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Extended Horn's hypergeometric function $\mathbf{H_{11}}$ *

M. S. Metwally[1], S. Abo-Hasha[2] and Karima Hamza[3,†]

1. Department of Mathematics, Faculty of Science (Suez),
Suez University, Egypt.
2,3. Department of Mathematics, Faculty of Science,
South Valley University, Qena, Egypt.
1. E-mail: met641958@yahoo.com , 2. E-mail: dr.shadyhasha@gmail.com
3. E-mail: karimahamza767@gmail.com

Abstract In this paper we introduce an extension of the Horn's hypergeometric function $\mathbf{H_{11}}$. Furthermore, we investigate the limit formulas, integral representations, differentiation formulas, infinite sums, recursion formulas, Laplace, Mellin and fractional Fourier transforms for the extended Horn's hypergeometric function $\mathbf{H_{11}}$. Finally, we discuss double Laplace and double Mellin transforms of this function.

Key words Horn's hypergeometric function $\mathbf{H_{11}}$, the generalization of the Pochhammer symbol, limit formulas, recursion formulas, Laplace transform, Mellin transform, Fourier transform, Upadhyaya transform.

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1 Introduction

Recently, special functions have become an active factor for the development of many scientific fields, so most mathematicians have obtained different expansions of some special functions by adding some extra parameters to their definitions, for example, see [1, 2]. The utility of special functions, functions of matrix arguments and orthogonal matrix polynomials in many diverse fields of science, engineering and other branches of knowledge is well established since decades and a huge amount of literature already exists in this field with which the researchers in this field are well versed, so we do not give here any exhaustive list of references in this direction. Directly discussing the contents of the paper in hand, we observe that in Subsection 1.1, we recall some important definitions which are used in this paper. In Section 2, we define the extended Horn's hypergeometric function $\mathbf{H_{11}}$. In Section 3, we obtain the limit formulas for this function. In Section 4, we establish the integral representations and the differentiation formulas are derived for this function in Section 5. We prove some infinite sums in Section 6 and the recursion formulas for this function are presented in Section 7. In Section 8 we give the Laplace, Mellin and fractional Fourier transforms for this function. The fractional integration of the extended Horn's hypergeometric function is discussed in Section 9. The double Laplace and the

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†Corresponding author Karima Hamza, E-mail: karimahamza767@gmail.com

double Mellin transforms for the introduced function are established in Section 10 and the conclusion is given in the last section.

1.1 Preliminaries

In this subsection we recall some important definitions which are used in this paper.

Definition 1.1. The generalization of the extended gamma function is defined as (see, [7])

$$\Gamma_{p,q}^{(\mu,\nu)}(z)=\int_0^\infty t^{z-1}\, exp(-\frac{t^\mu}{p}-\frac{q}{t^\nu})dt, \tag{1.1}$$

$(\Re(z)>0,\Re(p)>0,\Re(q)>0,\Re(\mu)>0,\Re(\nu)>0)$, $\Gamma_{1,0}^{(1,1)}(z)=\Gamma(z)$ and $\Gamma_{1,q}^{(1,1)}(z)=\Gamma_q(z)$.

Definition 1.2. Let $z,\mu\in\mathbb{C}$ and $\Re(p)>0,\Re(q)>0,\Re(\mu)>0,\Re(\nu)>0$, the generalization of the Pochhammer symbol is defined as (see [8])

$$(z;p,q;\mu,\nu)_n=\begin{cases}\frac{\Gamma_{p,q}^{(\mu,\nu)}(z+n)}{\Gamma(z)} & ,\Re(p)>0,\Re(q)>0,\Re(\mu)>0,\Re(\nu)>0,\\ \\ (z)_n & ,p=1,q=0,\mu=1,\nu=0\end{cases}, \tag{1.2}$$

$$(z;p,q;\mu,\nu)_{m-n}=\frac{(-1)^n(z;p,q;\mu,\nu)_m}{(1-z-m)_n},$$
$$(z;p,q;\mu,\nu)_{m-n+1}=z(z+1;p,q;\mu,\nu)_{m-n},$$
$$(z;p,q;\mu,\nu)_{m-n-1}=\frac{1}{z-1}(z-1;p,q;\mu,\nu)_{m-n}.$$

Definition 1.3. The Laplace transform of a function $f(z)$ for $\Re(s)>0$ is defined as (see [6])

$$\pounds\{f(z):s\}=\int_0^\infty e^{-sz}f(z)dz. \tag{1.3}$$

We also mention here that till date the Upadhyaya transform (see, Upadhyaya [11] and Upadhyaya et. al. [12]) is the most robust and the most powerful generalization of all the different kinds of variants of the above stated classical Laplace transform which are introduced by a number of researchers worldwide into the mathematics research literature during the past three decades!

Definition 1.4. The Mellin transform of $f(z)$ is defined as (see [5])

$$M\{f(z):s\}=\int_0^\infty z^{s-1}f(z)dz. \tag{1.4}$$

Definition 1.5. Let $f\in\Phi(R)$ then the fractional Fourier Transform of $f(x)$ of order α is defined as (see [3])

$$\Im_\alpha[f(x):\omega]=\int_R e^{i\omega^{\frac{1}{\alpha}}}f(x)dx,\ \omega>0, \tag{1.5}$$

where $0<\alpha\le 1$.

Definition 1.6. The double Laplace transform of a function $f(x,y)$ of two variables x and y, defined in the first quadrant of the $xy-$ plane, is defined for $\Re(r)>0,\Re(s)>0$ as (see [9])

$$\pounds_2\{\ f(x,y)\ \}=\int_0^\infty\int_0^\infty f(x,y)\ e^{-(rx+sy)}dxdy. \tag{1.6}$$

This is of utmost importance to further mention here in the context of the present definition that the double Laplace transform of a function defined by (1.6) *is a particular case of the Double Upadhyaya transform* introduced in 2019 by Upadhyaya (for more details we refer the interested reader to Upadhyaya [11]).

Definition 1.7. The double Mellin transform of a function $f(x,y)$ of two variables x and y, defined in the first quadrant of the $xy-$ plane, is defined as (see [10])

$$M_{xy}\{\ f(x,y):r,s\ \}=\int_0^\infty\int_0^\infty f(x,y)x^ry^sdxdy. \tag{1.7}$$

2 The extended Horn's hypergeometric $\mathbf{H_{11}}$

In this section, we define the extended Horn's hypergeometric function $\mathbf{H_{11}}$.

Definition 2.1. The extended Horn's hypergeometric function $\mathbf{H_{11}}$ is defined by

$$ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!} x^m y^n; |x| < \infty, |y| < 1, \tag{2.1} $$

where $(\alpha; p, q, \mu, \nu)_m$ is the extended Pochhammer symbol as defined in (1.2).

Special case: For $p = 1, q = 0, \mu = 1, \nu = 0$ we can directly see that (1.2) reduces to the conventional Horn's hypergeometric function H_{11} (see [13]) and the relation between the two is given by

$$ {}_{1,0}\mathbf{H}_{\mathbf{11}}^{(1,0)}(\alpha,\beta,\gamma;\delta;x,y) = H_{11}(\alpha,\beta,\gamma;\delta;x,y). $$

3 Limit formulas

In this section we present some limit formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$.

Theorem 3.1. *The following limit formulas for ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ hold true:*

$$ \lim_{\alpha\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x/\alpha\ ,\alpha y) = {}_0F_1(-;\delta;x)\ {}_2F_0(\beta,\gamma;-;y), \tag{3.1} $$

$$ \lim_{\beta\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{\beta}) = {}_{p,q}\ \mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\gamma;\delta;x,y), \tag{3.2} $$

$$ \lim_{\delta\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\delta x,y) = {}_1F_0((\alpha;p,q,\mu,\nu);-;x\ )\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y), \tag{3.3} $$

$$ \lim_{\gamma\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{\gamma}) = {}_{p,q}\ \mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\beta;\delta;x,y), \tag{3.4} $$

$$ \lim_{\varepsilon\to 0} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\frac{\alpha}{\varepsilon},\beta,\gamma;\delta;\varepsilon x,y/\varepsilon) = {}_0F_1(-;\delta;\alpha x){}_2F_0(\beta,\gamma;-;y/\alpha), \tag{3.5} $$

$$ \lim_{\varepsilon\to 0} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\frac{\delta}{\varepsilon};x/\varepsilon,y) = {}_1F_0((\alpha;p,q,\mu,\nu);-;x/\delta\ )\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y), \tag{3.6} $$

$$ \lim_{\varepsilon\to 0} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\frac{\beta}{\varepsilon},\gamma;\delta;x,\varepsilon y) = {}_{p,q}\ \mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\gamma;\delta;x,\beta y), \tag{3.7} $$

$$ \lim_{\varepsilon\to 0} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\frac{\gamma}{\varepsilon};\delta;x,\varepsilon y) = {}_{p,q}\ \mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\beta;\delta;x,\gamma y) \tag{3.8} $$

and

$$ \lim_{\varepsilon\to 0} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\frac{\beta}{\varepsilon},\frac{\gamma}{\varepsilon};\delta;x,\varepsilon^2 y) = {}_{p,q}\ \mathbf{H}_{\mathbf{5}}^{(\mu,\nu)}(\alpha;\delta;x,\beta\gamma y). \tag{3.9} $$

Proof. We have

$$ \begin{aligned} \lim_{\alpha\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x/\alpha\ ,\alpha y) &= \sum_{m,n=0}^{\infty} \frac{(\beta)_n(\gamma)_n x^m y^n}{(\delta)_m m!n!} \cdot \lim_{\alpha\to\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}}{(\alpha)^{m-n}} \\ &= H_{11}(-,\beta,\gamma;\delta;x,y) = {}_0F_1(-;\delta;x){}_2F_0(\beta,\gamma;-;y), \end{aligned} $$

which the desired result of (3.1). Similarly, we write

$$ \begin{aligned} \lim_{\beta\to\infty} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{\beta}) &= \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}(\gamma)_n x^m y^n}{(\delta)_m m!n!} \cdot \lim_{\beta\to\infty} \frac{(\beta)_n}{(\beta)^n} \\ &= {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,-,\gamma;\delta;x,y) = {}_{p,q}\mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\gamma;\delta;x,y), \end{aligned} $$

which gives the desired result of (3.2). By using the same technique we can prove (3.3) and (3.4).

Also, we can write

$$\lim_{\varepsilon\to 0}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\frac{\alpha}{\varepsilon},\beta,\gamma;\delta;\varepsilon x,y/\varepsilon)=\lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\frac{\alpha}{\varepsilon};p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n(\varepsilon x)^m(\frac{y}{\varepsilon})^n}{(\delta)_m m!n!}$$
$$=\lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\beta)_n(\gamma)_n x^m y^n}{(\delta)_m m!n!}\cdot\frac{(\frac{\alpha}{\varepsilon};p,q;\mu,\nu)_{m-n}}{(\frac{\alpha}{\alpha\varepsilon})^{m-n}}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\beta)_n(\gamma)_n(\alpha x)^m(\frac{y}{\alpha})^n}{(\delta)_m m!n!}\cdot\lim_{\varepsilon\to 0}\frac{(\frac{\alpha}{\varepsilon};p,q;\mu,\nu)_{m-n}}{(\frac{\alpha}{\varepsilon})^{m-n}}$$
$$=\ {}_0F_1(-;\delta;\alpha x){}_2F_0(\beta,\gamma;-;y/\alpha),$$

which proves the result of (3.5).
Similarly, we can write

$$\lim_{\varepsilon\to 0}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\frac{\delta}{\varepsilon};x/\varepsilon,y)=\lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n(\frac{x}{\varepsilon})^m y^n}{(\frac{\delta}{\varepsilon})_m m!n!}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n(\frac{x}{\delta})^m y^n}{m!n!}\cdot\lim_{\varepsilon\to 0}\frac{(\frac{\delta}{\varepsilon})_m}{(\frac{\delta}{\varepsilon})^m}$$
$$=\ {}_1F_0((\alpha;p,q,\mu,\nu);-;x/\delta\ )\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y).$$

thus showing the result of (3.6). In a similar manner we can prove (3.7), (3.8) and (3.9). □

Theorem 3.2. *The following limit formulas also hold true for* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$:

$$\lim_{\beta\to\infty}\ \lim_{\gamma\to\infty}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x\ ,y/(\beta\gamma))=\ {}_{p,q}\mathbf{H}_{\mathbf{5}}^{(\mu,\nu)}(\alpha;\delta;x,y)\tag{3.10}$$

and

$$\lim_{\alpha\to\infty}\ \lim_{\delta\to\infty}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x/\alpha\ ,\alpha\delta y)=exp(x)\ {}_2F_0(\beta,\gamma;-;y).\tag{3.11}$$

Proof. The proof of this theorem is parallel to that of Theorem 3.1. □

4 Integral representations

In this section,we establish several integral representations for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$.

Theorem 4.1. *The following integral representations for the extended Horn's hypergeometric function* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ *hold true:*

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}exp(-\frac{t^{\mu}}{p}-\frac{q}{t^{\nu}})\ {}_0F_1(-;\delta;xt){}_2F_0(\beta,\gamma;-;y/t)dt,\tag{4.1}$$

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)=\frac{1}{\Gamma(\beta)}\int_0^{\infty}t^{\beta-1}exp(-t)\ {}_{p,q}\mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\gamma;\delta;x,yt)\ dt,\tag{4.2}$$

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)=\frac{1}{\Gamma(\gamma)}\int_0^{\infty}t^{\gamma-1}exp(-t)\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_1F_1(\beta;1-\alpha-m;-yt)dt,\tag{4.3}$$

and

$$\int_0^{\infty}t^{s-1}exp(-\rho t)\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_1F_1(\beta;1-\alpha-m;-yt)dt=\frac{\Gamma(s)}{\rho^s}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,s;\delta;x,-\frac{y}{\rho}).\tag{4.4}$$

Proof. Using the integral formula of extended Pochhammer symbol $(\alpha; p, q; \mu, \nu)$ in the definition of the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$, we can write

$$
\begin{aligned}
{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) &= \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n x^m y^n}{(\delta)_m m!n!} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha+m-n-1} exp(-\frac{t^{\mu}}{p} - \frac{q}{t^{\nu}}) \sum_{m,n=0}^{\infty} \frac{(\beta)_n(\gamma)_n x^m y^n}{(\delta)_m m!n!} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} exp(-\frac{t^{\mu}}{p} - \frac{q}{t^{\nu}})\ {}_0F_1(-;\delta;xt) {}_2F_0(\beta,\gamma;-;y/t)dt,
\end{aligned}
$$

which is the required result of (4.1).
Also using integral formula of the Pochhammer symbol for the parameter β in the definition of the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$, we can write

$$
\begin{aligned}
{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) &= \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta+n-1} exp(-t)\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,-,\gamma;\delta;x,y)\ dt \\
&= \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} exp(-t)\ {}_{p,q}\mathbf{H}_{\mathbf{4}}^{(\mu,\nu)}(\alpha,\gamma;\delta;x,yt)\ dt.
\end{aligned}
$$

which is proves (4.2). The proof of (4.3) is parallel to that of (4.2).
From (4.3), we can obtain the equality

$$
\int_0^{\infty} t^{s-1} exp(-\rho t)\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_1F_1(\beta;1-\alpha-m;-yt)dt = \frac{\Gamma(s)}{\rho^s}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,s;\delta;x,-\frac{y}{\rho}).
$$

That can be regarded as the Laplace transform

$$
\pounds\{f(t):\rho\} = \int_0^{\infty} e^{-\rho t} f(t)dx
$$

Or the Mellin transform

$$
M\{f(t):s\} = \int_0^{\infty} t^{s-1} f(t)dt
$$

of the functions

$$
t^{s-1}\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_1F_1(\beta;1-\alpha-m;-yt)
$$

and

$$
exp(-\rho t)\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_1F_1(\beta;1-\alpha-m;-yt)
$$

,respectively. □

5 Differentiation formulas

In this section, we give a few differentiation formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$.

Theorem 5.1. *The following differentiation formulas for the function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ hold true:*

$$
D_x^r[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(\alpha)_r}{(\delta)_r}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+r,\beta,\gamma;\delta+r;x,y), \tag{5.1}
$$

$$
D_y^r[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(-1)^r(\beta)_r(\gamma)_r}{(1-\alpha)_r}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-r,\beta+r,\gamma+r;\delta;x,y), \tag{5.2}
$$

$$
D_q^r[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(-1)^{r(\nu+1)}}{(1-\alpha)_{r\nu}}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-r\nu,\beta,\gamma;\delta;x,y) \tag{5.3}
$$

and

$$
D_{x,y}^{(r,s)}[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(\alpha)_{r-s}(\beta)_s(\gamma)_s}{(\delta)_r}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+r-s,\beta+s,\gamma+s;\delta+r;x,y). \tag{5.4}
$$

Proof. Differentiating (2.1) with respect to x yields

$$D_x[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m(m-1)!n!} x^{m-1}y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m+1-n}(\beta)_n(\gamma)_n}{(\delta)_{m+1}m!n!} x^m y^n = \frac{\alpha}{\delta}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta+1;x,y).$$

Repeating this process, we obtain

$$D_x^r[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(\alpha)_r}{(\delta)_r}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+r,\beta,\gamma;\delta+r;x,y),$$

which is the desired result (5.1). The result of (5.2) follows in the same way as the result of (5.1). Differentiating (2.1) with respect to q yields

$$D_q\ [\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = D_q\ [\ \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!} x^m y^n\ ]$$

and

$$\frac{\partial}{\partial q}\ (\alpha;p,q;\mu,\nu)_{m-n} = \frac{1}{\Gamma(\alpha)}\ (\frac{\partial}{\partial q} \int_0^{\infty} t^{\alpha+m-n-1} exp(-\frac{t^{\mu}}{p} - \frac{q}{t^{\nu}})\ dt\ ).$$

Taking derivatives under the integral symbol by using the Leibnitz rule, we get

$$\frac{\partial}{\partial q}\ (\alpha;p,q;\mu,\nu)_{m-n} = \frac{-1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-\nu+m-n-1} exp(-\frac{t^{\mu}}{p} - \frac{q}{t^{\nu}})\ dt$$

$$= -\frac{\Gamma(\alpha-\nu)}{\Gamma(\alpha)\Gamma(\alpha-\nu)} \int_0^{\infty} t^{\alpha-\nu+m-n-1} exp(-\frac{t^{\mu}}{p} - \frac{q}{t^{\nu}}) = -\frac{\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\ (\alpha-\nu;p,q;\mu,\nu)_{m-n}.$$

Since

$$\frac{\Gamma(\alpha-\nu)}{\Gamma(\alpha)} = (\alpha)_{-\nu} = \frac{(-1)^{\nu}}{(1-\alpha)_{\nu}}.$$

Then, we can write

$$D_q\ [\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(-1)^{(\nu+1)}}{(1-\alpha)_{\nu}}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-\nu,\beta,\gamma;\delta;x,y).$$

Repeating this process, we obtain

$$D_q^r\ (\alpha;p,q;\mu,\nu)_{m-n} = (-1)^r(\alpha)_{-r\nu}\ (\alpha-r\nu;p,q;\mu,\nu)_{m-n} = \frac{(-1)^{r(\nu+1)}}{(1-\alpha)_{r\nu}}\ (\alpha-r\nu;p,q;\mu,\nu)_{m-n}.$$

Hence, we can write

$$D_q^r\ [\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \frac{(-1)^{r(\nu+1)}}{(1-\alpha)_{r\nu}}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-r\nu,\beta,\gamma;\delta;x,y),$$

which is (5.3). Also differentiating (2.1) with respect to x and y, then repeating differentiation with respect to x and y of orders r, s respectively and making some simple calculations yields (5.4). □

Theorem 5.2. *The following derivative formulas for (2.1) hold true:*

$$D_y^r[y^{\beta+r-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = (\beta)_r\ y^{\beta-1}\ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta+r,\gamma;\delta;x,y), \tag{5.5}$$

$$D_y^r[y^{\gamma+r-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = (\gamma)_r\ y^{\gamma-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma+r;\delta;x,y) \tag{5.6}$$

and

$$D_x^r[x^{\delta-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = (-1)^r(1-\delta)_r\ x^{\delta-r-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta-r;x,y). \tag{5.7}$$

Proof. Multiplying (2.1) by $y^{\beta+r-1}$ and taking the derivative of order r with respect to y, we have

$$D_y^r[y^{\beta+r-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ]$$

$$= D_y^r[\ y^{\beta+r-1}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!}x^m y^n\ ]$$

$$= \sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\beta+n)_r(\gamma)_n}{(\delta)_m m!n!}x^m y^{\beta+n-1}$$

$$= (\beta)_r\ y^{\beta-1}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta+r)_n(\gamma)_n}{(\delta)_m m!n!}x^m y^n.$$

Thus, we obtain (5.5). Similarly we can prove (5.6). Multiplying (2.1) by $x^{\delta-1}$ and taking the derivative of order r with to x, we can write

$$D_x^r[x^{\delta-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = D_x^r[\ x^{\delta-1}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m\ m!n!}x^m y^n\ ]$$

$$= \sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m(\delta+m)_{-r}m!n!}x^{\delta+m-r-1}y^n = (-1)^r(1-\delta)_r\ x^{\delta-r-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta-r;x,y),$$

which is (5.7). □

Theorem 5.3. *The following partial differential equation for ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ holds true:*

$$p^7 D_p^2[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)] + 2p^4 D_p[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)] - p^3(\alpha)_{2\mu}[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+2\mu,\beta,\gamma;\delta;x,y)] = 0. \tag{5.8}$$

Proof. Differentiating (2.1) with respect to p, we can write

$$D_p\ [\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = D_p\ [\ \sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!}x^m y^n\ ].$$

Since we have

$$\frac{\partial}{\partial p}\ (\alpha;p,q;\mu,\nu)_{m-n} = \frac{1}{\Gamma(\alpha)}\ (\frac{\partial}{\partial p}\int_0^{\infty} t^{\alpha+m-n-1}exp(-\frac{t^\mu}{p}-\frac{q}{t^\nu})\ dt\ ).$$

Taking derivatives under the integral symbol by using the Leibnitz rule, we get

$$\frac{\partial}{\partial p}\ (\alpha;p,q;\mu,\nu)_{m-n} = \frac{p^{-2}}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha+\mu+m-n-1}exp(-\frac{t^\mu}{p}-\frac{q}{t^\nu})\ dt = \frac{(\alpha)_\mu}{p^2}\ (\alpha+\mu;p,q;\mu,\nu)_{m-n}.$$

Then, we get

$$\frac{\partial}{\partial p}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) = \frac{(\alpha)_\mu}{p^2}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+\mu,\beta,\gamma;\delta;x,y)$$

and

$$\frac{\partial^2}{\partial p^2}\ (\alpha;p,q;\mu,\nu)_{m-n} = \frac{-2(\alpha)_\mu}{p^3}\ (\alpha+\mu;p,q;\mu,\nu)_{m-n} + \frac{(\alpha)_{2\mu}}{p^4}\ (\alpha+2\mu;p,q;\mu,\nu)_{m-n}.$$

Thus, we obtain

$$\frac{\partial^2}{\partial p^2}[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)] = \frac{-2(\alpha)_\mu}{p^3}[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+\mu,\beta,\gamma;\delta;x,y)] + \frac{(\alpha)_{2\mu}}{p^4}[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+2\mu,\beta,\gamma;\delta;x,y)].$$

Multiplying by p^7, we get

$$p^7 D_p^2[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + 2p^4\ D_p[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)] - p^3(\alpha)_{2\mu}[{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+2\mu,\beta,\gamma;\delta;x,y) = 0,$$

which is the desired result (5.8). □

6 Infinite sums

In this section we prove some infinite sum formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$.

Theorem 6.1. *For $|t| < 1$, the following infinite summation formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ hold good:*

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, t^k\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+k,\beta,\gamma;\delta;x,y) = (1-t)^{-\alpha}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\frac{x}{1-t},y(1-t)), \tag{6.1}$$

$$\sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}\, t^k\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta+k,\gamma;\delta;x,y) = (1-t)^{-\beta}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{1-t}) \tag{6.2}$$

and

$$\sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!}\, t^k\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma+k;\delta;x,y) = (1-t)^{-\gamma}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{1-t}). \tag{6.3}$$

Proof. Using the fact that

$$(1-t)^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, t^k,$$

we can write

$$(1-t)^{-\alpha}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\frac{x}{1-t},y(1-t)) = \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;k,\mu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!}\, x^m y^n (1-t)^{-\alpha+n-m}$$

$$= \sum_{k,m,n=0}^{\infty} \frac{(\alpha;p,q;k,\mu)_{m-n+k}(\beta)_n(\gamma)_n}{(\delta)_m k!m!n!}\, x^m y^n t^k = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, t^k \sum_{m,n=0}^{\infty} \frac{(\alpha+k;p,q;k,\mu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!}\, x^m y^n$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, t^k\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+k,\beta,\gamma;\delta;x,y),$$

which proves (6.1). Also we can write

$$(1-t)^{-\beta}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,\frac{y}{1-t}) = \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;k,\mu)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!n!}\, x^m y^n (1-t)^{-\beta-n}$$

$$= \sum_{k,m,n=0}^{\infty} \frac{(\alpha;p,q;k,\mu)_{m-n}(\beta)_{n+k}(\gamma)_n}{(\delta)_m k!m!n!}\, x^m y^n t^k = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}\, t^k \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;k,\mu)_{m-n}(\beta+k)_n(\gamma)_n}{(\delta)_m m!n!}\, x^m y^n$$

$$= \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} t^k\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta+k,\gamma;\delta;x,y),$$

which gives (6.2). The same technique can be used to prove (6.3). □

7 Recursion formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$

In this section we present some recursion formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$. We start with the following theorem:

Theorem 7.1. *For $n \in \mathbb{N}$ and $\delta \neq 0, -1, -2, \ldots$, the following two recursion formulas for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ hold good:*

$$\begin{aligned} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+n,\beta,\gamma;\delta;x,y) &= {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{x}{\delta}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+k,\beta,\gamma;\delta+1;x,y) \\ &- \beta\gamma y \sum_{k=1}^{n} \frac{{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+k-2,\beta+1,\gamma+1;\delta;x,y)}{(\alpha+k-1)(\alpha+k-2)}, \alpha \neq 1-k, \alpha \neq 2-k, k \in \mathbb{N}, \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-n,\beta,\gamma;\delta;x,y) &= {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{x}{\delta}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-k+1,\beta+1,\gamma;\delta+1;x,y) \\ &+ \beta\gamma y \sum_{k=1}^{n} \frac{{}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-k-1,\beta,\gamma+1;\delta;x,y)}{(\alpha-k)(\alpha-k-1)}, \alpha \neq k, \alpha \neq 1+k, k \in \mathbb{N}. \end{aligned} \tag{7.2}$$

Proof. Applying the transformation formula $(\alpha+1)_{m-n} = (\alpha)_{m-n}(1+\frac{m-n}{\alpha})$ in the definition of the extension of the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$, we get the contiguous formula:

$$\begin{aligned} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta;x,y) &= {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{x}{\delta}\, {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta+1;x,y) \\ &- \frac{\beta\gamma y}{\alpha(\alpha-1)}\, {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta,\gamma;\delta;x,y), d \neq 0, \alpha \neq 0, \alpha \neq 1. \end{aligned} \tag{7.3}$$

Calculating the function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ with the parameter $\alpha+n$ by (7.3) for n times, we obtain the required result (7.1). Replacing α by $\alpha-1$ in (7.3), we get

$$\begin{aligned} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta,\gamma;\delta;x,y) &= {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{x}{\delta}\, {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta+1;x,y) \\ &+ \frac{\beta\gamma y}{(\alpha-1)(\alpha-2)}\, {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-2,\beta,\gamma;\delta;x,y), d \neq 0, \alpha \neq 1, \alpha \neq 2. \end{aligned} \tag{7.4}$$

By calculating the function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ with the parameter $\alpha-n$ by (7.4) for n times, we obtain the required result of (7.2). □

Theorem 7.2. *For $\alpha \neq 1$ the following recursion relations hold good for the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$:*

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta+n,\gamma;\delta;x,y) = {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{\gamma y}{\alpha-1}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta+k,\gamma+1;\delta;x,y), \tag{7.5}$$

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta-n,\gamma;\delta;x,y) = {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{\gamma y}{\alpha-1}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta-k+1,\gamma+1;\delta;x,y), \tag{7.6}$$

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma+n;\delta;x,y) = {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{\beta y}{\alpha-1}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta+1,\gamma+k;\delta;x,y) \tag{7.7}$$

and

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma-n;\delta;x,y) = {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{\beta y}{\alpha-1}\sum_{k=1}^{n} {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha-1,\beta+1,\gamma-k+1;\delta;x,y). \tag{7.8}$$

Proof. Applying the transformation formula $(\beta+1)_n = (\beta)_n(1+\frac{n}{\beta})$ in the definition of the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$, we get the contiguous formula:

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta+1,\gamma;\delta;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{\gamma y}{\alpha-1}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha-1,\beta+1,\gamma+1;\delta;x,y). \quad (7.9)$$

Calculating the function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ with the parameter $\beta+n$ by (7.9) for n times, we obtain the required result (7.5). Replacing β by $\beta-1$ in (7.9),we get

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta-1,\gamma;\delta;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{\gamma y}{\alpha-1}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha-1,\beta,\gamma+1;\delta;x,y). \quad (7.10)$$

Calculating the function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ with the parameter $\beta-n$ by (7.10) for (7.6) n times, we obtain the required result. The proofs of (7.7) and (7.8) are similar to the proofs of (7.5) and (7.6). □

Theorem 7.3. *The extended Horn's hypergeometric function* ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ *satisfies the identities:*

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta-n;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \alpha x \sum_{k=1}^{n} \frac{{}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta-k+2;x,y)}{(\delta-k)(\delta-k+1)}, \quad \delta \neq k-1, k\in\mathbb{N} \quad (7.11)$$

and

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta+n;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \alpha x \sum_{k=1}^{n} \frac{{}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta+k+1;x,y)}{(\delta+k-1)(\delta+k)}, \quad \delta \neq 1-k, -k\in\mathbb{N}. \quad (7.12)$$

Proof. From the relation

$$\frac{1}{(\delta-1)_m} = \frac{1}{(\delta)_m} + \frac{m}{(\delta-1)(\delta)_m},$$

we can write

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta-1;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) + \frac{\alpha x}{(\delta-1)\delta}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta+1;x,y),\ \delta\neq 0,1. \quad (7.13)$$

Iterating this method on ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ with the parameter $\delta-n$ for n times, we obtain (7.11). Replacing δ by $\delta+1$ in (7.13), we obtain

$$ {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta+1;x,y) = {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y) - \frac{\alpha x}{\delta(\delta+1)}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha+1,\beta,\gamma;\delta+2;x,y),\ \delta\neq 0,-1. \quad (7.14)$$

Iterating this method on ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ with the parameter $\delta+n$ for n times, we obtain (7.12). □

8 Laplace, Mellin and Fractional Fourier Transforms

In this section we prove some results about the Laplace, Mellin and Fractional Fourier transforms [3,4] of the function ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$.

Theorem 8.1. *The Laplace transform of* ${}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}$ *is as follows :*

$$\mathcal{L}\{x^{a-1}\, {}_{p,q}\mathbf{H}_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,y) : s\} = \frac{\Gamma(a)}{s^a}\, {}_2F_1((\alpha;p,q;\mu,\nu),a;\delta;\frac{\omega}{s}).\, {}_2F_1(\beta,\gamma;1-\alpha-m;-y). \quad (8.1)$$

Proof. Applying the Laplace transform (1.3) on both sides of (2.1), we have

$$\mathcal{L}\{x^{a-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,y):s\}=\int_0^\infty x^{a-1}e^{-sx}[\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n(\omega x)^m y^n}{(\delta)_m m!n!}]dx$$

Putting $sx=u$, $dx=\frac{du}{s}$, $x=0$, $u=0$ and $x=\infty$, $u=\infty$, we get

$$\mathcal{L}\{x^{a-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,y):s\}=\sum_{m,n=0}^{\infty}[\frac{1}{s^{a+m}}(\int_0^\infty e^{-u}u^{a+m-1}du)\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\ \omega^m y^n}{(\delta)_m m!n!}]$$

$$=\frac{1}{s^a}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\Gamma(a+m)\ \omega^m y^n}{s^m(\delta)_m m!n!}=\frac{\Gamma(a)}{s^a}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(a)_m(\beta)_n(\gamma)_n\ \omega^m y^n}{s^m(\delta)_m m!n!}$$

$$=\frac{\Gamma(a)}{s^a}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_m(a)_m(\beta)_n(\gamma)_n\ \omega^m(-y)^n}{(1-\alpha-m)_n(\delta)_m m!n!}$$

$$=\frac{\Gamma(a)}{s^a}\ {}_2F_1((\alpha;p,q;\mu,\nu),a;\delta;\frac{\omega}{s})\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y),$$

which is the desired result of (8.1). □

Remark 8.2. We mention here in the context of Theorem 8.1 that the result of this theorem can be further generalized by invoking the properties of the Upadhyaya transform (see, Upadhyaya [11] and Upadhyaya et. al. [12]) and we propose to give that generalization of Theorem 8.1 in one of our future communications.

Theorem 8.3. *The Mellin transform of* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ *is as follows :*

$$M\{e^{-x}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,y):s\}=\Gamma(s)\ {}_2F_1((\alpha;p,q;\mu,\nu),s;\delta;\omega)\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y).\quad(8.2)$$

Proof. Applying the Mellin transform (1.4) on both sides of (2.1), we have

$$M\{e^{-x}\ {}_{p,q}H_{11}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,y):s\}$$

$$=\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\omega^m y^n}{(\delta)_m m!n!}\int_0^\infty x^{s-1}e^{-x}x^m dx$$

$$=\Gamma(s)\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_m(s)_m(\beta)_n(\gamma)_n\omega^m(-y)^n}{(1-\alpha-m)_n(\delta)_m m!n!}$$

$$=\Gamma(s)\ {}_2F_1((\alpha;p,q;\mu,\nu),s;\delta;\omega)\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y),$$

which is the desired result (8.2). □

Theorem 8.4. *The Fractional Fourier transform of* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ *of order* v *for* $x<0$ *is given by:*

$$\Im_v[\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ]=(i\omega^{\frac{1}{v}})^{-1}\ {}_2F_1((\alpha;p,q;\mu,\nu),1;\delta;-(i\omega^{\frac{1}{v}})^{-1})\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y).\quad(8.3)$$

Proof. Since, we have

$${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n x^m y^n}{(\delta)_m m!n!}$$

$$=\sum_{m,n=0}^{\infty}\frac{(-1)^n(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n x^m y^n}{(1-\alpha-m)_n(\delta)_m m!n!}.\quad(8.4)$$

Applying the Fractional Fourier transform (1.5) on both sides of (8.4), we have

$$\Im_v[\ _{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \sum_{m,n=0}^{\infty} \frac{(-1)^n(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n y^n}{(1-\alpha-m)_n(\delta)_m m!n!} \int_{-\infty}^{0} e^{i\omega^{\frac{1}{v}}x}\, x^m dx.$$

Putting $t = -i\omega^{\frac{1}{v}}x$, $dt = -i\omega^{\frac{1}{v}}dx$, $x = -\infty$, $t = \infty$ and $x = 0$, $t = 0$, we get

$$\Im_v[\ _{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n(-y)^n}{(1-\alpha-m)_n(\delta)_m m!n!} \int_{0}^{\infty} e^{-t}\,(-\frac{t}{i\omega^{\frac{1}{v}}})^m\, \frac{dt}{i\omega^{\frac{1}{v}}}$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n(-y)^n(-1)^m(i)^{-(m+1)}(\omega)^{-\frac{m+1}{v}}\Gamma(m+1)}{(1-\alpha-m)_n(\delta)_m m!n!}$$
$$= (i\omega^{\frac{1}{v}})^{-1}\ {}_2F_1((\alpha;p,q;\mu,\nu),1;\delta;-(i\omega^{\frac{1}{v}})^{-1})\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y),$$

which is the desired result (8.3).

Special Case: For $v = 1$ the result gives the conventional Fourier transform of $_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$:

$$\Im[\ _{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = (i\omega)^{-1}\ {}_2F_1((\alpha;p,q;\mu,\nu),1;\delta;-(i\omega)^{-1})\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y).$$

□

9 Fractional integration of the extended Horn's hypergeometric function

Now we give below a theorem about the fractional integration of the extended Horn's hypergeometric function $_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$.

Theorem 9.1. *Let $\xi, \varrho, \eta \in \mathbb{C}$ with $\Re(\xi) > 0$. Then the fractional integration of the extended Horn's hypergeometric function $_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ is as follows:*

$$I_{0,x}^{\xi,\varrho,\eta}[x^{\rho-1}\ _{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = x^{\rho-\varrho-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\varrho)}{\Gamma(\rho-\varrho)\Gamma(\rho+\eta+\xi)}$$
$$\times\ {}_3F_3((\alpha;p,q;\mu,\nu),\rho,\rho+\eta-\varrho;\delta,\rho-\varrho,\rho+\eta+\xi;x)\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y). \tag{9.1}$$

Proof. By Saigo [4],

$$I_{0,x}^{\xi,\varrho,\eta}x^{\lambda-1} = x^{\lambda-\varrho-1}\frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\varrho)}{\Gamma(\lambda-\varrho)\Gamma(\lambda+\eta+\xi)}.$$

Then we can write

$$I_{0,x}^{\xi,\varrho,\eta}[x^{\rho-1}\ _{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;x,y)\ ] = I_{0,x}^{\xi,\varrho,\eta}[x^{\rho-1}\ {}_1F_1((\alpha;p,q;\mu,\nu);\delta;x)\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y)\ ]$$
$$= \sum_{m,n=0}^{\infty} \frac{(-1)^n(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n y^n}{(1-\alpha-m)_n(\delta)_m m!n!}\ I_{0,x}^{\xi,\varrho,\eta}[x^{\rho+m-1}]$$
$$= \sum_{m,n=0}^{\infty} \frac{(-1)^n(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n y^n}{(1-\alpha-m)_n(\delta)_m m!n!}\ [\,x^{\rho-\varrho+m-1}\frac{\Gamma(\rho+m)\Gamma(\rho+\eta-\varrho+m)}{\Gamma(\rho-\varrho+m)\Gamma(\rho+\eta+\xi+m)}\,]$$
$$= x^{\rho-\varrho-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\varrho)}{\Gamma(\rho-\varrho)\Gamma(\rho+\eta+\xi)}\ {}_3F_3((\alpha;p,q;\mu,\nu),\rho,\rho+\eta-\varrho;\delta,\rho-\varrho,\rho+\eta+\xi;x)$$
$$\times\ {}_2F_1(\beta,\gamma;1-\alpha-m;-y),$$

which is the desired result (9.1). □

10 Double Laplace and double Mellin transforms

In this section we give two results concerning the double Laplace and the double Mellin transforms of ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$. We remark in advance that the result of Theorem 10.1 below can be generalized further by employing the double Upadhyaya transform (see, Upadhyaya [11]) which we shall do in a future work of ours.

Theorem 10.1. *The double Laplace transform of* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ *is as follows :*

$$\pounds_2\{x^{a-1}y^{b-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y)\}$$

$$=\frac{\Gamma(a)\Gamma(b)}{r^a s^b}\ {}_2F_1((\alpha;p,q;\mu,\nu),a;\delta;\frac{\omega}{r})\ {}_3F_1(\beta,\gamma,b;1-\alpha-m;-\frac{\rho}{s}). \tag{10.1}$$

Proof. Applying the double Laplace transform (1.6) on both sides of (2.1), we have

$$\begin{aligned}
&\pounds_2\{\ x^{a-1}y^{b-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y)\ \}\\
&=\pounds\{\ x^{a-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y):r\ \}\pounds\{\ y^{b-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y):s\ \}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\omega^m\rho^n}{(\delta)_m m!n!}\{\ (\int_0^\infty x^{a+m-1}e^{-rx}dx\ )(\int_0^\infty y^{b+n-1}e^{-sy}dy\ )\ \}.
\end{aligned}$$

Putting $rx=u$, $dx=\frac{du}{r}$, $x=0$, $u=0$, $x=\infty$, $u=\infty$ and $sy=z$, $dy=\frac{dz}{s}$, $y=0$, $z=0$, $y=\infty$, $z=\infty$, we get

$$\begin{aligned}
&\pounds_2\{\ x^{a-1}y^{b-1}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y)\ \}\\
&=\frac{1}{r^{a+m}s^{b+n}}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\omega^m\rho^n}{(\delta)_m m!n!}\{\ (\int_0^\infty u^{a+m-1}e^{-u}du\ )(\int_0^\infty z^{b+n-1}e^{-z}dz\ )\ \}\\
&=\frac{\Gamma(a)\Gamma(b)}{r^{a+m}s^{b+n}}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n(a)_m(b)_n\omega^m\rho^n}{(\delta)_m m!n!}\\
&=\frac{\Gamma(a)\Gamma(b)}{r^{a+m}s^{b+n}}\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_m(\beta)_n(\gamma)_n(a)_m(b)_n\omega^m(-\rho)^n}{(\delta)_m\ (1-\alpha-m)_n m!n!}\\
&=\frac{\Gamma(a)\Gamma(b)}{r^a s^b}\ {}_2F_1((\alpha;p,q;\mu,\nu),a;\delta;\frac{\omega}{r})\ {}_3F_1(\beta,\gamma,b;1-\alpha-m;-\frac{\rho}{s}),
\end{aligned}$$

which is the desired result (10.1). □

Theorem 10.2. *The double Mellin transform of* ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ *is as follows :*

$$M_{xy}\{e^{-(x+y)}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y):r,s\}$$

$$=\Gamma(r)\Gamma(s)\ {}_2F_1((\alpha;p,q;\mu,\nu),r;\delta;\omega)\ {}_3F_1(\beta,\gamma,s;1-\alpha-m;-\rho). \tag{10.2}$$

Proof. Applying the double Mellin transform (1.7) on both sides of (2.1), we have

$$\begin{aligned}
&M_{xy}\{e^{-(x+y)}\ {}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}(\alpha,\beta,\gamma;\delta;\omega x,\rho y):r,s\}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_{m-n}(\beta)_n(\gamma)_n\omega^m\rho^n}{(\delta)_m m!n!}\{\ (\int_0^\infty x^{r-1}e^{-x}x^m dx\ )(\int_0^\infty y^{s-1}e^{-y}y^n dy\ )\ \}\\
&=\Gamma(r)\Gamma(s)\sum_{m,n=0}^{\infty}\frac{(\alpha;p,q;\mu,\nu)_m(r)_m(\beta)_n(\gamma)_n(s)_n\ \omega^m(-\rho)^n}{(1-\alpha-m)_n(\delta)_m m!n!}\\
&=\Gamma(r)\Gamma(s)\ {}_2F_1((\alpha;p,q;\mu,\nu),r;\delta;\omega)\ {}_3F_1(\beta,\gamma,s;1-\alpha-m;-\rho),
\end{aligned}$$

which is the desired result (10.2). □

11 Concluding remarks

In this paper we explored some properties of the extended Horn's hypergeometric function ${}_{p,q}\mathbf{H}_{\mathbf{11}}^{(\mu,\nu)}$ which includes the limit formulas, integral representations, differentiation formulas, infinite sums, recursion formulas, the Laplace, Mellin, fractional Fourier transforms, the fractional integration formula and the double Laplace and the double Mellin transforms of this function. The results given in this paper can be applied to probability theory, astrophysics, biological sciences and engineering.

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References

[1] Chaudhry, M. A., Qadir, A., Raque, M. and Zubair, S. M. (1997). Extension of Euler's Beta function, *J. Comput. Appl. Math.*, 78, 19–32.

[2] Choi, J., Rathie, A. K. and Parmar, R. K. (2014). Extension of extended beta, hypergeometric and confluent hypergeometric, *Honam Mathematical Journal*, 36(2), 357–385.

[3] Romero, L., Cerutti, R. and Luque, L. (2011). A new fractional Fourier transform and convolutions products, *International Journal of Pure and Applied Mathematics*, 66, 397–408.

[4] Saigo, M. (1978). A remark on integral operators involving the Gauss hypergeometric functions, *Math. Rep. Kyushu Univ.*, XI-2, 135–143.

[5] Bertrand, J., Bertrand, P. and Ovarlez, J. (2000). The Mellin transform in The Transforms and Applications Handbook, Second Edition, D. Alexander,Ed., CRC Press, Boca Raton, FL, USA.

[6] Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications, Chapman and Hall, CRC Press, Boca Raton, FL, USA.

[7] Sahin, R., Yagci, O., Yagbasan, M. B., Cetinkaya, A. and Kiymaz, I. O. (2018). Further generalization of Gamma, Beta and related functions, *Journal of Inequalities and Special Functions*, 9, 1–7.

[8] Sahin, Recep and Yagci, Oguz (2020). A new generalization of Pochhammer symbol and its applications, *Applied Mathematics and Nonlinear Sciences*, 5(1), 255–266.

[9] Debnath Lokenath, The double Laplace transforms and their properties with applications to functional, integral and partial differential equation, *International Journal of Applied and Computational Mathematics*, 2(2), 223–241.

[10] Hassan, Eltayeb and Kilicman, Adel (2007). A note of Mellin transform and partial differential equations, *International Journal of Pure and Applied Mathematics*, 34(4), 457–467.

[11] Upadhyaya, L. M. (2019). Introducing the Upadhyaya integral transform, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 38(1), 471–510.

[12] Upadhyaya, L. M., Shehata, A. and Kamal, A. (2021). An update on the Upadhyaya transform, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 40(1), 26–44.

[13] Srivastava, H. M. and Karlsson, Per W. (1985). Multiple Gaussian Hypergeometric Series, John Wiley and Sons, New York.