Original Article

Available online at
www.bpasjournals.com

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.176-192
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00016.X

A CLASS OF ESTIMATORS FOR ESTIMATING THE POPULATION MEAN AND VARIANCE USING AUXILIARY INFORMATION UNDER ADOPTIVE CLUSTER SAMPLING IN SAMPLE SURVEYS

Housila P. Singh[1], Anita Yadav[2,*]

Author Affiliation:
[1] School of Studies in Statistics, Vikram University, Ujjain, Madhya Pradesh 456010, India
[2] School of Studies in Statistics, Vikram University, Ujjain, Madhya Pradesh 456010, India

***Corresponding Author:**
Anita Yadav, School of Studies in Statistics, Vikram University, Ujjain, Madhya Pradesh 456010, India

E-mail: yadavd.anita@gmail.com

Received on 05.08.2018 Revised on 20.03.2019 Accepted on 07.04.2019

Abstract: For estimating the mean of finite population using information on an auxiliary variable we define the classes of estimators under adoptive cluster sampling in this paper. Expressions for their biases and mean squared errors are obtained under large sample approximation. The minimum mean squared errors of each class of estimators are also given. A similar class of estimators is defined for the variance of the estimator of the mean. A condition is obtained under which the proposed class of estimators of the variance of the estimator is minimum.

Keywords: Finite population, study variable, auxiliary variable, Bias and Mean squared error, adoptive cluster sampling.

2010 AMS Mathematics Subject Classification: 62D05, 62D99.

1. INTRODUCTION

Adoptive cluster sampling due to Thompson [16] is an efficient procedure for sampling rare and hidden clustered populations. Examples of such clustered population includes animals and plants of rare and endangered species, fisheries, uneven minerals exploration, pollution concentrations, epidemiology of sporadic diseases, noise problems, drug users, HIV and AIDS patients, criminals and hot spot investigations. The environmental populations, such as plants and animals having patchy distribution of units have been the motivation for adoptive cluster sampling, see Chaudhry and Hanif [2]. In adoptive cluster sampling, an initial sample is selected by simple random sampling without replacement (SRSWOR) of units. If the value of the study variable from a sampled unit satisfies a pre-specified condition *C*, that is, $\{i,\ y_i \geq C\}$, then the unit's neighborhood will also be added to the sample. If any other units that are "adaptively" added also satisfy the condition *C*, then their neighbourhoods are also added to the sample. This process is continued until no more units that satisfy the condition are found. The set of all units selected and all neighbouring units that satisfy the condition is known as the network. The adoptive sample units, which do not satisfy the conditions, are called edge units. A network and its associated edge units are called a cluster. If a unit is selected in the initial sample and does not satisfy the condition *C*, then there is only one unit in the network. A neighbourhood must be defined such that if unit *i* is in the neighbourhood of unit *j* then unit *j* is in the neighborhood of unit *i*. In this study neighbourhood of a unit is defined as the four spatially adjacent units, that is to the left, right, top and bottom of that unit as demonstrated in Figure 1.1, see Chutiman [3].

0	0	0	0	0
0	7	**0**	0	0
0	**0**	2	**0**	0
0	2	4	2	**0**
0	1	5*	3	**0**
0	**0**	**0**	**0**	9
0	0	0	0	0

Figure 1: The example of network where a unit neighborhood is defined as four spatially adjacent units.

It is a well established fact that the use of auxiliary information at the estimation stage improves the efficiency of an estimator of the population mean of the study variable *y*. The use of auxiliary information might be constructive to obtain better the efficiency of an estimator in adaptive cluster sampling (ACS) case. For example, in an ornithological survey, it is likely to get improved results using ACS to group a rare and clustered species. The count of particular species in a locality is the study variable *y* then the changes in food availability, habitat, or temperature would be the auxiliary variables. Another illustration may be, the count of disease affected plants is the study variable in an agricultural survey and auxiliary variable(s) might be the fertility of the ground, the cultivable region or the climate conditions, see Chaudhry and Hanif [2, pp.553 -554]. Few authors including Chao [1], Dryver and Chao [5], Chutiman and Kumphon [4], Chutiman [3] and Chaudhry and Hanif [2] have made the use of auxiliary information in estimating the population mean of the study variable *y* under adaptive cluster sampling (ACS).

In this paper we make an effort to develop a class of estimators for population mean and the variance of the Hansen and Hurwitz's estimator of the population mean of the study variable *y* using information on auxiliary variable *x*. We obtain the bias and mean squared error of the suggested class of estimators up to the first order of approximation under ACS. Conditions are also given for the minimum mean squared errors of the proposed classes of estimators.

2. NOTATIONS

Let $U=(u_1,u_2,...,u_N)$ be a finite population of *N* units. Let $(y_i,x_i), i=1,2,...,N$ denote the observation on the (study, auxiliary) variables for the unit $u_i(i=1,2,...,N)$. Suppose an initial sample of *n* units is selected with a simple random sampling without replacement (SRSWOR) scheme. Let A_i denote the network that includes unit *i* and m_i be the number of units in that network. Let w_{yi} and w_{xi} denote the average *y*-value and the average *x*- value in the network which includes unit *i* such that

$$w_{yi}=\frac{1}{m_i}\sum_{j\in A_i} y_j \text{ and } w_{xi}=\frac{1}{m_i}\sum_{j\in A_i} x_j$$

respectively. Adaptive cluster sampling can be considered as simple random sampling without replacement when the averages of networks are considered see, Thompson [15] and Dryver and Chao [5]. Let us use the notations $\overline{w}_y$ and $\overline{w}_x$ to denote the sample means of the study and auxiliary variables in the transformed population respectively, such that $\overline{w}_y=\frac{1}{n}\sum_{i=1}^{n} w_{yi}$ and $\overline{w}_x=\frac{1}{n}\sum_{i=1}^{n} w_{xi}$.

For simplicity, we assume that population size *N* is large enough as compared to sample size *n* so that finite population correction (fpc) terms are ignored i.e. $\left(1-\frac{n}{N}\right)=(1-f)\cong 1$.

We write

$s_{wy}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(w_{yi} - \bar{w}_y\right)^2$ and $s_{wx}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(w_{xi} - \bar{w}_x\right)^2$ are unbiased estimators of

$S_{wy}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(w_{yi} - \bar{Y}\right)^2$ and $S_{wx}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(w_{xi} - \bar{X}\right)^2$ respectively,

$\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}\left(w_{yi} - \bar{Y}\right)^r\left(w_{xi} - \bar{X}\right)^s, (r,s)$ being non negative integers;

$$C_{wy}^2 = \frac{S_{wy}^2}{\bar{Y}^2} = \frac{\mu_{20}}{\bar{Y}^2}, C_{wx}^2 = \frac{S_{wx}^2}{\bar{X}^2} = \frac{\mu_{02}}{\bar{X}^2}$$

$\rho_{wxy} = \frac{\mu_{11}}{S_{wy}S_{wx}}$: Correlation coefficient between w_y and w_x,

$$\lambda_{wyx} = \frac{\mu_{12}}{\bar{Y}S_{wx}^2}, \gamma_1(w_x) = \frac{\mu_{03}}{S_{wx}^3}, \beta_1(wx) = \gamma_1^2(w_x), \beta_2(wx) = \frac{\mu_{04}}{S_{wx}^4}.$$

Let $a_{wy} = \frac{\bar{w}_y}{\bar{Y}}, u_{wx} = \frac{\bar{w}_x}{\bar{X}}$ and $v_{wx} = \frac{s_{wx}^2}{S_{wx}^2}$.

$$E(a_{wy}) = E(u_{wx}) = E(v_{wx}) = 1,$$

$$E\left\{(a_{wy} - 1)^2\right\} = n^{-1}C_{wy}^2, E\left\{(u_{wx} - 1)^2\right\} = n^{-1}C_{wx}^2, E\left\{(a_{wy} - 1)(u_{wx} - 1)\right\} = n^{-1}\rho_{wyx}C_{wy}C_{wx},$$

$$E\left\{(v_{wx} - 1)^2\right\} = n^{-1}\{\beta_2(w_x) - 1\}, E\left\{(a_{wy} - 1)(u_{wx} - 1)\right\} = n^{-1}\lambda_{wyx},$$

$$E\left\{(u_{wx} - 1)(v_{wx} - 1)\right\} = n^{-1}\gamma_1(w_x)C_{wx}.$$

3. THE SUGGESTED CLASS OF ESTIMATORS BASED ON $\left(\bar{w}_x, \bar{X}\right)$

Whatever be the sample chosen, let u_{wx} assume values in the bounded, closed convex subset, Q, of one dimensional real space containing the point 'unity'. Following the procedure similar to that adopted by Srivastava [10,8] we define a class of estimators for population mean $\bar{Y}$ under ACS as

$$\bar{y}_g = \bar{w}_y g(u_{wx}) \tag{3.1}$$

where (.) is a parametric function such that $g(1) = 1$ and which satisfies the following conditions:

3(i). The function $g(u_{wx})$ is continuous and bounded in Q.

3(ii). The first and second order partial derivatives of $g(u_{wx})$ exist and are continuous and bounded in Q.

Since there are only a finite number of samples, the expectation and mean squared error (MSE) of the estimator $\bar{y}_g$ exist under the condition 3(i). Expanding the function $g(u_{wx})$ through the point 'unity' in a second order Taylor's series, we obtain

$$\bar{y}_g = \bar{w}_y\left[g(1) + (u_{wx} - 1)g_1(1) + \frac{1}{2}(u_{wx} - 1)^2 g_{11}\left(u_{wx}^*\right)\right] \tag{3.2}$$

where $u_{wx}^* = 1 + \theta(u_{wx} - 1), 0 < \theta < 1$ and $g_1(1)$ and $g_{11}(1)$ denote the first and second order partial derivatives of the function $g(u_{wx})$.

Noting in (3.2) that $g(1) = 1$ and $\bar{w}_y = \bar{Y}(1 + \in)$ and $\eta = (u_{wx} - 1)$ respectively, we have

$$\bar{y}_g = \bar{Y}(1 + \in)\left[1 + \eta g_1(1) + \frac{\eta^2}{2}g_{11}\left(u_{wx}^*\right)\right]$$

$$= \bar{Y}\left[1+\in +\eta g_1(1)+\in \eta g_1(1)+\frac{1}{2}(1+\in)\eta^2 g_{11}\left(u^*_{wx}\right)\right] \tag{3.3}$$

Taking the expectation and noting that the second order partial derivative of $g(u_{wx})$ is bounded, we obtain

$$E(\bar{y}_g) = \bar{Y} + O(n^{-1}) \tag{3.4}$$

To the first degree of approximation, the MSE of $\bar{y}_g$ is

$$MSE(\bar{y}_g) = E(\bar{y}_g - \bar{Y})^2$$

$$= \bar{Y}^2\ E\left[\in^2 + 2\in \eta g_1(1) + \eta^2 g_1^2(1)\right]$$

$$= \frac{\bar{Y}^2}{n}\left[C_{wy}^2 + C_{wx}^2 g_1^2(1) + 2\rho_{wyx} C_{wy} C_{wx} g_1(1)\right] \tag{3.5}$$

The MSE of $\bar{y}_g$ is minimized for

$$g_1(1) = -\rho_{wyx}\left(C_{wy}/C_{wx}\right) \tag{3.6}$$

$$= -k_{wyx}$$

where, $k_{wyx} = \rho_{wyx}\dfrac{C_{wy}}{C_{wx}}$.

Substitution of (3.6) in (3.5) yields the minimum MSE of $\bar{y}_g$ as

$$MSE_{min}(\bar{y}_g) = \left(\frac{S_{Wy}^2}{n}\right)\left(1-\rho_{wyx}^2\right) \tag{3.7}$$

which is the same as the approximate MSE of the linear regression estimator $\bar{y}_{wlr} = \bar{w}_y + \hat{\beta}_{wyx}\left(\bar{X} - \bar{w}_x\right)$,

where $\hat{\beta}_{wyx} = \dfrac{s_{wyx}}{s_{wx}^2}$ is the estimate of the population regression coefficient β_{wyx}, in

which $s_{wyx} = \dfrac{1}{(n-1)}\sum_{i=1}^{n}\left(w_{yi} - \bar{w}_y\right)\left(w_{xi} - \bar{w}_x\right)$ and $s_{wx}^2 = \dfrac{1}{(n-1)}\sum_{i=1}^{n}\left(w_{xi} - \bar{w}_x\right)^2$ are

unbiased estimators of $S_{wyx} = \dfrac{1}{(N-1)}\sum_{i=1}^{n}\left(w_{yi} - \bar{Y}\right)\left(w_{xi} - \bar{X}\right)$ and $S_{wx}^2 = \dfrac{1}{N-1}\sum_{i=1}^{n}\left(w_{xi} - \bar{X}\right)^2$

respectively. Thus no estimator of the proposed class of estimators $\bar{y}_g$ can have MSE up to first order of approximation smaller than the asymptotic variance/MSE of the linear regression estimator.

Any parametric function $g(u_{wx})$ satisfying the conditions 3(i) and 3(ii) above can define an estimator of $\bar{y}_g$ .The class of such estimators is very vast. The following estimators:

$$\bar{y}_1 = \bar{w}_y u_{wx}^{\alpha}$$

$$\bar{y}_2 = \bar{w}_y \frac{1}{\{1+\alpha(u_{wx}-1)\}}$$

$$\bar{y}_3 = \bar{w}_y\left[\alpha + (1-\alpha)u_{wx}^{-1}\right]$$

$$\bar{y}_4 = \bar{w}_y\left[\alpha + (1-\alpha)u_{wx}\right]$$

$$\bar{y}_5 = \bar{w}_y\ exp\left\{\alpha\left(u_{wx}^{-1}\right)\right\}$$

$$\bar{y}_6 = \bar{w}_y \, exp\left\{\frac{\alpha(u_{wx}-1)}{(u_{wx}+1)}\right\}$$

$$\bar{y}_7 = \bar{w}_y \left[\alpha \, u_{wx} + (1-\alpha)u_{wx}^{2}\right]$$

$$\bar{y}_8 = \bar{w}_y \left[\alpha u_{wx}^{-1} + (1-\alpha)u_{wx}^{-2}\right]$$

$$\bar{y}_9 = \bar{w}_y \left(\frac{a+b^*}{au_{wx}+b^*}\right)^{\alpha}$$

$$\bar{y}_{10} = \bar{w}_y \, exp\left\{\frac{a\,(1-u_{wx})}{a\,(1+u_{wx})+b^*}\right\}$$

$$\bar{y}_{11} = \bar{w}_y \, exp\left\{\frac{\alpha\,(1-u_{wx})}{1+(a-1)\,u_{wx}}\right\}$$

Etc. are members of the proposed class of estimators $\bar{y}_g$ and (α, a, b^*) $\{\text{with } b^* = b/\bar{X}\}$ are suitable chosen constants.

4. BIAS OF THE CLASS OF ESTIMATORS $\bar{y}_g$

To obtain the bias of the class of estimators $\bar{y}_g$, we will have to strengthen, the conditions on $g(u_{wx})$ assuming that the third order partial derivative also exists and is continuous and bounded. Then expanding $g(u_{wx})$ about the point 'unity' in a third order Taylor's series.

$$\bar{y}_g = \bar{Y}(1+\in)\left[1+(u_{wx}-1)g_1(1)+\frac{(u_{wx}-1)^2}{2}g_{11}(1)+\frac{(u_{wx}-1)^3}{6}g_{111}\left(u_{wx}^*\right)\right]$$

$$= \bar{Y}(1+\in)\left[1+\eta g_1(1)+\frac{\eta^2}{2}g_{11}(1)+\frac{\eta^3}{6}g_{111}\left(u_{wx}^*\right)\right]$$

$$= \bar{Y}\left[1+\in+\eta g_1(1)+\in\eta g_1(1)+\frac{\eta^2}{2}g_{11}(1)+\in\frac{\eta^2}{2}g_{11}(1)+\frac{\eta^3}{6}g_{111}\left(u_{wx}^*\right)+\in\frac{\eta^3}{6}g_{111}\left(u_{wx}^*\right)\right]$$

or

$$\left(\bar{y}_g-\bar{Y}\right)=\bar{Y}\left[\begin{array}{l}\in+\eta g_1(1)+\in\eta g_1(1)+\dfrac{\eta^2}{2}g_{11}(1)\\ +\dfrac{\in\eta^2}{2}g_{11}(1)+\dfrac{\eta^3}{6}g_{111}\left(u_{wx}^*\right)+\dfrac{\in\eta^3}{6}g_{111}\left(u_{wx}^*\right)\end{array}\right], \tag{4.1}$$

where $g^*(u_{wx})$ denotes the third order partial derivative of the function $g(u_{wx})$ about the point $u_{wx} = u_{wx}^*$.

Taking the expectation of both sides of (4.1) and retaining the terms up to the terms of order n^{-1},
we obtain the bias of $\bar{y}_g$ as

$$B\left(\bar{y}_g\right)=E\left(\bar{y}_g-\bar{Y}\right)$$

$$=\left(\frac{\overline{Y}}{n}\right)C_{wx}^{2}\left[k_{wyx}\,g_{1}(1)+\frac{1}{2}g_{11}(1)\right] \tag{4.2}$$

If we set $g_{11}(1)=2g_{1}^{2}(1)$ in (4.2) we have

$$B(\overline{y}_{g})=\frac{\overline{Y}}{n}C_{wx}^{2}g_{1}(1)\left[k_{wyx}+g_{1}(1)\right]$$

which is zero,

$$\text{if either } g_{1}(1)=0 \text{ or, } g_{1}(1)=-k_{wyx} \tag{4.3}$$

The meaning of $g_{1}(1)=0$ is that there is no involvement of u_{wx} in the estimator. The case $g_{1}(1)=-k_{wyx}$ is that value of $g_{1}(1)$ which minimizes the MSE of the estimator belonging to the class of estimators $\overline{y}_{g}$.

Thus we arrive at the following Theorem.

Theorem 4.1: If the function $g(u_{wx})$ satisfies the relation $g_{11}(1)=2g_{1}^{2}(1)$, then the estimator belonging to the class of estimators $\overline{y}_{g}$ would be almost unbiased (i.e. unbiased up to first order of approximation) with the minimum MSE.

To illustrate this result we consider the following estimator for population mean $\overline{Y}$ as

$$\overline{y}_{2}=\overline{w}_{y}\frac{\overline{X}}{\{\overline{X}+\alpha(\overline{w}_{x}-\overline{X})\}}=\overline{w}_{y}\left[1+\alpha(u_{wx}-1)\right]^{-1}$$

Here, $g(u_{wx})=\left[1+\alpha(u_{wx}-1)\right]^{-1}$,

For this function we have

$$g_{1}(1)=-\alpha\text{, and } g_{11}(1)=2\alpha^{2}$$

$$\Rightarrow g_{11}(1)=2g_{(1)}^{2}$$

which satisfies the condition stated in the Theorem 4.1.

To the first degree of approximation the bias and MSE of $\overline{y}_{2}$ are respectively given by

$$B(\overline{y}_{2})=\frac{\overline{Y}}{n}C_{wx}^{2}\left[k_{wyx}+\alpha^{2}\right] \tag{4.4}$$

$$MSE(\overline{y}_{w})=\frac{\overline{Y}^{2}}{n}\left[C_{wy}^{2}+C_{wx}^{2}\alpha^{2}-2\rho_{wyx}C_{wx}C_{wy}\alpha\right]. \tag{4.5}$$

The $MSE(\overline{y}_{w})$ is minimum when

$$\alpha=-k_{wyx} \tag{4.6}$$

Substitution of (4.6) in (4.4) and (4.5) respectively yields

$$B(\overline{y}_{2})\cong 0$$

and

$$MSE(\overline{y}_{w})=\left(S_{wy}^{2}/n\right)\left(1-\rho_{wyx}^{2}\right)$$
$$=MSE(\overline{y}_{wlr}).$$

Remark 4.1: The following known estimators are also the members of the suggested class of estimators $\overline{y}_{g}$:

$$\overline{y}_{1}^{*}=\overline{w}_{y}\text{ (usual unbiased estimator)}$$

$$\overline{y}_{2}^{*}=\overline{w}_{y}\frac{\overline{X}}{\overline{w}_{x}}=\overline{w}_{y}u_{wx}^{-1}\text{ (ratio estimator due to Dryver and Chao [5])}$$

$$\overline{y}_{3}^{*}=\overline{w}_{y}\frac{\overline{w}_{x}}{\overline{X}}=\overline{w}_{y}u_{wx}\text{ (product estimator due to Chaudhary and Hanif [2])}$$

$$\bar{y}_4^* = \bar{w}_y\left(\frac{1+C_{wx}^*}{u_{wx}+c_{wx}^*}\right) \quad \text{(Chutiman [3] estimator)}$$

$$\bar{y}_5^* = \bar{w}_y\left(\frac{\beta_2(w_x)+C_{wx}^*}{\beta_2(w_x)u_{wx}+C_{wx}^*}\right) \quad \text{(Chutiman [3] estimator)}$$

$$\bar{y}_6^* = \bar{w}_y\left(\frac{1+\beta_2^*(w_x)}{u_{wx}+\beta_2^*(w_x)}\right) \quad \text{(Chutiman [3] estimator)}$$

$$\bar{y}_7^* = \bar{w}_y\left(\frac{1+\rho_{wxwy}^*}{u_{wx}+\rho_{wxwy}^*}\right) \quad \text{(Yadav et al. [17] estimator)}$$

$$\bar{y}_8^* = \bar{w}_y\left(\frac{\bar{X}^2}{\bar{w}_x^2}\right) = \bar{w}_y u_{wx}^{-2} \quad \text{(Yadav et al. [17] estimator)}$$

$$\bar{y}_9^* = \bar{w}_y\left(\frac{\beta_2(w_x)+\beta_1^*(w_x)}{\beta_2(w_x)u_{wx}+\beta_1^*(w_x)}\right) \quad \text{(Yadav et al. [17] estimator)}$$

$$\bar{y}_{10}^* = \bar{w}_y\left(\frac{\beta_1(w_x)+\beta_2^*(w_x)}{\beta_1(w_x)u_{wx}+\beta_2^*(w_x)}\right) \quad \text{(Yadav et al. [17] estimator)}$$

$$\bar{y}_{11}^* = \bar{w}_y\, exp(1-u_{wx}) \quad \text{(Chaudhry and Hanif [2] estimator)}$$

$$\bar{y}_{12}^* = \bar{w}_y\, exp\left(\frac{1-u_{wx}}{1+u_{wx}}\right) \quad \text{(Chaudhry and Hanif [2] estimator)}$$

$$\bar{y}_{13}^* = \bar{w}_y\, exp(u_{wx}-1) \quad \text{(Chaudhry and Hanif [2] estimator)}$$

$$\bar{y}_{14}^* = \bar{w}_y\, exp\left(\frac{u_{wx}-1}{u_{wx}+1}\right) \quad \text{(Chaudhry and Hanif [2] estimator),}$$

where, $C_{wx}^* = \dfrac{C_{wx}}{\bar{X}}$, $\beta_1^*(w_x) = \dfrac{\beta_1(w_x)}{\bar{X}}$, $\beta_2^*(x) = \dfrac{\beta_2(w_x)}{\bar{X}}$, $\rho_{wxwy}^* = \dfrac{\rho_{wxwy}}{\bar{X}}$,
$\left(\bar{X}, C_{wx}, \beta_1(w_x), \beta_2(w_x), \rho_{wxwy}\right)$ are known.

Some more estimators:

$$\bar{y}_{15}^* = \bar{w}_y\left(\frac{N-nu_{wx}}{N-n}\right) \quad \text{(Srivenkataramna [14] – type estimator)}$$

$$\bar{y}_{16}^* = \bar{w}_y\left(\frac{N-n}{N-nu_{wx}}\right) \quad \text{(Srivenkataramna [14] – type estimator)}$$

$$\bar{y}_{17}^* = \bar{w}_y\left(\frac{L^*-u_{wx}}{L^*-1}\right) \quad \text{(Srivenkataramna and Tracy [13] – type estimator)}$$

$$\bar{y}_{18}^{*} = \bar{w}_y \left(\frac{L^* + 1}{L^* + u_{wx}} \right)$$ (Srivenkataramna and Tracy [13] – type estimator)

$$\bar{y}_{19}^{*} = \left(\bar{w}_y + a\right) \left(\frac{1 + L^*}{L^* + u_{wx}} \right)$$

$$\bar{y}_{20}^{*} = \left(\bar{w}_y + a\right) \left(\frac{1 + L^*}{L^* + u_{wx}} \right) - a$$

$$\bar{y}_{21}^{*} = \left(\bar{w}_y + \theta_1\right) \left(\frac{d\bar{w}_x + \theta_2}{d\bar{X} + \theta_2} \right)^{\alpha} - \theta_1$$

$$= \left(\bar{w}_y + \theta_1\right) \left(\frac{du_{wx} + \theta_2^*}{d + \theta_2^*} \right)^{\alpha} - \theta_1$$

$$\bar{y}_{22}^{*} = \bar{w}_y \frac{1}{\left(2 - u_{wx}\right)}$$ (Reddy [7] -type estimator)

etc. are members of the proposed class of estimators $\bar{y}_g$ at (3.1), where $L^* = \frac{L}{\bar{X}}, \theta_2^* = \frac{\theta_2}{\bar{X}}$ and $\left(d, \theta_1, \theta_2, L\right)$ are suitably chosen constants. The biases and MSEs of the estimators $\bar{y}_j^*$ $\left(j = 1, \ldots, 22\right)$ can easily be obtained from (4.2) and (3.5) respectively just by putting the values of $g_1(1)$ and $g_{11}(1)$.

It is easily shown that if we consider a wider class of estimators

$$\bar{y}_G = G\left(\bar{w}_y, u_{wx}\right) \tag{4.7}$$

of population mean $\bar{Y}$, where the function $G(.)$ satisfies $G\left(\bar{Y}, 1\right) = \bar{Y}$ and $G_1\left(\bar{Y}, 1\right) = 1$, $G_1\left(\bar{Y}, 1\right)$ denoting the first partial derivative of $G\left(\bar{w}_y, u_{wx}\right)$ with respect to $\bar{w}_y$, the minimum mean squared error of $\bar{y}_G$ is equal to (3.7) and not reduced. The difference type estimator $\bar{y}_d = \bar{w}_y + d\left(\bar{X} - \bar{w}_x\right)$ is a member of the class of estimators $\bar{y}_G$ but not of the class $\bar{y}_g$ defined by (3.1). In the following section we extend the class of estimators $\bar{y}_g$ defined by (3.1) to the ones which depend also upon the ratio $\left(s_{wx}^2 / S_{wx}^2\right) = v_{wx}$ and show that the asymptotic mean squared error can be lower than that attained by the class $\bar{y}_g$ defined by (3.1).

5. THE PROPOSED CLASS OF ESTIMATORS BASED ON $\left(u_{wx}, v_{wx}\right) = \left(\frac{\bar{w}_x}{\bar{X}}, \frac{s_{wx}^2}{S_{wx}^2} \right)$

Let $u_{wx} = \bar{w}_x / \bar{X}$ and $v_{wx} = \left(s_{wx}^2 / S_{wx}^2\right)$. Whatever be the sample chosen, let the values of $\left(u_{wx}, v_{wx}\right)$ lie in a bounded, closed convex subset, *S*, of the two dimensional real space containing the point (1,1). We propose a class of estimators for the population mean $\bar{Y}$ as

$$\bar{y}_h = \bar{w}_y h\left(u_{wx}, v_{wx}\right), \tag{5.1}$$

where $h\left(u_{wx}, v_{wx}\right)$ is a function of u_{wx} and v_{wx} such that $h(1,1) = 1$ such that it satisfies the following conditions:

5(i). The function $h\left(u_{wx}, v_{wx}\right)$ is continuous and bounded in *S*.

5(ii). The first and second order partial derivatives of $h\left(u_{wx}, v_{wx}\right)$ exist and are continuous and bounded in *S*.

Expanding $h(u_{wx}, v_{wx})$ about the point (1,1) in a second order Taylor's series we have that $E(\bar{y}_h) = \bar{Y} + O(n^{-1})$, and so the bias of $\bar{y}_h$ is of the order n^{-1}.

The bias and mean squared error of the class of estimators $\bar{y}_h$ to the first degree of approximation are respectively given by

$$B(\bar{y}_h) = \frac{\bar{Y}}{2n}\left[\begin{array}{l} 2\rho_{wxwy} C_{wx} C_{wy} h_1(1,1) + 2\lambda_{wyx} h_2(1,1) + C_{wx}^2 h_{11}(1,1) \\ + 2C_{wx}\gamma_1(w_x) h_{12}(1,1) + 2(\beta_2(w_x) - 1) h_{22}(1,1) \end{array}\right] \tag{5.2}$$

$$MSE(\bar{y}_h) = \frac{\bar{Y}^2}{n}\left[\begin{array}{l} C_{wy}^2 + 2\rho_{wxwy} C_{wx} C_{wy} h_1(1,1) + 2\lambda_{wyx} h_2(1,1) + C_{wx}^2 h_1^2(1,1) \\ + (\beta_2(w_x) - 1) h_2^2(1,1) + 2C_{wx}\gamma_1(w_x) h_1(1,1) h_2(1,1) \end{array}\right] \tag{5.3}$$

where

$$h_1(1,1) = \left.\frac{\partial h(u_{wx}, v_{wx})}{\partial u_{wx}}\right|_{(1,1)}, \; h_2(1,1) = \left.\frac{\partial h(u_{wx}, v_{wx})}{\partial v_{wx}}\right|_{(1,1)}, \; h_{11}(1,1) = \left.\frac{\partial^2 h(u_{wx}, v_{wx})}{\partial^2 u_{wx}}\right|_{(1,1)}$$

$$h_{12}(1,1) = \left.\frac{\partial^2 h(u_{wx}, v_{wx})}{\partial v_{wx}\partial u_{wx}}\right|_{(1,1)} \text{ and } h_{12}(1,1) = \left.\frac{\partial^2 h(u_{wx}, v_{wx})}{\partial^2 v_{wx}}\right|_{(1,1)}.$$

The $MSE(\bar{y}_h)$ at (5.3) is minimized for

$$\left.\begin{array}{l} h_1(1,1) = \dfrac{\left[\lambda_{wyx}\gamma_1(w_x) - \rho_{wxwy} C_{wy}(\beta_2(w_x) - 1)\right]}{C_{wx}(\beta_2(w_x) - \beta_1(w_x) - 1)} = h_{1(opt)}^{(1,1)} \\ h_2(1,1) = \dfrac{(\rho_{wxwy} C_{wy}\gamma_1(w_x) - \lambda_{wyx})}{(\beta_2(w_x) - \beta_1(w_x) - 1)} = h_{2(opt)}^{(1,1)} \end{array}\right\}, \text{ say} \tag{5.4}$$

Thus the resulting minimum mean squared error of $\bar{y}_h$ is given by

$$MSE_{min}(\bar{y}_h) = (\bar{Y}^2/n)\left[C_{wy}^2(1 - \rho_{wxwy}^2) - \frac{\{\rho_{wxwy} C_{wy}\gamma_1(w_x) - \lambda_{wyx}\}^2}{(\beta_2(w_x) - \beta_1(w_x) - 1)}\right] \tag{5.5}$$

Now we state the following theorem

Theorem 5.1: Up to the terms of order n^{-1},

$$MSE(\bar{y}_h) \geq (\bar{Y}^2/n)\left[C_{wy}^2(1 - \rho_{wxwy}^2) - \frac{\{\rho_{wxwy} C_{wy}\gamma_1(w_x) - \lambda_{wyx}\}^2}{(\beta_2(w_x) - \beta_1(w_x) - 1)}\right]$$

with equality holding if $h_1(1,1) = h_{1(opt)}(1,1)$ and $h_2(1,1) = h_{2(opt)}(1,1)$ where $h_{i(opt)}(1,1), i = 1,2$ is given by (5.5).

From (3.7) and (5.5) we have

$$MSE_{min}(\bar{y}_g) - MSE_{min}(\bar{y}_h) = (\bar{Y}^2/n)\frac{\{\rho_{wxwy} C_{wy}\gamma_1(w_x) - \lambda_{wyx}\}^2}{(\beta_2(w_x) - \beta_1(w_x) - 1)} \tag{5.6}$$

which is positive.

Thus the proposed class of estimators $\bar{y}_h$ is better than the class of estimators $\bar{y}_g$, with equality if and only if $\rho_{wxwy} C_{wy}\gamma_1(w_x) = \lambda_{wyx}$ .So we conclude that the proposed class of estimators $\bar{y}_h$ would be worth using when the relationship between w_y and w_x is markedly nonlinear and $(\beta_2(w_x) - \beta_1(w_x) - 1)$ is small.

Any parametric function $h(u_{wx}, v_{wx})$ such that $h_1(1,1) = 1$ and satisfying the conditions 5(i) and 5(ii) can generate an estimator of the class $\bar{y}_h$. The class of such estimators is very large. The following functions for example give some simple estimators of this class:

$$g(u_{wx}, v_{wx}) = u_{wx}^{\alpha}\ v_{wx}^{\beta}$$

$$g(u_{wx}, v_{wx}) = \frac{1+\alpha(u_{wx}-1)}{1+\beta(v_{wx}-1)}$$

$$g(u_{wx}, v_{wx}) = [1+\alpha(u_{wx}-1)+\beta(v_{wx}-1)]$$

$$g(u_{wx}, v_{wx}) = [1-\alpha(u_{wx}-1)-\beta(v_{wx}-1)]^{-1}$$

$$g(u_{wx}, v_{wx}) = a_1 u_{wx}^{\alpha} + a_2 v_{wx}^{\beta}, \qquad a_1 + a_2 = 1$$

$$g(u_{wx}, v_{wx}) = \frac{1}{1+\alpha\left(u_{wx}^{\beta}\ v_{wx}^{\delta}-1\right)}$$

The minimum values of (α, β, δ) in the estimators defined by the above functions are determined from the conditions (5.4) and with these optimum values, the asymptotic MSE is given by (5.5).

Remark 5.1. It may be noted that the MSE in (3.7) and (5.5) of the classes of estimators $\bar{y}_g$ and $\bar{y}_h$ defined by (3.1) and (5.1) respectively are attained when the optimum values of the parameters, which are functions of the unknown population values, are used. When the parameters are estimated from the data from which $\bar{w}_y$, $\bar{w}_x$ and s_{wx}^2 have been calculated, there is an extra component of mean squared errors of the estimators which have not so far been considered. To use such estimators in practice one has to use some guessed values of population parameters $(\rho_{wxwy}, C_{wy}, C_{wx}, \beta_2(w_x), \beta_1(w_x), \lambda_{wyx})$, either through the past experience or through a pilot sample survey. It may be noted that even if the values of the population parameters used in the estimators are not exactly equal to their optimum values as given by (3.6) (and (5.4)) but are close enough, the resulting estimator will be better than usual estimators, for instance see Das and Tripathi [18]. For a discussion on this point in connection with the estimation of population mean using multivariate ratio estimator, the reader is referred to Srivastava [12]. In this connection we also mention the works of Srivastava and Jhajj [9,11].

Remark 5.2. The class of estimators $\bar{y}_h$ defined by (5.1) does not include even the simple difference type estimators such as

$$\bar{y}_{h(1)} = \bar{w}_y + \alpha_1(\bar{X} - \bar{w}_x) + \alpha_2(S_{wx}^2 - s_{wx}^2) \tag{5.7}$$

$$\bar{y}_{h(2)} = \bar{w}_y + \alpha_1(S_{wx}^2 - s_{wx}^2) + \alpha_2(C_{wx}^2 - \hat{C}_{wx}^2) \tag{5.8}$$

where $\hat{C}_{wx}^2 = s_{wx}^2 / \bar{w}_x^2$. However it can be easily seen that the following class of estimators

$$\bar{y}_H = H(\bar{w}_y, u_{wx}, v_{wx}) \tag{5.9}$$

includes the estimators $\bar{y}_{h(1)}$ and $\bar{y}_{h(2)}$, where $H(\bar{w}_y, u_{wx}, v_{wx})$ is a function of $\bar{w}_y, u_{wx}, v_{wx}$ such that

$$H_1(\bar{Y},1,1) = \bar{Y} \tag{5.10}$$

$$\Rightarrow H_1(\bar{Y},1,1) = \left.\frac{\partial H(\bar{w}_y, u_{wx}, v_{wx})}{\partial\, \bar{w}_y}\right|_{(\bar{Y},1,1)} = 1.$$

It can be easily shown that to the first degree of approximation the minimum asymptotic mean squared error of $\bar{y}_H$ is equal to (5.5) and is not reduced. We note the estimators $\bar{y}_{h(1)}$ and $\bar{y}_{h(2)}$ defined by (5.7) and (5.8) respectively are members of the class of estimators $\bar{y}_H$ but not of the class of estimators $\bar{y}_h$ defined by (5.1). For the optimum values of the constants involved in the estimators $\bar{y}_{h(1)}$ and $\bar{y}_{h(2)}$, they attain the minimum mean squared error given by (5.5).

Thus we conclude that up to the first order of approximation one cannot improve upon the estimators such as

$$\bar{y}_1 = \bar{w}_y u_{wx}^{\alpha}\ v_{wx}^{\beta},$$
$$\bar{y}_2 = \bar{w}_y \frac{\{1+\alpha(u_{wx}-1)\}}{\{1+\beta(v_{wx}-1)\}},$$
$$\bar{y}_3 = \bar{w}_y\left[1+\alpha(u_{wx}-1)+\beta(v_{wx}-1)\right],$$
$$\bar{y}_4 = \bar{w}_y\left[1-\alpha(u_{wx}-1)-\beta(v_{wx}-1)\right]^{-1},$$
$$\bar{y}_5 = \left[a_1 u_{wx}^{\alpha} + a_2 v_{wx}^{\beta}\right],\quad a_1 + a_2 = 1,$$
$$\bar{y}_6 = \bar{w}_y\left[1+\alpha\left(U_{wx}^{\beta} V_{wx}^{\delta} - 1\right)\right]^{-1},$$

$\bar{y}_{h(1)}$ or $\bar{y}_{h(2)}$ by choosing more functions of u_{wx} and v_{wx} satisfying (5.10). Thus, for example, if one takes a linear combination of the exponential estimators $\bar{y}_1$ and the estimator $\bar{y}_3$,

$$\bar{y}_{h(3)} = \eta\left(\frac{\bar{w}_x}{\bar{X}}\right)^{\alpha}\left(\frac{s_{wx}^2}{S_{wx}^2}\right)^{\beta} + (1-\eta)\left\{\bar{w}_y + \alpha(u_{wx}-1)+\beta(v_{wx}-1)\right\} \tag{5.11}$$

the mean squared error will not be reduced below the expression (5.5), the estimator $\bar{y}_{h(3)}$ being a member of the class $\bar{y}_H$ defined by (5.9), see Srivastava and Jhajj [9, p.94].

6. ESTIMATION OF THE VARIANCE OF HANSEN AND HURWITZ ESTIMATOR OF THE POPULATION MEAN USING INFORMATION ON (u_{wx}, v_{wx})

Consider the notations $\bar{w}_y$ and $\bar{w}_x$ for the sample means of the study and auxiliary variables in the transformed population respectively, such that

$$\bar{w}_y = \frac{1}{n}\sum_{i=1}^{n} w_{yi} \quad \text{and} \quad \bar{w}_x = \frac{1}{n}\sum_{i=1}^{n} w_{x_i} \quad .$$

Thompson [16] envisaged an unbiased estimator for population mean $\bar{Y}$ in ACS based on the modification of the Hansen- Hurwitz estimator:

$$t_0 = \frac{1}{n}\sum_{i=1}^{n} w_{yi} = \bar{w}_y \tag{6.1}$$

The MSE/variance of $t_0 = \bar{w}_y$ is given by

$$MSE(t_0) = \frac{1-f}{n(N-1)}\sum_{i=1}^{N}\left(w_{yi} - \bar{Y}\right)$$
$$= \left(\frac{1}{n} - \frac{1}{N}\right) S_{wy}^2.$$

When N is sufficiently large so that $f \cong 0 \Rightarrow \frac{1}{N} \cong 0$ so, under this situation, the MSE/ variance of the estimator t_0 under ACS is given by

$$Var(t_0) = MSE(\bar{w}_y) \cong \frac{1}{n} S_{wy}^2 = V \ \text{(say)} \tag{6.2}$$

An unbiased estimator of $Var(t_0)$ in (6.2) is given by

$$\hat{Var}(t_0) = \frac{s_{wy}^2}{n} = \hat{V}_0 \ \text{(say)} \tag{6.3}$$

where, $s_{wy}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(w_{y_i} - \bar{w}_y\right)^2$.

The mean squared error of $\hat{V}$ to the second order approximation is given by

$$MSE\left(\hat{V}_0\right)=\frac{V^2}{n}\left(\beta_2\left(w_y\right)-1\right)=\frac{V^2}{n}\left(\lambda_{40}-1\right)=\frac{S_{wy}^2}{n^3}\left(\lambda_{40}-1\right) \tag{6.4}$$

Let $u_{wx}=\overline{w}_x/\overline{X}$ and $v_{wx}=s_{wx}^2/S_{wx}^2$. Whatever be the sample chosen, and let $\left(u_{wx},v_{wx}\right)$ assume values in a bounded closed convex subset, S of the two dimensional real space containing the point (1,1).
Motivated by Srivastava and Jhajj [9] we propose the class of estimators of the population variance V as

$$\hat{V}_t=\hat{V}_0 t\left(u_{wx},v_{wx}\right) \tag{6.5}$$

where $t\left(u_{wx},v_{wx}\right)$ is a function of u_{wx} and v_{wx} such that $t\left(1,1\right)=1$ and such that it is satisfies the following conditions:

6(i). The function $t\left(u_{wx},v_{wx}\right)$ is continuous and bounded in S.

6(ii). The first and second order partial derivatives of $t\left(u_{wx},v_{wx}\right)$ exist and are continuous and bounded in S.

To the first degree of approximation, the MSE of V_t is given by

$$MSE\left(\hat{V}_t\right)=\left(V^2/n\right)\begin{bmatrix}\left(\lambda_{40}-1\right)+C_{wx}^2t_1^2\left(1,1\right)+\left(\lambda_{04}-1\right)t_2^2\left(1,1\right)+2\lambda_{03}C_{wx}t_1\left(1,1\right)t_2\left(1,1\right)\\+2\lambda_{21}C_{wx}t_1\left(1,1\right)+2\left(\lambda_{22}-1\right)t_2\left(1,1\right)\end{bmatrix} \tag{6.6}$$

$$=\frac{S_{wy}^2}{n^3}\begin{bmatrix}\left(\lambda_{40}-1\right)+C_{wx}^2t_1^2\left(1,1\right)+\left(\lambda_{04}-1\right)t_2^2\left(1,1\right)+2\lambda_{03}C_{wx}t_1\left(1,1\right)t_2\left(1,1\right)\\+2\lambda_{21}C_{wx}t_1\left(1,1\right)+2\left(\lambda_{22}-1\right)t_2\left(1,1\right)\end{bmatrix}$$

where

$$\mu_{rs}=\frac{1}{N}\sum_{i=1}^{N}\left(w_{yi}-\overline{Y}\right)^r\left(w_{xi}-\overline{X}\right)^s$$

$$\lambda_{rs}=\frac{\mu_{rs}}{\mu_{20}^{r/2}\ \mu_{02}^{s/2}}$$, (r,s) being non negative integers.

The MSE of $\hat{V}_t$ thus is a function of $t_1\left(1,1\right)$ and $t_2\left(1,1\right)$ which is minimized for

$$\left.\begin{aligned}t_1\left(1,1\right)&=\frac{\left[\left(\lambda_{22}-1\right)\lambda_{03}-\lambda_{21}\left(\lambda_{04}-1\right)\right]}{\left(\lambda_{04}-\lambda_{03}^2-1\right)C_{wx}}\\ t_2\left(1,1\right)&=\frac{\left[\lambda_{21}\lambda_{03}-\left(\lambda_{22}-1\right)\right]}{\left(\lambda_{04}-\lambda_{03}^2-1\right)}\end{aligned}\right\} \tag{6.7}$$

Thus the resulting minimum MSE of the class of $\hat{V}_t$ is given by

$$MSE_{min}\left(\hat{V}_t\right)=\left(V^2/n\right)\left[\left(\lambda_{40}-1\right)-\frac{\left\{\left(\lambda_{22}-1\right)^2-2\lambda_{21}\lambda_{03}\left(\lambda_{22}-1\right)+\lambda_{21}^2\left(\lambda_{04}-1\right)\right\}}{\left(\lambda_{04}-\lambda_{03}^2-1\right)}\right] \tag{6.8}$$

$$=\left(S_{wy}^2/n\right)\left[\left(\lambda_{40}-1\right)-\frac{\left\{\left(\lambda_{22}-1\right)^2-2\lambda_{21}\lambda_{03}\left(\lambda_{22}-1\right)+\lambda_{21}^2\left(\lambda_{04}-1\right)\right\}}{\left(\lambda_{04}-\lambda_{03}^2-1\right)}\right]$$

Thus we state the following theorem.

Theorem 6.1: Up to the terms of order n^{-1},

$$MSE\left(\hat{V}_t\right) \geq \left(S_{wy}^2/n^2\right)\left[(\lambda_{40}-1)-\frac{\left\{(\lambda_{22}-1)^2-2\lambda_{21}\lambda_{03}(\lambda_{22}-1)+\lambda_{21}^2(\lambda_{04}-1)\right\}}{\left(\lambda_{04}-\lambda_{03}^2-1\right)}\right]$$

with equality holding if

$$t_1(1,1)=\frac{[(\lambda_{22}-1)\lambda_{03}-\lambda_{21}(\lambda_{04}-1)]}{\left(\lambda_{04}-\lambda_{03}^2-1\right)C_{wx}}$$

$$t_2(1,1)=\frac{[\lambda_{21}\lambda_{03}-(\lambda_{22}-1)]}{\left(\lambda_{04}-\lambda_{03}^2-1\right)}.$$

Any parametric function $t\left(u_{wx}, v_{wx}\right)$ satisfying the conditions 6(i) and 6(ii) can define acceptable estimator of $Var\left(t_0\right)=V$. The class of estimators $\hat{V}_t$ is very vast. Some members of the class of estimators represented by $\hat{V}_t$ are

$$\hat{V}_{t(1)}=\hat{V}_0 \quad u_{wx}^{\alpha} v_{wx}^{\beta}$$

$$\hat{V}_{t(2)}=\hat{V}_0 \quad [1+\alpha(u_{wx}-1)+\beta(v_{wx}-1)]$$

$$\hat{V}_{t(3)}=\hat{V}_0 \quad [1-\alpha(u_{wx}-1)-\beta(v_{wx}-1)]^{-1}$$

$$\hat{V}_{t(4)}=\hat{V}_0 \quad \left\{a_1 u_{wx}^{\alpha}+a_2 v_{wx}^{\beta}\right\},\ a_1+a_2=1$$

$$\hat{V}_{t(5)}=\hat{V}_0 \quad \left\{\alpha u_{wx}+(1-\alpha)\, v_{wx}^{\beta}\right\}$$

$$\hat{V}_{t(6)}=\hat{V}_0 \quad exp\{\alpha(u_{wx}-1)+\beta(v_{wx}-1)\}$$

$$\hat{V}_{t(7)}=\hat{V}_0 \quad \frac{\{1+\alpha(u_{wx}-1)\}}{\{1-\beta(v_{wx}-1)\}}$$

etc. If α and β in the above estimators are respectively given by the right hand side of the equations (6.7) then, $V_{t(j)},\ j=1,\ldots,7$ attain the lower bound of the mean squared error given by (6.8).

6.1. Bias of the Suggested Class of Estimators

To derive the bias of the estimator $\hat{V}_t$, we will have to strengthen the conditions on $t\left(u_{wx}, v_{wx}\right)$ of section 6, supposing that its third order partial derivatives also exist and are continuous and bounded. Then expanding $t\left(u_{wx}, v_{wx}\right)$ about the point (1,1) in a third order Taylor's series, taking expectation of $\hat{V}_t$ and retaining terms up to the terms of order n^{-1}, we obtain

$$B\left(\hat{V}_t\right)=E\left(\hat{V}_t\right)-V$$

$$=\frac{V}{2n}\left[\begin{array}{l}2\lambda_{21}C_{wx}t_1(1,1)+2(\lambda_{22}-1)t_2(1,1)+C_{wx}^2 t_{11}(1,1)\\ +(\lambda_{04}-1)\,t_{22}(1,1)+2\lambda_{03}C_{wx}t_{12}(1,1)\end{array}\right] \tag{6.9}$$

$$=\frac{S_{wy}^2}{2n^2}\left[\begin{array}{l}2\lambda_{21}C_{wx}t_1(1,1)+2(\lambda_{22}-1)t_2(1,1)+C_{wx}^2 t_{11}(1,1)\\ +(\lambda_{04}-1)\,t_{22}(1,1)+2\lambda_{03}C_{wx}t_{12}(1,1)\end{array}\right].$$

Thus it is observed that the bias of $\hat{V}_t$ depends also upon the second order partial derivatives of the function $t\left(u_{wx}, v_{wx}\right)$ at the point (1, 1), and hence will be different for different estimators of the class. So the biases and MSEs of the estimators belonging to the proposed class of estimators from (6.8) and (6.9) can be had by just putting the suitable values of $t_1(1,1)$, $t_2(1,1)$, $t_{11}(1,1)$ and $t_{22}(1,1)$. Further, if we consider a larger class of estimators $\hat{V}_T = T(\hat{V}_0, u_{wx}, v_{wx})$ of $Var(t_0) = V$, where $T(V,1,1) = V$ and $T_1(V,1,1) = 1$, $T_1(V,1,1)$ denoting the first partial derivative of $T(\hat{V}_0, u_{wx}, v_{wx})$ with respect to $\hat{V}_0$, the minimum mean squared error of $\hat{V}_T$ is equal to (6.8) and it is not reduced. Also the difference type estimator $\hat{V}_d = \hat{V}_0 + \alpha(u_{wx} - 1) + \beta(v_{wx} - 1)$ is a member of the class represented by $\hat{V}_T$ but not of $\hat{V}_t$ defined by (6.5). Thus, the minimum MSE of the difference type estimator is attained by estimators from the class $\hat{V}_t$.

Remark 6.1: It may sometimes happen that the information only on population mean $\overline{X}$ of the auxiliary variable *x* is available. In such a case we define a class of estimators of $Var(t_0) = V$ as

$$\hat{V}_t^* = \hat{V}_0 \; t^*(u_{wx}) \tag{6.10}$$

where $t^*(u_{wx})$ is a function of u_{wx} such that $t^*(1) = 1$ and satisfies certain regularity conditions as given for the class of estimators $\hat{V}_t$ i.e., 6(i) and 6(ii) as mentioned earlier.

To the first degree of approximation, the bias and MSE of $\hat{V}_t^*$ are respectively given by

$$B(\hat{V}_t^*) = \frac{V}{2n}\left[2\lambda_{21} C_{wx} t_1^*(1) + C_{wx}^2 t_{11}^*(1)\right] \tag{6.11}$$

$$= \frac{S_{wy}^2}{2n^2}\left[2\lambda_{21} C_{wx} t_1^*(1) + C_{wx}^2 t_{11}^*(1)\right]$$

$$MSE(\hat{V}_t^*) = \frac{V^2}{n}\left[(\lambda_{40} - 1) + 2\lambda_{21} C_{wx} t_1^*(1) + C_{wx}^2 t_1^{*2}(1)\right] \tag{6.12}$$

$$= \frac{S_{wy}^4}{n^3}\left[(\lambda_{40} - 1) + 2\lambda_{21} C_{wx} t_1^*(1) + C_{wx}^2 t_1^{*2}(1)\right]$$

where $t_1^*(1) = \left.\frac{\partial t^*(u_{wx})}{\partial u_{wx}}\right|_{u_{wx}=1}$ and $t_{11}^*(1) = \left.\frac{\partial^2 t^*(u_{wx})}{\partial u_{wx}^2}\right|_{u_{wx}=1}$.

The $MSE(\hat{V}_t^*)$ is minimum when

$$t_1^*(1) = \frac{\lambda_{21}}{C_{wx}} \tag{6.13}$$

Thus the minimum MSE of $\hat{V}_t^*$ is given by

$$MSE_{min}(\hat{V}_t^*) = \frac{V^2}{n}\left[(\lambda_{40} - 1) - \lambda_{21}^2\right] \tag{6.14}$$

Now we state the following corollary.

Corollary 6.1: Up to the terms of order n^{-1},

$$MSE(\hat{V}_t^*) \geq \frac{V^2}{n}\left[(\lambda_{40} - 1) - \lambda_{21}^2\right]$$

with equality holding if $t_{(1)}^* = -\lambda_{21}$.

The following estimators:

$$\hat{V}_{t(1)}^* = \hat{V}_0 \; u_{wx}^{\alpha}$$

$$\hat{V}_{t(2)}^* = \hat{V}_0 \; \left[1 + \alpha(u_{wx} - 1)\right]$$

$$\hat{V}_{t(3)}^{*} = \frac{\hat{V}_0}{\{1+\alpha(u_{wx}+1)\}}$$
$$\hat{V}_{t(4)}^{*} = \hat{V}_0 \ \{\alpha+(1-\alpha)u_{wx}\}$$
$$\hat{V}_{t(5)}^{*} = \hat{V}_0 \ \{\alpha+(1-\alpha)\,u_{wx}^{-1}\}$$
$$\hat{V}_{t(6)}^{*} = \hat{V}_0 \ exp\{\alpha(u_{wx}-1)\}$$

etc. are the members of the proposed class of estimators $\hat{V}_t^{*}$.

It is easily seen that the optimum values of the parameter α in all the estimators are given by the right hand side of (6.12).

The biases and MSEs of the estimators belonging to the proposed class of estimators $\hat{V}_t^{*}$ can be obtained from (6.11) and (6.12) respectively just by putting the appropriate values of $t_1(1)$ and $t_{11}(1)$.

It can be easily proved that if we consider a wider class of estimators of

$$\hat{V}_t^{*} = T^{*}(\hat{V}_0, u_{wx})$$

of the $Var(t)=V$, where the function $T^{*}(\hat{V}_0, u_{wx})$ satisfies $T^{*}(V,1)=V$ and $T_1^{*}(V,1)=1$, $T_1(V,1)$ denoting the first order partial derivative of $T^{*}(\hat{V}_0, u_{wx})$ with respect to $\hat{V}_0$, the minimum MSE of $\hat{V}_t^{*}$ is equal to (6.14) and it is not reduced. The difference type estimator $\hat{V}_d^{*} = \hat{V}_0 + \alpha(u_{wx}-1)$ is a member of the class $\hat{V}_T^{*}$ but not of the class $\hat{V}_t^{*}$.

From (6.4), (6.8) and (6.14) we have

$$MSE(\hat{V}_0) - MSE_{min}(\hat{V}_t^{*}) = \frac{V^2\lambda_{21}^2}{n} \tag{6.15}$$

$$MSE_{min}(\hat{V}_t^{*}) - MSE_{min}(\hat{V}_t) = \frac{V^2}{n}\frac{\{\lambda_{21}\lambda_{03}-(\lambda_{22}-1)^2\}}{(\lambda_{04}-\lambda_{03}^2-1)} \tag{6.16}$$

From (6.15) and (6.16) we have the inequality

$$MSE_{min}(\hat{V}_t) \le MSE_{min}(\hat{V}_t^{*}) \le MSE(\hat{V}_0) \tag{6.17}$$

It follows from (6.17) that the proposed class of estimators $\hat{V}_t$ is more efficient than the usual unbiased estimator $\hat{V}_0$ and the class of estimators $\hat{V}_t^{*}$.

Remark 6.2: When the variance S_{wx}^2 is known only, we define a family of estimators for population variance $Var(t_0)=V$ as

$$\hat{V}_t^{**} = \hat{V}_0 t^{**}(v_{wx}) \tag{6.18}$$

where $t^{**}(v_{wx})$ is a function of v_{wx} such that $t^{**}(1)=1$ and it is also satisfies the same regularity conditions 6(i) and 6(ii) as given above for the class of estimators $\hat{V}_t$.

To the first degree of approximation, the bias and MSE of the class of estimators $\hat{V}_t^{**}$ are respectively given by

$$B(\hat{V}_t^{**}) = \frac{V}{2n}\left[2(\lambda_{22}-1)t_1^{**}(1)+(\lambda_{04}-1)t_{11}^{*}(1)\right] \tag{6.19}$$
$$= \frac{S_{wy}^2}{2n^2}\left[2(\lambda_{22}-1)t_1^{**}(1)+(\lambda_{04}-1)t_{11}^{*}(1)\right]$$

$$MSE\left(\hat{V}_t^{**}\right)=\frac{V^2}{n}\left[(\lambda_{40}-1)+2(\lambda_{22}-1)t_1^{**}(1)+(\lambda_{04}-1)\,t_1^{*2}(1)\right] \tag{6.20}$$

$$=\frac{S_{wy}^4}{n^3}\left[(\lambda_{40}-1)+2(\lambda_{22}-1)t_1^{**}(1)+2(\lambda_{04}-1)\,t_1^{**2}(1)\right]$$

The $MSE\left(\hat{V}_t^{**}\right)$ at (6.20) is minimized for

$$t_1^{**}(1)=-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)} \tag{6.21}$$

Thus the resulting minimum MSE of $\hat{V}_t^{**}$ is given by

$$MSE_{min}\left(\hat{V}_t^{**}\right)=\frac{V^2}{n}\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right] \tag{6.22}$$

$$=\frac{S_{wy}^4}{n^3}\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]$$

Now we state the following corollary.

Corollary 6.2: Up to the first order of approximation,

$$MSE\left(\hat{V}_t^{**}\right)\geq\frac{V^2}{n}\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]$$

with the equality holding if

$$t_1^{**}(1)=-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}$$

Some members of the class of estimators represented by $\hat{V}_t^{**}$ are:

$$\hat{V}_{t(1)}^{**}=\hat{V}_0\ \ v_{ux}^{\alpha}$$

$$\hat{V}_{t(2)}^{**}=\hat{V}_0\ \ [1+\alpha(v_{ux}-1)]$$

$$\hat{V}_{t(3)}^{**}=\hat{V}_0\ \ [1-\alpha(v_{ux}-1)]^{-1}$$

$$\hat{V}_{t(4)}^{**}=\hat{V}_0\ \ exp\{\alpha(v_{ux}-1)\}$$

$$\hat{V}_{t(5)}^{**}=\hat{V}_0\ \ [\alpha v_{ux}+(1-\alpha)\,exp\{(v_{ux}-1)\}]$$

$$\hat{V}_{t(6)}^{**}=\hat{V}_0\ \ \left[\alpha v_{ux}^{-1}+(1-\alpha)\,exp\{(1-v_{ux})\}\right]\text{, etc.}$$

If α in the above estimators is given by the right hand side of the equation in (6.21), $\hat{V}_{t(j)}^{*}$, $j=1,\ldots,6$ attain the lower bound of the mean squared error given by (6.22).

Next, if we consider a wider class of estimators $\hat{V}_T^{**}=T^{**}\left(\hat{V}_0, v_{ux}\right)$ of the variance t_0 (i.e. V), where $T^{**}(V,1)=V$ and $T_1^{**}(V,1)=1$, $T_1^{**}(V,1)$ being the first partial derivative of $T^{**}(.)$ with respect to $\hat{V}_0$, the minimum MSE of $\hat{V}_T^{**}$ is the same as that of $\hat{V}_t^{**}$ given by (6.22) and it is not reduced. It is easily seen that the difference type estimator $\hat{V}_d^{**}=\hat{V}_0+\alpha(v_{ux}-1)$ is a member of the class represented by $\hat{V}_T^{**}$ but not of $\hat{V}_t^{**}$.

From (6.4), (6.8) and (6.22) we have

$$MSE\left(\hat{V}_0\right) - MSE_{min}\left(\hat{V}_t^{**}\right) = \frac{V^2(\lambda_{22}-1)^2}{n(\lambda_{04}-1)} \geq 0 \tag{6.23}$$

$$MSE\left(\hat{V}_t^{**}\right) - MSE_{min}\left(\hat{V}_t\right) = \frac{V^2}{n}\frac{\left[(\lambda_{22}-1)\lambda_{03} - \lambda_{21}(\lambda_{04}-1)\right]^2}{\left(\lambda_{04}-\lambda_{03}^2-1\right)\ (\lambda_{04}-1)} \geq 0 \tag{6.24}$$

Thus from (6.23) and (6.24) we have the inequality:

$$MSE_{\min}\left(\hat{V}_t\right) \leq MSE_{\min}\left(\hat{V}_t^{**}\right) \leq MSE\left(\hat{V}_0\right). \tag{6.25}$$

It follows from (6.25) that the suggested class of estimators is better than the estimators $\hat{V}_t^{**}$ and $\hat{V}_0$.

REFERENCES

[1]. Chao, C.T. (2004). Ratio estimation on adaptive cluster sampling, *Journal of Chinese Statistical Association,* 42, 307-327.

[2]. Chaudhry, M.S. and Hanif, M. (2015)**.** Generalized exponential-cum-exponential estimator in adaptive cluster sampling, *Pakistan Journal of Statistics and Operation Research*, 11(4), 553-574.

[3]. Chutiman, N. (2013). Adaptive cluster sampling using auxiliary variable, *Journal of Mathematics and Statistics,* 9(3), 249-255.

[4]. Chutiman, N. and Kumphon, B. (2008). Ratio estimator using two auxiliary variables for adaptive cluster sampling, *Thailand Statistician*, 6(2), 241-256.

[5]. Dryver, A.L. and Chao, C.T. (2007). Ratio estimators in adaptive cluster sampling, *Environmetrics*, 18, 607-620.

[6]. Hansen, M.H. and Hurwitz, W.N. (1946). The problem of non response in sample surveys, *Journal of American Statistical Association*, 41(236), 517-529.

[7]. Reddy, V.N. (1974). On a transformed ratio method of estimation, *Sankhaya C*, 36(1), 59- 70.

[8]. Srivastava, S.K. (1980). A class of estimators using auxiliary information in sample surveys, *Canad. Jour. Statist*, 8(2): 253-254.

[9]. Srivastava, S.K. and Jhajj, H.S. (1980). A class of estimators using auxiliary information for estimating finite population variance, *Sankhya C*, 42 (1-2), 87-96.

[10]. Srivastava, S.K. (1971). A generalized estimator for the mean of a finite population using multi auxiliary information, *Journal of American Statistical Association*, 66, 404-407.

[11]. Srivastava and Jhajj (1983). A Class of Estimators of the Population Mean Using Multi Auxiliary Information, *Calcutta Statistical Association*, Vol 32, issue 1-2.

[12]. Srivastava, S. K. (1966). Product estimator, *Journal of Indian Statistical Association*, 4, 29- 33.

[13]. Srivenkataramana, T. and Tracy, D.S. (1980). An alternative to ratio method in sample Surveys, *Ann. Inst. Statist. Math.*, 32(1), 111-120.

[14]. Srivenkatramna, T., (1980). A dual to ratio estimator in sample surveys, *Biometrika*, 67(1), 199-204.

[15]. Thompson (2002). Sampling, Wiley, New York.

[16]. Thompson, S.K. (1990): Adaptive cluster sampling, *Journal of American Statistical Association*, 85, 1050-1059.

[17]. Yadav, S.K., Mishra, S., Mishra, S.S. and Chutiman, N. (2016). Improved ratio estimators of population mean in adaptive cluster sampling, *Journal of Statistics Applications and Probability Letters,* 3(1), 1-6.

[18]. Das, A.K. , Tripathi, T. . (1978): Use of auxiliary information in estimating the finite population variance . *Sankhya C,* 40: 139 – 148.