Original Article

Available online at
www.bpasjournals.com

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.320-323
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00032.8

FUZZY INDEPENDENT SETS AND FUZZY COVERING

V.T. Chandrasekaran[1,*], R. Sangeetha[2]

Authors Affiliation:
[1]Associate Professor, Department of Mathematics, Jawahar Science College, Neyveli, Tamil Nadu 607803, India.
E-mail: vtcvtc14@gmail.com
[2]Assistant Professor, Department of Mathematics, Jawahar Science College, Neyveli, Tamil Nadu 607803, India.
E-mail: sangeethashriram2015@gmail.com

***Corresponding Author:**
V.T. Chandrasekaran, Associate Professor, Department of Mathematics, Jawahar Science College, Neyveli, Tamil Nadu 607803, India.

E-mail: vtcvtc14@gmail.com

Received on 20.01.2019 Revised on 30.05.2019 Accepted on 02.06.2019

Abstract:
In this paper, we define fuzzy vertex independent sets, fuzzy vertex covering, fuzzy arc independent set, fuzzy arc covering of a fuzzy graph. Some properties of maximum fuzzy independent set and minimum fuzzy covering are discussed.

Keywords: Fuzzy vertex independent set, fuzzy vertex independence number, fuzzy vertex covering, fuzzy vertex covering number, fuzzy arc independent set, fuzzy arc independence number, fuzzy arc covering, fuzzy arc covering number, saturated fuzzy vertex, M -alternating paths, M -augmenting path.

2010 Mathematics Subject Classification: 05C72.

1. INTRODUCTION

Bhutani and Rosenfeld [1] introduced the notion of fuzzy graph and several fuzzy analogues of graph theoretic concepts such as paths, cycles and connectedness. Nagoorgani and Chandrasekaran [2] discussed fuzzy independent sets and fuzzy bipartite graph in fuzzy graphs. Bondy and Murthy [3] have given a brief idea about saturation of a vertex using matching in crisp graphs. In this paper we define the concepts of fuzzy vertex covering and fuzzy arc independent sets. We also discuss the relation between the fuzzy vertex covering and fuzzy vertex independent set, fuzzy arc independent set and fuzzy vertex covering.

2. PRELIMINARIES

We summarize some basic definitions in fuzzy graphs. A fuzzy set of a nonempty set V is a mapping $\sigma : V \to [0,1]$. A fuzzy relation on V is a fuzzy subset of $V \times V$. If μ and λ are fuzzy relations, then $(\mu \circ \lambda)(u,w) = \sup\{\mu(u,v) \wedge \lambda(v,w) : v \in V\}$ where $\wedge$ means "inf". The powers of fuzzy relations are defined as $\mu^1 = \mu, \mu^2 = \mu \circ \mu, \mu^3 = \mu \circ \mu^2$ and so on. It is also defined that $\mu^\infty = \sup_{k=1,2,3...} \mu^k$. A fuzzy graph $G(\sigma, \mu)$ is a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where for all u, v in V, we have $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$. The fuzzy graph $H(\tau,\eta)$ is called a fuzzy sub graph of $G(\sigma,\mu)$ if $\tau(u) \le \sigma(u)$ for all $u \in V$ and $\eta(u,v) \le \mu(u,v)$ for all $u, v \in V$ [4]. A path ρ in a fuzzy graph G is a sequence of distinct vertices $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0, 1 \le i \le n$; here $n \ge 0$ is called the length of the path ρ. The consecutive pairs (u_{i-1}, u_i) are called the arcs of the path. The strength of a path is defined as $\wedge_{i=1}^{n} \mu(u_{i-1}, u_i)$. In other words, the strength of a path is defined to be the weight of the weakest arc of the path. Two vertices which are joined by a path are said to be connected. Clearly u and v is connected iff $\mu^{\infty}(u,v) > 0$. A strongest path joining any two vertices u and v has strength $\mu^{\infty}(u,v)$. We shall sometimes refer to this as the strength of connectedness between the vertices. The underlying crisp graph of the fuzzy graph $G(\sigma,\mu)$ is denoted by the pair of set $G^* : (\sigma^*, \mu^*)$ where, $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$. Throughout this paper we assume that $G(\sigma,\mu)$ is a finite fuzzy graph, i.e., σ^* is finite and by G, we mean the fuzzy graph $G(\sigma,\mu)$.

3. BASIC DEFINITIONS

Let us recollect the definition of fuzzy subset of a fuzzy set. The function α is said to be a fuzzy subset of σ if $\alpha, \sigma : V \to [0,1]$ and $\alpha(x) \le \sigma(x)\ \forall x \in X$ [4]. Further α is said to be a proper fuzzy subset of σ if $\alpha(x) < \sigma(x)$, when $\sigma(x) > 0$. Also the cardinality of a fuzzy set σ is defined as $|\sigma| = \sum \sigma(u)$ for all $u \in V$. An arc (u,v) is said to be strong if $\mu(u,v) \ge \mu^{\infty}(u,v)$. The strong neighborhood of u is $N_s(u) = \{v \in \sigma^* : (u,v)$ is a strong arc$\}$. We use the notation, $u \in \sigma$, we mean that $\sigma(u) > 0$ and otherwise $u \notin \sigma$. A fuzzy vertex $u \in G(\sigma,\mu)$ is said to be an effective vertex if $\sigma(u) = \max_{v \in N_s(u)} \mu(u,v)$. A fuzzy arc $(u,v) \in G(\sigma,\mu)$ is said to be an effective arc if $\min(\sigma(u), \sigma(v)) = \mu(u,v)$. A fuzzy graph G having every arc as an effective arc is called an effective fuzzy graph. A fuzzy graph G having every vertex as an effective vertex is called a vertex effective fuzzy graph. A fuzzy graph G is fuzzy bipartite if the vertex set σ can be partitioned into two nonempty sets σ_1 and σ_2 such that σ_1 and σ_2 are fuzzy independent sets. These σ_1 and σ_2 are called fuzzy bipartitions of σ. Thus every strong arc of G has one end in σ_1 and other end in σ_2. A fuzzy bipartite graph G with fuzzy bipartition σ_1 and σ_2 is said to be a complete fuzzy bipartite graph if for each vertex of σ_1, every vertex of σ_2 is a strong neighbor.

4. FUZZY INDEPENDENT SETS AND FUZZY COVERING

Definition 4.1:
Two fuzzy vertices of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them. A fuzzy subset S of σ is said to be fuzzy vertex independent set of G if any two fuzzy vertices of S are fuzzy independent. A fuzzy vertex independent set S is called maximal fuzzy independent set if no superset of S is a fuzzy vertex independent set. The cardinality of any maximum fuzzy independent set of G is called its fuzzy vertex independence number and is denoted by ξ.

Definition 4.2:
A fuzzy subset K of σ such that every strong arc of G has at least one end in K is called a fuzzy vertex covering of G. A fuzzy vertex covering K of G is called a minimal fuzzy vertex covering if no fuzzy subset of K is a fuzzy vertex covering. The cardinality of any minimum fuzzy vertex covering of G is called its fuzzy vertex covering number and is denoted by β **.**

Definition 4.3:
Two strong arcs are fuzzy independent if there is no common vertex between them. A strong arc subset M of μ is said to be fuzzy arc independent set of G if any two strong arcs of M are fuzzy independent. A fuzzy arc independent set M is called maximal fuzzy arc independent set if no superset of M is a fuzzy arc independent set. The cardinality of any maximum fuzzy arc independent set of G is called its fuzzy arc independence number and is denoted by η.

Definition 4.4:
A fuzzy arc covering of G is a fuzzy subset C of μ such that each vertex of G is an end of some strong arc in C. A fuzzy arc covering C is called a minimal fuzzy arc covering if no proper subset of C is a fuzzy arc covering.
Note: For any fuzzy graph with pendent vertices, fuzzy arc covering number cannot be determined.

Definition 4.5:
A fuzzy arc independent set M saturates a fuzzy vertex v or v is said to be saturated, if some strong arc of M is incident with v, otherwise v is called as unsaturated.

Definition 4.6:
Let M be a fuzzy arc independent set of G. An M-alternating path in G is a path whose strong arcs are alternately in $\mu \setminus \mathrm{M}$ ($\mu \setminus \mathrm{M} = (\mu^* - M^*) \wedge \mu$) and M. An M-augmenting path is an M-alternating path whose origin and terminus are M-unsaturated.

5. MAIN RESULTS

Theorem 5.1:
A fuzzy set $S \subseteq \sigma$ is a fuzzy vertex independent set of G iff $\sigma \setminus \mathrm{S}$ is a fuzzy vertex covering of G where $\sigma \setminus \mathrm{S} = (\sigma^* - S^*) \wedge \sigma.$
Proof. By definition, S is a fuzzy vertex independent set of G iff no strong arc of G has both ends in S, equivalently, iff each strong arc has at least one end is $\sigma \setminus \mathrm{S}$. But this is so iff $\sigma \setminus \mathrm{S}$ is a fuzzy covering of G.

Corollary 5.2:
If G is an effective fuzzy graph, then $\xi + \beta = |\sigma|$.
Proof. Let S be a maximum fuzzy vertex independent set of G and K be a minimum fuzzy vertex covering of G. Then by theorem (5.1), $\sigma \setminus \mathrm{S}$ is a fuzzy vertex covering and $\sigma \setminus \mathrm{K}$ is a fuzzy vertex independent set. Now, $|\sigma| - \xi = |\sigma \setminus \mathrm{S}| \le \beta$ and $|\sigma| - \beta = |\sigma \setminus \mathrm{K}| \ge \xi$ together imply that $\xi + \beta = |\sigma|$.

Theorem 5.3:
Let G be an effective and complete fuzzy bipartite graph, then $\xi(G) = \max\{|\sigma_1|, |\sigma_2|\}$ where σ_1 and σ_2 are fuzzy bipartition of G.
Proof. Since G be a complete fuzzy bipartite graph with bipartitions σ_1 and σ_2, both are fuzzy vertex independent sets, this completes the proof.

Lemma 5.4:
Let M be a fuzzy arc independent set and K be a fuzzy vertex covering such that $|M| = |K|$. Then M is a maximum fuzzy arc independent set and K is a minimum fuzzy vertex covering.
Proof. If M' be a maximum fuzzy arc independent set and K' be a minimum fuzzy vertex covering then $|M| \le |\mathrm{M}'| \le |K'| \le |K|$. Since $|M| = |K|$, it follows that $|K| = |K'|$ and $|M| = |M'|$.

Theorem 5.5:
Let G be an effective fuzzy bipartite graph. Suppose $M = \{M_1, M_2, \ldots, M_l\}$ be a maximum fuzzy arc independent set and $K = \{K_1, K_2, \ldots K_m\}$ be a minimum fuzzy vertex covering, then $\beta = \eta + \sum [\sigma(K_i) - \mu(M_j)]$ for all i, where the fuzzy arc M_j incidents on the fuzzy vertex K_i.
Proof: Let G be an effective fuzzy bipartite graph and η, β be the fuzzy arc independence number and fuzzy vertex covering number respectively.

Since K is a fuzzy vertex covering, each M_i of M has at least one of its ends in K. Consider K_1 with M_j for some j, which incident on K_1. Since G is an effective bipartite graph, then either $\sigma(K_1) = \mu(M_j)$ or $\sigma(K_1) > \mu(M_j)$. If $\sigma(K_1) = \mu(M_j)$ then $\sigma(K_1) = \mu(M_j) + 0$ and if $\sigma(K_1) >$ $\mu(M_j)$, then

$\sigma(K_1) = \mu(M_j) + [\sigma(K_1) - \mu(M_j)]$. This is true for all other fuzzy vertices $i = 2,3,\ldots,m.$ Taking sum on both sides, we have

$\sum_{i=1}^{m} \sigma(K_i) = \sum \mu(M_j) + \sum [\sigma(K_i) - \mu(M_j)]$ where, M_j are incident on K_i.

Thus $\beta = \eta + \sum [\sigma(K_i) - \mu(M_j)]$ for all i, thus proving the theorem.

6. NUMERICAL EXAMPLE

Let $V = \{u_1, u_2, u_3, u_4, u_5\}$ and σ on V is defined as $\sigma(u_1)$ = 0.8, $\sigma(u_2)$ = 0.8, $\sigma(u_3)$ = 0.3, $\sigma(u_4)$ = 0.6, $\sigma(u_5)$ = 0.9, and μ on $V \times V$ is defined as $\mu(u_1, u_2)$ = 0.6, $\mu(u_2, u_3)$ = 0.2, $\mu(u_1, u_4)$ = 0.6, $\mu(u_2, u_5)$ = 0.7, $\mu(u_4, u_5)$ = 0.5. Then $G(\sigma, \mu)$ is a fuzzy graph on V.

Then the fuzzy subset { $u_1(0.8)$, $u_2(0.8), u_5(0.9)$ }is a fuzzy vertex covering. The fuzzy subset { $u_1(0.6), u_2(0.7), u_5(0.7)$ } is a minimal fuzzy vertex covering. The fuzzy subset { $u_2(0.7), u_4(0.6)$ } is a minimum fuzzy vertex covering. Also, if β = 1.3 then a fuzzy subset M = { $(u_1, u_4)(0.6)$, $(u_2, u_5)(0.7)$} is a fuzzy arc independent set. The fuzzy vertices in the fuzzy set { $u_1(0.8)$, $u_2(0.8)$, $u_4(0.6)$, $u_5(0.9)$} are M - saturated fuzzy vertices.

REFERENCES

[1]. Bhutani, K.R. and Rosenfeld, A. (2003). Strong arcs in fuzzy graphs, *Information Sciences*, Vol. 152, 319-322.

[2]. Nagoorgani, A. and Chandrasekaran, V.T. (2010) Fuzzy Graph Theory, Allied Publishers, New Delhi.

[3]. Bondy, J.A. and Murthy, U.S.R. (1976). Graph theory with Applications, MacMillan Press Ltd., London and Basingstoke.

[4]. Chandrasekaran, V.T. and Sangeetha, R. (2018). Fuzzy Dominating Set, *International Journal of Mathematics and its Application,* Vol. 6, 301-305.