow \theta_1(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\left\{5 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!}\right\}\right\}$$

$$\Rightarrow \theta_1(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\left[5\mathcal{U}\{1\} + \mathcal{U}\{x\} + \frac{1}{2!}\mathcal{U}\{x^2\} + \frac{1}{3!}\mathcal{U}\{x^3\} + \frac{1}{4!}\mathcal{U}\{x^4\} - \frac{4}{5!}\mathcal{U}\{x^5\}\right]\right\}\right]$$

$$\Rightarrow \theta_1(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\left[5\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha\gamma}{\beta^2}\right) + \left(\frac{\alpha\gamma^2}{\beta^3}\right) + \left(\frac{\alpha\gamma^3}{\beta^4}\right) + \left(\frac{\alpha\gamma^4}{\beta^5}\right) - 4\left(\frac{\alpha\gamma^5}{\beta^6}\right)\right]\right\}\right]$$

$$\Rightarrow \theta_1(x) = 5\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^5}{\beta^6}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^6}{\beta^7}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^7}{\beta^8}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^8}{\beta^9}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^9}{\beta^{10}}\right)\right\} - 4\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{10}}{\beta^{11}}\right)\right\}$$

$$\Rightarrow \theta_1(x) = 5\left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^9}{9!}\right) - 4\left(\frac{x^{10}}{10!}\right) \tag{4.26}$$

$$\theta_2(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\{\theta_1(x)\}\right\}$$

$$\Rightarrow \theta_2(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\mathcal{U}\left\{5\left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^9}{9!}\right) - 4\left(\frac{x^{10}}{10!}\right)\right\}\right\}\right]$$

$$\Rightarrow \theta_2(x) = \mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\left[\left(\frac{5}{5!}\right)\mathcal{U}\{x^5\} + \left(\frac{1}{6!}\right)\mathcal{U}\{x^6\} + \left(\frac{1}{7!}\right)\mathcal{U}\{x^7\} + \frac{1}{8!}\mathcal{U}\{x^8\} + \frac{1}{9!}\mathcal{U}\{x^9\} - \left(\frac{4}{10!}\right)\mathcal{U}\{x^{10}\}\right]\right\}$$

$$\Rightarrow \theta_2(x) = \left[\mathcal{U}^{-1}\left\{\left(\frac{\gamma^5}{\beta^5}\right)\left[5\left(\frac{\alpha\gamma^5}{\beta^6}\right) + \left(\frac{\alpha\gamma^6}{\beta^7}\right) + \left(\frac{\alpha\gamma^7}{\beta^8}\right) + \left(\frac{\alpha\gamma^8}{\beta^9}\right) + \left(\frac{\alpha\gamma^9}{\beta^{10}}\right) - 4\left(\frac{\alpha\gamma^{10}}{\beta^{11}}\right)\right]\right\}\right]$$

$$\Rightarrow \theta_2(x) = 5\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{10}}{\beta^{11}}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{11}}{\beta^{12}}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{12}}{\beta^{13}}\right)\right\} + \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{13}}{\beta^{14}}\right)\right\}$$
$$+ \mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^{14}}{\beta^{15}}\right)\right\} - 4\mathcal{U}^{-1}\left\{\frac{\alpha\gamma^{15}}{\beta^{16}}\right\}$$
$$\Rightarrow \theta_2(x) = 5\left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{11}}{11!}\right) + \left(\frac{x^{12}}{12!}\right) + \left(\frac{x^{13}}{13!}\right) + \left(\frac{x^{14}}{14!}\right) - 4\left(\frac{x^{15}}{15!}\right) \tag{4.27}$$

Using (4.25), the required solution of (4.19) with (4.20) is given by

$$\theta(x) = 4 + \left\{1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \left(\frac{x^4}{4!}\right) + \left(\frac{x^5}{5!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^7}{7!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^9}{9!}\right) + \left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{11}}{11!}\right) + \ldots\right\},$$

that converges to the exact solution $\theta(x) = 4 + e^x$.

5 Conclusion

This study effectively determines the solution to the linear Volterra integro-differential equations of the second kind by using the Upadhyaya decomposition method. The solutions to the problems under consideration show that the Upadhyaya decomposition approach can solve the linear Volterra integro-differential equations of the second kind quickly and with lesser computational efforts. In future we propose to solve the system of simultaneous linear Volterra integro-differential equations by employing the Upadhyaya decomposition method.

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