

COMPLETENESS OF HAUSDORFF INDUCED 2-FUZZY METRIC SPACE

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Abstract

In this paper the concept of completeness induced by the Hausdorff metric is studied and the 2-fuzzy compactness induced by the Hausdorff metric is also discussed.

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1. INTRODUCTION

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has earlier been investigated by many different authors from different viewpoints. In 1975, Kramosil and Michalek [13] introduced the concept of fuzzy metric space. George and Veeramani [4] modified the concept of fuzzy metric space. Later, Gregori and Romaguera[8] proved that the topological space induced by the fuzzy metric is metrizable. The new version of fuzzy metric is more restrictive, but it determines the class of spaces that are connected with the class of metrizable topological spaces. So it is interesting to study the new version of the fuzzy metric. Many contributions to the study of fuzzy metric spaces can be found in [5]-[7], [10], [12], [14]-[18], [20], [21]. Rodriguez- Lo'pez and Romaguera [19] gave a construction of the Hausdorff fuzzy metric on the set of non- empty compact sets.

Recently Ahmed et al. [1] introduced the definition of 2-fuzzy metric space. A natural problem that arises in this field is to introduce an appropriate notion of completeness and construct a satisfactory theory of completion of 2-fuzzy metric spaces.

In this paper we give a satisfactory notion of completeness and propose a construction of completion of the Hausdorff induced 2-fuzzy metric spaces on the family of non empty finite sets. Moreover we obtain necessary

and sufficient condition for the Hausdorff 2-fuzzy metric spaces on the family of non empty compact sets to be complete.

2. PRELIMINARIES

Definition 2.1:

A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t -norm if it satisfies the following conditions

- (i) $*$ is associative and commutative.
- (ii) $*$ is continuous.
- (iii) $a * 1 = a$ for all $a \in [0,1]$.
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2:

A 3-tuple $(\mathcal{F}(X), M, *)$ is said to be a 2-fuzzy metric space if $\mathcal{F}(X)$ is the set of all fuzzy sets on X , $*$ is the continuous t -norm and M is a fuzzy set on $\mathcal{F}(X) \times \mathcal{F}(X) \times (0, \infty)$ satisfying the following conditions for all $f, g, h \in \mathcal{F}(X)$ and $s, t \in (0, \infty)$.

- (i) $M(f, g, t) > 0$.
- (ii) $M(f, g, t) = 1$ if and only if $f = g$.
- (iii) $M(f, g, t) = M(g, f, t)$.
- (iv) $M(f, g, t) * M(g, h, s) \leq M(f, h, t + s)$.
- (v) $M(f, g, \cdot)$ is a continuous function from $(0, \infty)$ to $[0,1]$.

Definition 2.3:

Let $(\mathcal{F}(X), M, *)$ be a 2-fuzzy metric space and $r \in (0,1)$, the open ball $B(f, r, t)$ is defined as a set of all $g \in \mathcal{F}(X)$ such that $M(f, g, t) > 1 - r$ where $t > 0$.

Definition 2.4:

- (i) Let $(\mathcal{F}(X), M, *)$ be a 2-fuzzy metric space. A sequence $\{f_n\}$ is said to be a Cauchy sequence if for each $r \in (0,1)$ and $t > 0$ there exists a natural number N such that $M(f_n, f_m, t) > 1 - r$ for all $n, m \geq N$.
- (ii) Let $(\mathcal{F}(X), M, *)$ be a 2-fuzzy metric space. A sequence $\{f_n\}$ is said to converge to f if for each $r \in (0,1)$ and $t > 0$ there exists N such that $M(f_n - f, t) > 1 - r$ for all $n \geq N$.
- (iii) $(\mathcal{F}(X), M, *)$ is said to be complete if every Cauchy sequence in $\mathcal{F}(X)$ is convergent.

3. HAUSDORFF FUZZY METRIC ON $\mathcal{F}(X)$

Theorem 3.1:

Given a 2-fuzzy metric space $(\mathcal{F}(X), M, *)$. Let $\{f_n\}$ and $\{g_n\}$ converge to f and g respectively then $M(f_n, g_n, t) = M(f, g, t)$.

Theorem 3.2:

Let $(\mathcal{F}(X), M, *)$ be a 2-fuzzy metric space. Let A be a closed subset of $\mathcal{F}(X)$. If $\{f_n\}$ converges to f and if $f_n \in A$ for all n then $f \in A$.

Definition 3.3:

Let $(\mathcal{F}(X), M, *)$ be a complete 2-fuzzy metric space and let $\mathcal{K}(X)$ be the set of all non empty 2-fuzzy compact subsets of $\mathcal{F}(X)$.

Define $M(f, B, t) = \sup_{g \in B} M(f, g, t)$

$M(B, f, t) = \sup_{g \in B} M(g, f, t)$ for all $f \in \mathcal{F}(X)$ and B a subset of $\mathcal{F}(X)$.

For $A, B \in \mathcal{K}(X)$, define the 2-fuzzy hausdorff metric on $\mathcal{K}(X)$ as a function

$$H: \mathcal{K}(X) \times \mathcal{K}(X) \times (0, \infty) \rightarrow [0, 1] \text{ by}$$

$$H(A, B, t) = \min\{\inf_{f \in A} M(f, B, t), \inf_{g \in B} M(A, g, t)\}.$$

Definition 3.4:

$$\text{Define } M'(f, B, t) = \sup_{g \in B} M(f, g, t)$$

$$M^*(A, B, t) = \inf_{f \in A} M'(f, B, t)$$

and the 2-fuzzy Hausdorff metric on $\mathcal{K}(X)$ is a mapping

$$H: \mathcal{K}(X) \times \mathcal{K}(X) \times (0, \infty) \rightarrow [0, 1] \text{ defined by}$$

$$H(A, B, t) = \min\{M^*(A, B, t), M^*(B, A, t)\}.$$

Theorem 3.5:

Let $f \in \mathcal{F}(X)$ and A, B, C be elements in $\mathcal{K}(X)$ then

- (1) $M'(f, B, t) = 1$ iff $f \in B$.
- (2) $M^*(A, B, t) = 1$ iff $A \subseteq B$.
- (3) There exists $g_f \in B$ such that $M'(f, B, t) = M(f, g_f, t)$.
- (4) There exists $f^* \in A$ and $g^* \in B$ such that $M^*(A, B, t) = M'(f^*, g^*, t)$.
- (5) If $A \subseteq B$ then $M'(f, B, t) \geq M'(f, A, t)$.
- (6) If $B \subseteq C$ then $M^*(A, C, t) \geq M^*(A, B, t)$.
- (7) $M^*(A \cup B, C, t) = \max\{M^*(A, C, t), M^*(B, C, t)\}$
- (8) $M^*(A, B, t) \geq M^*(A, C, t) * M^*(C, B, t)$.

Theorem 3.6:

The set $\mathcal{K}(X)$ with the Hausdorff metric H defines a fuzzy metric space.

Theorem 3.7:

Given $A \in \mathcal{K}(X)$ and ϵ a positive number in $(0, 1)$, then $A + \epsilon$ defined by

$$A + \epsilon = \{f \in \mathcal{F}(X) : M'(f, A, t) > 1 - \epsilon\} \text{ is closed.}$$

Proof:

Let $A \in \mathcal{F}(X)$ and $\epsilon \in (0, 1)$. If f is a limit point of $A + \epsilon$ then there exists a sequence $\{f_n\}$ in $A + \epsilon$ converging to f .

By definition, $M'(f_n, A, t) > 1 - \epsilon$ for all n .

By property (3) of theorem (3.5) for each 'n' there exist $g_n \in A$ such that

$$M'(f_n, A, t) = M'(f_n, g_n, t) > 1 - \epsilon, \forall n.$$

Since $A \in \mathcal{K}(X)$, A is sequentially compact and so $\{g_n\}$ in A has a convergent subsequence $\{g_{n_k}\}$ converging to g and so the subsequence $\{f_{n_k}\}$ converges to f . By theorem (3.1), $M'(f_{n_k}, g_{n_k}, t) = M'(f, g, t) > 1 - \epsilon$ as $f_{n_k} \rightarrow f$ and $g_{n_k} \rightarrow g$. By definition of $M'(f, A, t)$, we conclude that $M'(f, A, t) > 1 - \epsilon$.

Thus $f \in A + \epsilon$ and since f is an arbitrary limit point of $A + \epsilon$, it follows that $A + \epsilon$ is closed and it contains all its limit points.

Theorem 3.8:

Suppose that $A, B \in \mathcal{K}(X)$ and $\epsilon > 0$. Then $H(A, B, t) > 1 - \epsilon$ if and only if $A \subseteq B + \epsilon$ and $B \subseteq A + \epsilon$.

Proof:

Suppose $B \subseteq A + \epsilon$, then $M'(g, A, t) > 1 - \epsilon$ for every $g \in B$ which implies $M^*(B, A, t) > 1 - \epsilon$.

Suppose $M^*(B, A, t) > 1 - \epsilon$ then for every $g \in B$ by definition,

$$M^*(B, A, t) = \inf_{g \in B} M'(g, A, t) > 1 - \epsilon$$

i.e., $M'(g, A, t) > 1 - \epsilon$.

From the definition of $A + \epsilon$ we conclude that $B \subseteq A + \epsilon$ and by repeating the same argument as above we get $H(A, B, t) > 1 - \epsilon$ if and only if $A \subseteq B + \epsilon$.

Lemma 3.9:

Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}(X)$. If $\{f_{n_k}\}$ is a Cauchy sequence in $\mathcal{F}(X)$ for $f_{n_k} \in A_{n_k}$, then there exists a Cauchy sequence $\{g_n\}$ in $\mathcal{F}(X)$ such that $g_n \in A_n$ for all n and $f_{n_k} = g_{n_k}$ for all k .

Proof:

Let $\{f_{n_k}\}$ be a Cauchy sequence in $\mathcal{F}(X)$, where $f_{n_k} \in A_{n_k}$ for all k .

By property (3) of theorem (3.5) choose $g_n \in A_n$ such that

$$M'(f_{n_k}, A_n, t) = M(f_{n_k}, g_n, t) \text{ for } n_{k-1} < n < n_k.$$

By definition of M and M' ,

$$\begin{aligned} M(f_{n_k}, g_n, t) &= M'(f_{n_k}, A_n, t) \\ &> M^*(A_{n_k}, A_n, t) \\ &> H(A_{n_k}, A_n, t). \end{aligned}$$

Since, $f_{n_k} \in A_{n_k}$, $M(f_{n_k}, g_{n_k}, t) = M'(f_{n_k}, A_{n_k}, t) = 1$.

It follows that $f_{n_k} = g_{n_k}$ for all k . Let $\epsilon \in (0, 1)$, since $\{f_{n_k}\}$ is a Cauchy sequence in $\mathcal{F}(X)$, there exists a natural number K such that

$$M(f_{n_k}, f_{n_j}, t) > (1 - \epsilon) \text{ for all } k, j \geq K.$$

Again since $\{A_n\}$ is a Cauchy sequence in $\mathcal{K}(X)$, there exists a positive integer $N \geq n_k$ such that $H(A_n, A_m, t) > (1 - \epsilon)$ for all $m, n \geq N$.

Consider $M(g_n, g_m, t) \geq M(g_n, f_{n_k}, t) * M(f_{n_k}, f_{n_j}, t) * M(f_{n_j}, g_m, t)$

$$\begin{aligned} &= M'(f_{n_k}, A_n, t) * M(f_{n_k}, f_{n_j}, t) * M'(f_{n_j}, A_m, t) \\ &> M^*(A_{n_k}, A_n, t) * M(f_{n_k}, f_{n_j}, t) * M^*(A_{n_j}, A_m, t) \end{aligned}$$

$$> H(A_{n_k}, A_n, t) * M(f_{n_k}, f_{n_j}, t) * H(A_{n_j}, A_m, t) > (1 - \epsilon) * (1 - \epsilon) * (1 - \epsilon) = 1 - \epsilon$$

Hence $\{g_n\}$ is a Cauchy sequence in $\mathcal{F}(X)$ such that $g_n \in A_n$ for all n and $g_{n_k} = f_{n_k}$ for all k .

Lemma 3.10:

Let $\{A_n\}$ be a sequence in $\mathcal{K}(X)$. Let A be the set of all points $f \in \mathcal{F}(X)$ such that there is a sequence $\{f_n\}$ that converges to f and satisfies $f_n \in A_n$ for all n . If $\{A_n\}$ is a Cauchy sequence then the set A is closed and empty.

Proof:

Suppose A is non empty. Since $\{A_n\}$ is a Cauchy sequence there exists an integer n_1 such that $(\epsilon \in (0,1), \epsilon = \frac{1}{2})$.

$$H(A_n, A_m, t) > 1 - \frac{1}{2} \text{ for all } m, n \geq n_1. \text{ Choose } n_2 > n_1 \text{ such that}$$

$$H(A_n, A_m, t) > 1 - \epsilon^2 \text{ for all } m, n \geq n_2.$$

Continuing in this way there exists an increasing sequence $\{n_k\}$ such that

$$H(A_n, A_m, t) > 1 - \epsilon^k \text{ for all } m, n \geq n_k.$$

Let f_{n_1} be a fixed point in A_{n_1} .

By property (3) of the Theorem (3.3) choose $f_{n_2} \in A_{n_2}$ such that

$$M'(f_{n_1}, A_{n_2}, t) = M(f_{n_1}, f_{n_2}, t).$$

Again by definition of M, M', M^* and H we get

$$\begin{aligned} M(f_{n_1}, f_{n_2}, t) &= M'(f_{n_1}, A_{n_2}, t) \\ &> M^*(A_{n_1}, A_{n_2}, t) \\ &\geq H(A_{n_1}, A_{n_2}, t) \\ &> 1 - \epsilon. \end{aligned}$$

Similarly, choose $f_{n_3} \in A_{n_3}$ such that

$$\begin{aligned} M(f_{n_2}, f_{n_3}, t) &= M'(f_{n_2}, A_{n_3}, t) \\ &\geq M^*(A_{n_2}, A_{n_3}, t) \\ &> H(A_{n_2}, A_{n_3}, t) \\ &> 1 - \epsilon^2. \end{aligned}$$

Likewise construct a sequence $\{f_{n_k}\}$ for each $f_{n_k} \in A_{n_k}$ for all k .

$$\begin{aligned} M(f_{n_k}, f_{n_{k+1}}, t) &= M'(f_{n_k}, A_{n_{k+1}}, t) \\ &\geq M^*(A_{n_k}, A_{n_{k+1}}, t) \\ &> H(A_{n_k}, A_{n_{k+1}}, t) \\ &> 1 - \epsilon^k. \end{aligned}$$

$$\text{As } \epsilon^k = \frac{1}{2^k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ we get,}$$

$M(f_{n_k}, f_{n_{k+1}}, t) \rightarrow 1$ as a result $\{f_{n_k}\}$ is a Cauchy sequence.

By lemma (3.9), since $\{f_{n_k}\}$ is a Cauchy sequence and $f_{n_k} \in A_{n_k}$ there exists a Cauchy sequence $\{g_n\}$ in $\mathcal{F}(X)$ such that $g_n \in A_n$ for all n and $f_{n_k} = g_{n_k}$ for all k .

As $(\mathcal{F}(X), M, *)$ is a complete 2-fuzzy metric space, the Cauchy sequence $\{g_n\}$ converges to g in $\mathcal{F}(X)$. Since $g_n \in A_n$ and $\{g_n\}$ converges to g , by the definition, $g \in A$ which indicates that A is non empty.

It is left to prove that A is closed. Assume f is a limit point of A there exists a sequence $\{f_j\}$ in A converging to f .

Since $f_j \in A$, there exists a sequence $\{g_n\}$ such that $\{g_n\}$ converges to f_j and $g_n \in A_n$ for each n .

Hence there exists n_1 such that $h_{n_1} \in A_{n_1}$ and $M(h_{n_1}, f_1, t) > 1 - r$ where $r \in (0, 1)$ in particular, $r = \frac{1}{2}$.

Again for $n_2 > n_1$ choose $h_{n_2} \in A_{n_2}$ such that $M(h_{n_2}, f_2, t) > 1 - r^2$.

On repeating this procedure choose an increasing sequence of integers $\{n_k\}$ such that

$$M(h_{n_k}, f_k, t) > 1 - r^k.$$

$$\begin{aligned} \text{Thus } M(h_{n_k}, f, t) &\geq M(h_{n_k}, f_k, t) * M(f_k, f, t) \\ &> (1 - r^k) * M(f_k, f, t) \end{aligned}$$

As $\{f_k\}$ converges to f , it follows that $M(f, f, t) > 1 - r_0$.

Hence $\{h_{n_k}\}$ converges to f , as every convergent sequence is a Cauchy sequence, $\{h_{n_k}\}$ is a Cauchy sequence where $h_{n_k} \in A_{n_k}$.

By lemma (3.9), there exists a Cauchy sequence, $\{g_n\}$ in $\mathcal{F}(X)$ such that $g_n \in A_n$ and $g_{n_k} = h_{n_k}$. Thus $f \in A$ and so A is closed.

Lemma 3.11:

Let $\{C_n\}$ be a sequence of 2- fuzzy totally bounded sets in $\mathcal{F}(X)$ and A be a subset of $\mathcal{F}(X)$. If for $\epsilon \in (0, 1)$ there exists a natural number N such that $A \subseteq C_N + \epsilon$ then A is 2-fuzzy totally bounded.

Proof:

Let $\epsilon \in (0, 1)$, choose N such that $A \subseteq C_N + \epsilon$. Given that C_N is a 2-fuzzy totally bounded then

$$C_N \subseteq \bigcup_{i=1}^n B(f_i, \epsilon, t).$$

On reordering f_i , assume $B(f_i, \epsilon, t) \cap A$ is nonempty for $1 \leq i \leq m$ and $B(f_i, \epsilon, t) \cap A$ is empty for $m < i$.

So choose $g_i \in B(f_i, \epsilon, t) \cap A$ for $1 \leq i \leq m$.

Suppose $f \in A$, then $f \in D_N + \epsilon$ and so $M'(f, D_N, t) > 1 - \epsilon$, by property (3) of Theorem (3.3) there exists $h \in D_N$ such that

$$M(h, f, t) = M'(f, D_N, t)$$

and $M(f, f_i, t) \geq M(f, h, t) * M(h, f_i, t)$

$$\begin{aligned} &> (1 - \epsilon) * (1 - \epsilon) \\ &= 1 - \epsilon \end{aligned}$$

Hence, $f \in B(f_i, \epsilon, t)$ for some $1 \leq i \leq m$.

Also $g_i \in B(f_i, \epsilon, t) \cap A$ implies that $M(f_i, g_i, t) > 1 - \epsilon$.

Therefore, $M(f, g_i, t) \geq M(f, f_i, t) * M(f_i, g_i, t)$

$$> (1 - \epsilon) * (1 - \epsilon)$$

$$= 1 - \epsilon.$$

Hence for each $f \in A$, we get g_i such that $f \in B(g_i, \epsilon, t)$ for $1 \leq i \leq m$.

We conclude that

$$A \subseteq \bigcup_{i=1}^m B(g_i, \epsilon, t)$$

which implies A is 2-fuzzy totally bounded.

Theorem 3.12:

If $(\mathcal{F}(X), M, *)$ is complete then $(\mathcal{K}(X), H, *)$ is complete.

Proof:

Consider a Cauchy sequence $\{C_n\}$ in $\mathcal{K}(X)$ and define a 2-fuzzy set C to consist of all $f \in \mathcal{F}(X)$ such that there exists $\{f_n\}$ converging to f such that $f_n \in C_n$ for all n . Our aim is to prove that $C \in \mathcal{K}(X)$ and $\{C_n\}$ converges to C . By lemma (3.9), C is closed and nonempty.

For given $\epsilon \in (0, 1)$, $t \in (0, 1)$ there exists a positive integer N such that

$$H(C_n, C_m, t) > 1 - \epsilon \text{ for all } m, n \geq N.$$

By theorem (3.6), $H(C_n, C_m, t) > 1 - \epsilon$ if and only if $C_m \subseteq C_n + \epsilon$ for all $m > n \geq N$.

Let $f \in C$, there exists $\{f_i\}$ in C_i converging to f . By the Theorem (3.5) $C_n + \epsilon$ is closed and $f_i \in C_n + \epsilon$ implies that $f \in C_n + \epsilon$. Hence $C \subseteq C_n + \epsilon$. By Lemma (3.9) C is a 2-fuzzy totally bounded. C is a complete fuzzy metric space as C is a closed. Since C is nonempty, 2-fuzzy complete and totally bounded thus C is 2-fuzzy compact and thus $C \in \mathcal{K}(X)$.

Let $\epsilon \in (0, 1)$, to show that $\{C_n\}$ converges to $C \in \mathcal{K}(X)$.

i.e., $H(C_n, C, t) > 1 - \epsilon$ for all $n \geq N$.

In order to prove this, we have to show that $C \subseteq C_n + \epsilon$ and $C_n \subseteq C + \epsilon$ by above discussion, $C \subseteq C_n + \epsilon$ for all $n \geq N$.

To prove the reverse inclusion let $\epsilon \in (0, 1)$ and N be a positive integer such that

$$H(C_m, C_n, t) > 1 - \epsilon \text{ for all } m, n \geq N \text{ since } \{C_n\} \text{ is a Cauchy sequence.}$$

There exists a strictly increasing sequence $\{n_i\}$ of positive integers such that

$$H(A_m, A_n, t) > \frac{1}{2^{i+1}} (1 - \epsilon) \text{ for all } m, n \geq n_i \text{ by property (3) of theorem (3.5) we get,}$$

$$C_n \subseteq C_{n_1} + \epsilon, \text{ there exists } f_{n_1} \in C_{n_1} \text{ such that } M(g, f_{n_1}, t) > 1 - \epsilon.$$

As $C_{n_1} \subseteq C_{n_2} + \epsilon$, there exists $f_{n_2} \in C_{n_2}$ such that $M(f_{n_1}, f_{n_2}, t) > 1 - \epsilon$.

Again, $C_{n_2} \subseteq C_{n_3} + \epsilon$, there exists $f_{n_3} \in C_{n_3}$ such that $M(f_{n_2}, f_{n_3}, t) > 1 - \epsilon$.

By continuing this process we obtain a sequence $\{f_{n_i}\}$ such that for all positive integers i then $f_{n_i} \in C_{n_i}$ and $M(f_{n_i}, f_{n_{i+1}}, t) > \frac{1}{2^{i+1}}(1 - \epsilon)$, by the Theorem (3.2) we find $\{f_{n_i}\}$ is a Cauchy sequence the limit ' f ' of the sequence $\{f_{n_i}\}$ is in C , by Lemma (3.9).

$$\begin{aligned} \text{Also } M(g, f_{n_i}, t) &\geq M(g, f_n, t) * M(f_{n_1}, f_{n_2}, t) * \dots * M(f_{n_{i-1}}, f_{n_i}, t) \\ &> (1 - \epsilon) * (1 - \epsilon) * \dots * (1 - \epsilon) > (1 - \epsilon) \end{aligned}$$

It follows that $M(g, f, t) > (1 - \epsilon)$ and therefore, $g \in C + \epsilon$. Thus $C_n \in C + \epsilon$ and there exists N such that $H(C_n, C, t) > (1 - \epsilon)$ for all $n \geq N$. Thus $\{C_n\}$ converges to $C \in \mathcal{K}$. Hence if $(\mathcal{F}(X), M, *)$ is complete then $(\mathcal{K}(X), H, *)$ is 2-fuzzy complete metric space.

Theorem 3.13:

If $(\mathcal{F}(X), M, *)$ is 2-fuzzy compact then $(\mathcal{K}(X), H, *)$ is 2-fuzzy compact.

Proof:

By the Theorem (3.12), $\mathcal{K}(X)$ is 2-fuzzy complete, it is enough to show that $\mathcal{K}(X)$ is 2-fuzzy totally bounded. For $\epsilon \in (0, 1)$ there exist $f_i \in \mathcal{F}(X)$ such that

$$\mathcal{F}(X) \subseteq \bigcup_{i=1}^n B(f_i, \epsilon, t).$$

Consider $\{D_K: 1 \leq k \leq 2^p - 1\}$ a collection of all balls $B(f_i, \epsilon, t)$. Since $\mathcal{F}(X)$ is compact, each D_K is compact and so $D_K \in \mathcal{K}(X)$.

To prove that

$$\mathcal{K}(X) \subseteq \bigcup_{k=1}^{2^p-1} B(D_K, \epsilon, t).$$

Let $C \in \mathcal{K}(X)$, to show that $C \in B(D_K, \epsilon, t)$.

Choose $T_C = \{i: C \cap D_K \neq \emptyset\}$ and for j choose

$$D_j = \bigcup_{K \in T_C} D_K$$

By property (2) of the Theorem (3.3), $M^*(C, D_j, t) = 1$.

Let $k \in D_j$ then there exists some $i \in T_C$ and $h \in C$ such that $k, h \in D_K$ hence, $M'(k, C, t) \geq 1 - \epsilon$. As k is arbitrary, $M'(D_j, C, t) > 1 - \epsilon$.

Hence $H(C, D_j, t) = M^*(D_j, C, t) > 1 - \epsilon$ and thus $C \subseteq D_j$, so $\mathcal{K}(X)$ is totally bounded. Therefore $\mathcal{K}(X)$ is complete and totally bounded, hence $(\mathcal{K}(X), H)$ is 2-fuzzy compact.

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