Original Article

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.271-280

Available online at
www.bpasjournals.com

Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00026.2

SPECIAL CHARACTERIZATIONS OF POLYGONAL NUMBERS THROUGH PELL EQUATIONS

S. Vidhyalakshmi[1], S. Aarthy Thangam[2,*]

Authors Affiliation:
[1]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu 620 002, India.
[2]Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu 620 002, India.

***Corresponding Author:**
S. Aarthy Thangam, Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu 620 002, India.

E mail: aarthythangam@gmail.com

Received on 29.01.2019 Revised on 22.04.2019 Accepted on 28.05.2019

Abstract:
In this paper, different choices of positive and negative Pell equations are considered. Employing the non-zero integer solutions of each of the above choices of positive and negative Pell equations, the relations among the special polygonal numbers are exhibited.

Keywords: Positive Pell equation, Negative Pell equation, Polygonal numbers, Integer solutions.

2010 Mathematics Subject Classification: 11D09

1. INTRODUCTION

Every researcher in Number Theory is familiar with the subject of Diophantine equations. In fact, Number theory is the great and rich intellectual heritage of man-kind and essentially a man-made world to meet his ideals of intellectual perfection. No doubt that number is the essence of mathematical calculations and one may discover beautiful patterns in numbers. Recognizing number patterns is also an important problem solving skill. It is worth to quote the remark "There is strength in numbers, but organizing those numbers is one of the great challenges" by the mathematician John C. Mather and one may call "Mathematics as the science of patterns" as remarked by Ronald Graham.

The numbers that can be represented by a regular geometric arrangement of equally spaced points are called Figurate numbers [1]. In [2], the relations among the pairs of special m-gonal numbers generated through the solutions of the binary quadratic equation $y^2 = 2x^2 - 1$ are determined. In [3], the relations among special figurate numbers through the equation $y^2 = 10x^2 + 1$ are obtained. In [4], employing the solutions of the Pythagorean equation, the relations between the pairs of special polygonal numbers such that the difference in each pair is a perfect square is obtained. Also, Bert Miller [5] has defined a number known as Nasty number as follows: A positive integer n is a Nasty number if $n = ab = cd$ and $a+b = c-d$ or $a-b = c+d$ where a, b, c and d are non-zero distinct positive integers.

In this paper, a few interesting relations among the polygonal and centered polygonal numbers are obtained through employing the distinct integer solutions of the corresponding Pell equations.

2. NOTATIONS

- Polygonal number of rank n with size m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Centered Polygonal number of rank n with size m

$$Ct_{m,n}=\frac{mn(n+1)}{2}+1$$

3. METHOD OF ANALYSIS

Relation: 1

Consider the Pell equation

$$y^2=2x^2+1 \tag{1}$$

The general solution of (1) is

$$y_n=\frac{1}{2}f_n \quad , \quad x_n=\frac{1}{2\sqrt{2}}g_n$$

where, $f_n=\left(3+2\sqrt{2}\right)^{n+1}+\left(3-2\sqrt{2}\right)^{n+1}$, $g_n=\left(3+2\sqrt{2}\right)^{n+1}-\left(3-2\sqrt{2}\right)^{n+1}$

Observation:

Let $\{n_s\}$ be a sequence of positive integers defined by

$$n_s=\frac{1}{8}(f_s+2) \quad , \quad s=0,2,4,6.....$$

Note that $4t_{6,n_s}$ is a perfect square.

Relation: 2

Consider the Pell equation

$$y^2=3x^2+4 \tag{2}$$

The least positive solution of (2) is $x_0=2$, $y_0=4$.

To find the other solutions of (2), consider the positive Pell equation

$$y^2=3x^2+1$$

whose general solution $(\tilde{x}_n,\tilde{y}_n)$ is given by

$$\tilde{y}_n=\frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1}+\left(2-\sqrt{3}\right)^{n+1}\right]=\frac{1}{2}f_n$$

$$\tilde{x}_n=\frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1}-\left(2-\sqrt{3}\right)^{n+1}\right]=\frac{1}{2\sqrt{3}}g_n\,, \quad n=-1,0,1,2,.....$$

where, $f_n=\left(2+\sqrt{3}\right)^{n+1}+\left(2-\sqrt{3}\right)^{n+1}$ and $g_n=\left(2+\sqrt{3}\right)^{n+1}-\left(2-\sqrt{3}\right)^{n+1}$.

Applying the Brahmagupta lemma between the solutions (x_0,y_0) and $(\tilde{x}_n,\tilde{y}_n)$, the sequence of integer solutions to (2) are given by

$$x_{n+1}=x_0\tilde{y}_n+y_0\tilde{x}_n=f_n+\frac{2}{\sqrt{3}}g_n$$

$$y_{n+1}=y_0\tilde{y}_n+Dx_0\tilde{x}_n=2f_n+\sqrt{3}g_n$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1}=\frac{1}{6}\left(2f_s+\sqrt{3}g_s+2\right), \quad s=-1,1,3,5,....$$

Note that $4t_{8,n_{s+1}}$ is a Perfect square.

Relation: 3

Consider the Pell equation

$$y^2 = 6x^2 + 1 \tag{3}$$

The general solution of (3) is

$$y_n = \frac{1}{2} f_n \quad , \quad x_n = \frac{1}{2\sqrt{6}} g_n$$

where, $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1} \quad , \quad g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$.

Observation:

Let $\{n_s\}$ be a sequence of positive integers defined by

$$n_s = \frac{1}{12}\left(f_s + 2\right) \quad , \quad s = 0, 2, 4, 6.....$$

Note that $4t_{5,n_s}$ is a Perfect square.

Relation: 4

Consider the Pell equation

$$y^2 = 6x^2 + 9 \tag{4}$$

The least positive solution of (4) is $x_0 = 6$, $y_0 = 15$.

To find the other solutions of (4), consider the positive Pell equation

$$y^2 = 6x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2\sqrt{6}} g_n \,, \quad n = -1, 0, 1, 2, \ldots$$

with $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$ and $g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (4) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = 3 f_n + \frac{15}{2\sqrt{6}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{15}{2} f_n + 3\sqrt{6} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{12}\left(15 f_s + 6\sqrt{6} g_s - 6\right), \quad s = -1, 0, 1, 2, 3, ...$$

Note that $6Ct_{12,n_{s+1}} - 6$ is a Nasty number.

Relation: 5

Consider the Pell equation

$$y^2 = 20x^2 + 9 \tag{5}$$

The least positive solution of (5) is $x_0 = 6$, $y_0 = 27$.

To find the other solutions of (5), consider the positive Pell equation

$$y^2 = 20x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{5}}\left[\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right] = \frac{1}{4\sqrt{5}} g_n\,, \quad n=-1,0,1,2,\ldots$$

with, $f_n = \left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}$ and $g_n = \left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (5) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = 3f_n + \frac{27}{4\sqrt{5}} g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{27}{2} f_n + 6\sqrt{5} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{20}\left(27f_s + 12\sqrt{5}g_s + 6\right), \quad s=-1,1,3,5,\ldots.$$

Note that $2t_{7,n_{s+1}}$ is a Perfect square.

Relation: 6

Consider the Pell equation

$$y^2 = 24x^2 + 9 \tag{6}$$

The least positive solution of (6) is $x_0 = 3$, $y_0 = 15$.

To find the other solutions of (6), consider the positive Pell equation

$$y^2 = 24x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{4\sqrt{6}} g_n\,, \quad n=-1,0,1,2,\ldots$$

with $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$ and $g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (6) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = \frac{3}{2} f_n + \frac{15}{4\sqrt{6}} g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{15}{2} f_n + 3\sqrt{6} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{4}\left(5f_s + 2\sqrt{6}g_s - 2\right), \quad s=-1,0,1,2,3,\ldots.$$

Note that $3t_{3,n_{s+1}}$ is a Perfect square.

Relation: 7

Consider the Pell equation

$$y^2 = 24x^2 + 25 \tag{7}$$

The least positive solution of (7) is $x_0 = 1$, $y_0 = 7$.

To find the other solutions of (7), consider the positive Pell equation

$$y^2 = 24x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{4\sqrt{6}} g_n, \quad n = -1,0,1,2,\dots$$

with $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$ and $g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (7) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = \frac{1}{2} f_n + \frac{7}{4\sqrt{6}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{7}{2} f_n + \sqrt{6} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{4}\left(7 f_s + 2\sqrt{6} g_s - 2\right), \quad s = -1,0,1,2,3,\dots.$$

Note that $2t_{3,n_{s+1}} - 6$ is a Nasty Number.

Relation: 8

Consider the Pell equation

$$y^2 = 40x^2 + 25 \tag{8}$$

The least positive solution of (8) is $x_0 = 15$, $y_0 = 95$.

To find the other solutions of (8), consider the positive Pell equation

$$y^2 = 40x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(19+6\sqrt{10}\right)^{n+1} + \left(19-6\sqrt{10}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{10}}\left[\left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1}\right] = \frac{1}{4\sqrt{10}} g_n, \quad n = -1,0,1,2,\dots$$

with $f_n = \left(19+6\sqrt{10}\right)^{n+1} + \left(19-6\sqrt{10}\right)^{n+1}$ and $g_n = \left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1}$.

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (8) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = \frac{15}{2} f_n + \frac{95}{4\sqrt{10}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{95}{2} f_n + 15\sqrt{10} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{4}\left(19 f_s + 6\sqrt{10} g_s - 2\right), \quad s = -1, 0,1,2,3,\dots.$$

Note that $5t_{3,n_{s+1}}$ is a Perfect square.

Relation: 9

Consider the Pell equation

$$y^2 = 2x^2 - 4 \tag{9}$$

The least positive solution of (9) is $x_0 = 2$, $y_0 = 2$

To find the other solutions of (9), consider the positive Pell equation

$$y^2 = 2x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right] = \frac{1}{2\sqrt{2}} g_n , \quad n = -1, 0, 1, 2, \ldots$$

with $f_n = \left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}$ and $g_n = \left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (9) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = f_n + \frac{1}{2\sqrt{2}} g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = f_n + \sqrt{2} g_n .$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{4}\left(f_s + \sqrt{2} g_s - 2\right), \quad s = 0, 1, 2, 3, \ldots.$$

Note that $2Ct_{8,n_{s+1}} + 2$ is a Perfect square.

Relation: 10

Consider the Pell equation

$$y^2 = 3x^2 - 3 \tag{10}$$

The least positive solution of (10) is $x_0 = 2$, $y_0 = 3$.

To find the other solutions of (10), consider the positive Pell equation

$$y^2 = 3x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] = \frac{1}{2\sqrt{3}} g_n , \quad n = -1, 0, 1, 2, \ldots$$

with $f_n = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}$ and $g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (10) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = f_n + \frac{\sqrt{3}}{2} g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{3}{2} f_n + \sqrt{3} g_n .$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{12}\left(3f_s + 2\sqrt{3} g_s - 6\right), \quad s = 1, 3, 5, \ldots.$$

Note that $4Ct_{6,n_{s+1}}$ is a Perfect square.

Relation: 11

Consider the Pell equation

$$y^2 = 10x^2 - 15 \qquad (11)$$

The least positive solution of (11) is $x_0 = 2$, $y_0 = 5$.

To find the other solutions of (11), consider the positive Pell equation

$$y^2 = 10x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(19+6\sqrt{10}\right)^{n+1} + \left(19-6\sqrt{10}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{10}}\left[\left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1}\right] = \frac{1}{2\sqrt{10}} g_n, \quad n = -1, 0, 1, 2, \ldots$$

with $f_n = \left(19+6\sqrt{10}\right)^{n+1} + \left(19-6\sqrt{10}\right)^{n+1}$ and $g_n = \left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (11) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = f_n + \frac{5}{2\sqrt{10}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{5}{2} f_n + \sqrt{10} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{20}\left(5 f_s + 2\sqrt{10} g_s - 10\right), \quad s = 0, 1, 2, 3, \ldots.$$

Note that $2Ct_{10, n_{s+1}} + 2$ is a Perfect square.

Relation: 12

Consider the Pell equation

$$y^2 = 15x^2 - 6 \qquad (12)$$

The least positive solution of (12) is $x_0 = 1$, $y_0 = 3$.

To find the other solutions of (12), consider the positive Pell equation

$$y^2 = 15x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{15}}\left[\left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}\right] = \frac{1}{2\sqrt{15}} g_n, \quad n = -1, 0, 1, 2, \ldots$$

with $f_n = \left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}$ and $g_n = \left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}$.

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (12) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = \frac{1}{2} f_n + \frac{3}{2\sqrt{15}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{3}{2} f_n + \frac{\sqrt{15}}{2} g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{12}\left(3f_s + \sqrt{15}g_s - 6\right), \quad s = 0,1,2,3,....$$

Note that $10Ct_{12,n_{s+1}} + 15$ is a Perfect square.

Relation: 13
Consider the Pell equation

$$y^2 = 20x^2 - 71 \tag{13}$$

The least positive solution of (13) is $x_0 = 2$, $y_0 = 3$.
To find the other solutions of (13), consider the positive Pell equation

$$y^2 = 20x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right] = \frac{1}{2}f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{5}}\left[\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right] = \frac{1}{4\sqrt{5}}g_n, \quad n = -1,0,1,2,\dots$$

with $f_n = \left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}$ and $g_n = \left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (13) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = f_n + \frac{3}{4\sqrt{5}}g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{3}{2}f_n + 2\sqrt{5}g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{20}\left(3f_s + 4\sqrt{5}g_s + 6\right), \quad s = 0,2,4,....$$

Note that $2t_{7,n_{s+1}} + 4$ is a Perfect square.

Relation: 14
Consider the Pell equation

$$y^2 = 24x^2 - 15 \tag{14}$$

The least positive solution of (14) is $x_0 = 1$, $y_0 = 3$.
To find the other solutions of (14), consider the positive Pell equation

$$y^2 = 24x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2}f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{4\sqrt{6}}g_n, \quad n = -1,0,1,2,\dots$$

with $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$ and $g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (14) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = \frac{1}{2}f_n + \frac{3}{4\sqrt{6}}g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{3}{2}f_n + \sqrt{6}g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{12}\left(3f_s + 2\sqrt{6}g_s - 6\right), \quad s = 0,1,2,3,\ldots.$$

Note that $Ct_{3,n_{s+1}}$ is a Perfect square.

Relation: 15

Consider the Pell equation

$$y^2 = 24x^2 - 23 \tag{15}$$

The least positive solution of (15) is $x_0 = 1$, $y_0 = 1$.

To find the other solutions of (15), consider the positive Pell equation

$$y^2 = 24x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2}f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right] = \frac{1}{4\sqrt{6}}g_n, \quad n = -1,0,1,2,\ldots$$

with $f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$ and $g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$. Applying the Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (15) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = \frac{1}{2}f_n + \frac{1}{4\sqrt{6}}g_n$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = \frac{1}{2}f_n + \sqrt{6}g_n.$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{4}\left(f_s + 2\sqrt{6}g_s - 2\right), \quad s = 0,1,2,3,\ldots.$$

Note that $2t_{3,n_{s+1}} + 6$ is a Nasty Number.

Relation: 16

Consider the Pell equation

$$y^2 = 30x^2 - 21 \tag{16}$$

The least positive solution of (16) is $x_0 = 1$, $y_0 = 3$.

To find the other solutions of (16), consider the positive Pell equation

$$y^2 = 30x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(11+2\sqrt{30}\right)^{n+1} + \left(11-2\sqrt{30}\right)^{n+1}\right] = \frac{1}{2}f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{30}}\left[\left(11+2\sqrt{30}\right)^{n+1} - \left(11-2\sqrt{30}\right)^{n+1}\right] = \frac{1}{2\sqrt{30}}g_n, \quad n = -1,0,1,2,\ldots$$

where, $f_n = \left(11+2\sqrt{30}\right)^{n+1} + \left(11-2\sqrt{30}\right)^{n+1}$ and $g_n = \left(11+2\sqrt{30}\right)^{n+1} - \left(11-2\sqrt{30}\right)^{n+1}$.

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of integer solutions to (16) are given by

$$x_{n+1} = x_0 \tilde{y}_n + y_0 \tilde{x}_n = \frac{1}{2} f_n + \frac{3}{2\sqrt{30}} g_n$$

$$y_{n+1} = y_0 \tilde{y}_n + D x_0 \tilde{x}_n = \frac{3}{2} f_n + \frac{\sqrt{30}}{2} g_n .$$

Observation:

Let $\{n_{s+1}\}$ be a sequence of positive integers defined by

$$n_{s+1} = \frac{1}{12}\left(3 f_s + \sqrt{30} g_s - 6\right), \quad s = 0,1,2,3,\ldots.$$

Note that $10 Ct_{6,n_{s+1}} + 15$ is a Perfect square.

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