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Solution of the Diophantine equation $783^x + 85^y = z^2$ *

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Abstract In this paper we consider the Diophantine equation $783^x + 85^y = z^2$, where x, y, z are non-negative integers and determine the non-negative integer solutions of this equation. Our result shows that $(x, y, z) = (1,\ 0,\ 28)$ is a unique non-negative integer solution of this equation.

Key words Catalan's Conjecture, Diophantine Equation, Solution.

2020 Mathematics Subject Classification 11D61.

1 Introduction

Diophantine equations provide the answers of the critical problems of Astrology; Mathematics; Chemistry; Cryptography and Algebra. There are various methods available in the literature for solving Diophantine equations (linear or non-linear) but there is no universal method that solves all the Diophantine equations. There are numerous methods which are very well documented and are helpful in solving linear Diophantine equations [1–5]. Aggarwal et al. [6] studied the Diophantine equation $223^x + 241^y = z^2$ and proved that this equation has no solution in the set of non negative integers. Aggarwal et al. [7] examined completely the Diophantine equation $181^x + 199^y = z^2$.
Aggarwal and Sharma [8] proved that the non-linear Diophantine equation $379^x + 397^y = z^2$ has no non-negative integer solution. The Diophantine equation $193^x + 211^y = z^2$ was examined by Aggarwal [9]. Aggarwal and Kumar [10] studied the exponential Diophantine equation $\left(13^{2m}\right) + (6r + 1)^n = z^2$and proved that this equation is not solvable in the set of non-negative integers. Aggarwal and Upadhyaya [11] considered the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$ in their study and proved that this Diophantine equation has a unique non-negative integer solution. Gupta et al. [12] studied the Diophantine equation ${M_5}^p + {M_7}^q = r^2$ with the help of arithmetic modular method.
Bhatnagar and Aggarwal [13] proved that the Diophantine equation $421^p + 439^q = r^2$ has no solution in non-negative integers. Gupta et al. [14] examined the non-linear exponential Diophantine equation $(x^a + 1)^m + \left(y^b + 1\right)^n = z^2$ and proved that this equation has no non-negative integer solution. Gupta et al. [15] studied the non-linear exponential Diophantine equation $x^\alpha + (1 + my)^\beta = z^2$. Hoque and

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Kalita [16] studied the Diophantine equation $(p^q-1)^x+p^{qy}=z^2$. Kumar et al. [17] examined the Diophantine equation $601^p+619^q=r^2$ for non-negative integer solutions.
Kumar et al. [18] studied the Diophantine equation $\left(2^{2m+1}-1\right)+(6^{r+1}+1)^n=\omega^2$ and determined that this equation has no non-negative integer solution. Kumar et al. [19] proved that the Diophantine equation $\left(7^{2m}\right)+(6r+1)^n=z^2$ is not solvable in the set of non-negative integers. The Diophantine equation $211^\alpha+229^\beta=\gamma^2$ was examined by Mishra et al. [20] and they proved that this equation is not solvable in the set of non-negative integers. Sroysang [22–26] examined various Diophantine equations such as$323^x+325^y=z^2$,$3^x+45^y=z^2$, $143^x+145^y=z^2$, $3^x+85^y=z^2$ and $4^x+10^y=z^2$ for non-negative integer solutions.
Aggarwal et al. [27] considered the Diophantine equation $143^x+45^y=z^2$ in their study and proved that this equation has a unique solution in the set of non-negative integers. Aggarwal et al. [28] studied the Diophantine equations $143^x+485^y=z^2$ for its non-negative integer solutions. Recently, Aggarwal et al. [29] proved that the Diophantine equation $143^x+85^y=z^2$ has a unique solution in the set of non-negative integers.
The main aim of this manuscript is to study the Diophantine equation $783^x+85^y=z^2$, where x, y, z are non-negative integers and to determine non-negative integer solutions of this equation.

2 Preliminaries

Proposition 2.1. *Catalan's Conjecture [21]: The Diophantine equation $l^x-m^y=1$, where $l,\ m,\ x$ and y are integers such that $\min\{l,\ m,\ x, y\}>1$, has a unique solution $(l,\ m,\ x, y)=(3,\ 2,\ 2,\ 3)$.*

Lemma 2.2. *The Diophantine equation $783^x+1=z^2$, where x, z are non-negative integers, has a unique solution $(x,\ z)=(1,\ 28)$.*

Proof. Suppose that x, z are non-negative integers such that $783^x+1=z^2$. If $x=0$, then $z^2=2$ which is impossible. Then $x\geq 1$. Now $z^2=783^x+1\geq 783^1+1=784$. Thus $z\geq 28$. Now, we consider the equation $z^2-783^x=1$. By Proposition 2.1, we have $x=1$. It follows that $z^2=784$. Hence $z=28$. □

Lemma 2.3. *The Diophantine equation $85^y+1=z^2$, where y, z are non-negative integers, has no non-negative integer solution.*

Proof. Suppose that y, z are non-negative integers such that $85^y+1=z^2$. If $y=0$, then $z^2=2$ which is impossible. Then $y\geq 1$. Now $z^2=85^y+1\geq 85^1+1=86$. Thus $z\geq 10$. Now, we consider the equation $z^2-85^y=1$. By Proposition 2.1, we have $y=1$. It follows that $z^2=86$. This is a contradiction. Hence, the Diophantine equation $85^y+1=z^2$, where y, z are non-negative integers, has no non-negative integer solution. □

3 Main results

Theorem 3.1. $(x, y, z)=(1,\ 0,\ 28)$ *is the unique non-negative integer solution of the Diophantine equation $783^x+85^y=z^2$, where x, y, z are non-negative integers.*

Proof. Let x, y, z be non-negative integers such that $783^x+85^y=z^2$. By Lemma 2.3, we have $x\geq 1$. Note that z is even. Then $z^2\equiv 0 \pmod 4$. Since $85^y\equiv 1 \pmod 4$, it follows that $783^x\equiv 3 \pmod 4$. We obtain that x is odd. Now, we consider the following two cases:

Case 1. When $y=0$, by Lemma 2.2 we obtain that $x=1$ and $z=28$.

Case 2. When $y\geq 1$, then $85^y\equiv 0 \pmod 5$. Note that $783^x\equiv 3 \pmod 5$ or $783^x\equiv 2 \pmod 5$. Then $z^2\equiv 2 \pmod 5$ or $z^2\equiv 3 \pmod 5$. In fact, $z^2\equiv 0 \pmod 5$, or, $z^2\equiv 1 \pmod 5$ or $z^2\equiv 4 \pmod 5$. This is a contradiction.

Hence, $(x, y, z)=(1,\ 0,\ 28)$ is the unique non-negative integer solution of the Diophantine equation $783^x+85^y=z^2$, where x, y, z are non-negative integers. □

Corollary 3.2. *The Diophantine equation $783^x+85^y=w^4$, where $x,\ y, w$ are non-negative integers, has no non-negative integer solution.*

Proof. Let x, y, w be non-negative integers such that $783^x + 85^y = w^4$. Let $z = w^2$. Then the equation $783^x + 85^y = w^4$ becomes $783^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1,\ 0,\ 28)$. Then $w^2 = z = 28$. This is a contradiction. Hence, the Diophantine equation $783^x + 85^y = w^4$, where x, y, w are non-negative integers, has no non-negative integer solution. □

Corollary 3.3. $(x, y, s) = (1,\ 0,\ 14)$ *is the unique non-negative integer solution of the Diophantine equation* $783^x + 85^y = 4s^2$, *where* x, y, s *are non-negative integers.*

Proof. Let x, y, s be non-negative integers such that $783^x + 85^y = 4s^2$. Let $z = 2s$. Then the equation $783^x + 85^y = 4s^2$ becomes $783^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1,\ 0,\ 28)$. Then $2s = z = 28$. Thus $s = 14$. Hence, $(x, y, s) = (1,\ 0,\ 14)$ is the unique non-negative integer solution of the Diophantine equation $783^x + 85^y = 4s^2$, where x, y, s are non-negative integers. □

Corollary 3.4. *The Diophantine equation* $783^x + 85^y = 9u^2$, *where* x, y, u *are non-negative integers, has no non-negative integer solution.*

Proof. Let $x,\ y, u$ be non-negative integers such that $783^x + 85^y = 9u^2$. Let $z = 3u$. Then the equation $783^x + 85^y = 9u^2$ becomes $783^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1,\ 0,\ 28)$. Then $3u = z = 28$.
This is a contradiction. Hence the Diophantine equation $783^x + 85^y = 9u^2$, where x, y, u are non-negative integers, has no non-negative integer solution. □

Corollary 3.5. *The Diophantine equation* $783^x + 85^y = 4v^4$, *where* x, y, v *are non-negative integers, has no non-negative integer solution.*

Proof. Let x, y, v be non-negative integers such that $783^x + 85^y = 4v^4$. Let $z = 2v^2$. Then the equation $783^x + 85^y = 4v^4$ becomes $783^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1,\ 0,\ 28)$. Then $2v^2 = z = 28$. Thus $v^2 = 14$. This is a contradiction. Hence the Diophantine equation $783^x + 85^y = 4v^4$, where x, y, v are non-negative integers, has no non-negative integer solution. □

Corollary 3.6. *The Diophantine equation* $783^x + 85^y = 9t^4$, *where* x, y, t *are non-negative integers, has no non-negative integer solution.*

Proof. Let x, y, t be non-negative integers such that $783^x + 85^y = 9t^4$. Let $z = 3t^2$. Then the equation $783^x + 85^y = 9t^4$ becomes $783^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1,\ 0,\ 28)$. Then $3t^2 = z = 28$. This is a contradiction. Hence, the Diophantine equation $783^x + 85^y = 9t^4$, where x, y, t are non-negative integers, has no non-negative integer solution. □

4 Conclusion

In this manuscript the authors successfully examined the Diophantine equation $783^x + 85^y = z^2$, where x, y, z are non-negative integers for its non-negative integer solutions and proved that $(x, y, z) = (1,\ 0,\ 28)$ is a unique non-negative integer solution of this Diophantine equation. The methodology discussed in this manuscript can be applied to solve other Diophantine equations and their systems in future.

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