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On the Exton's triple hypergeometric function X_2 of matrix arguments *

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Abstract We define the Exton's triple hypergeometric function X_2 of matrix arguments and establish some integral representations for this function which generalize the corresponding results of Choi, Hasanov and Turaev (Choi, Junesang, Hasanov, Anvar and Turaev, Mamasali, Certain integral representations of Euler type for the Exton function X_2, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 17(4), (2010), 347–354) for the matrix arguments case.

Key words Hypergeometric functions, Exton's triple hypergeometric function, matrix argument, matrix transform, real positive definite, Hermitian positive definite.

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1 Introduction

Multiple hypergeometric functions attract a lot of interest in mathematics research on account of their broad area of application in mathematical sciences and also due to their frequent applications in other phsical and engineering sciences. The study of multiple hypergeometric functions of scalar arguments has a rich history as can be seen by going through the monographs of Slater [5], Exton [13]

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and Srivastava and Karlsson [16], besides others. A number of multiple hypergeometric functions are introduced by a number of researchers at different times, an account of some of these may be found in the above works [5, 13, 16]. Although considerable work exists in the literature for multiple hypergeometric functions of scalar arguments, yet little has been achieved in the direction of studying multiple hypergeometric functions of matrix arguments. Towards the study of multiple hypergeometric functions of matrix arguments the pioneering work of Mathai [4,7] deserves a special mention. Motivated by these works of Mathai [4, 7] and the fact that the functions of matrix arguments play a crucial role in statistical distribution theory besides other fields of engineering sciences, the first author wrote his doctoral dissertation [2] (see also, [1]) more than a decade and a half earlier wherein, he made the first systematic attempt to define and study the properties of a number of multiple hypergeometric functions of matrix arguments in succession to the works of Mathai [4, 7–9]. Our study of multiple hypergeometric functions of matrix arguments continues and herein we aim to define the Exton's triple hypergeometric function X_2 of matrix arguments (see, Exton [6]) and establish some integral representations for the same, which generalize the corresponding results recently deduced by Choi et al. [14] for this function. We aim to prove our results for the case of functions of real symmetric positive definite matrix arguments by invoking the Mathai's matrix transform technique [15] and then also state without proof the corresponding results for functions of complex (Hermitiain positive definite) matrix arguments.

Talking about the organization of the paper we state the preliminary results and definitions in the first section of the paper for functions of real symmetric positive definite matrix arguments. The Exton's X_2 function of real symmetric positive definite matrix arguments is defined in section 2 and a number of integral representations are proven for this function in section 3 of the paper and the corresponding results for functions of complex matrix arguments are stated in section 4.

We now explain the notations used by us in this paper. $A > 0$ means that the matrix A is positive definite, $A^{1/2}$ represents the symmetric square root of the matrix A, A' denotes the transpose of the matrix A, Re(.) the real part of (.), while $|A|$ denotes the determinant of the matrix A. $0 < X < I$ means that $X > 0$ and $I-X > 0$, i.e.. all the eigenvalues of X lie between 0 and 1 (see, Mathai [4, p.3]). The matrix transform (M-transform) of a function $f(X)$ of a $(p \times p)$ real symmetric positive definite matrix X is defined by Mathai [15] as follows:

$$M_f(\rho) = \int_{X>0} |X|^{\rho-(p+1)/2} f(X) dX \tag{1.1}$$

for $X > 0$ and $\text{Re}(\rho) > (p-1)/2$, whenever $M_f(s)$ exists. We state below the necessary preliminary definitions and results which we will frequently use in section 3 to esatblish our results:

Theorem 1.1. *Mathai [3, (2.24), p.23] - Let X and Y be $(p \times p)$ symmetric matrices of functionally independent real variables and A a $(p \times p)$ non singular matrix of constants. Then,*

$$Y = AXA' \Rightarrow dY = |A|^{p+1} dX \tag{1.2}$$

and

$$Y = aX \Rightarrow dY = a^{p(p+1)/2} dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2. *Gamma integral (Mathai [4, (2.1.3), p.33 and (2.1.2), p. 32]) -*

$$\int_{X>0} |X|^{\alpha-(p+1)/2} e^{-tr(BX)} dX = |B|^{-\alpha} \Gamma_p(\alpha) \tag{1.4}$$

for $\text{Re}(\alpha) > (p-1)/2$*, where,*

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \Gamma(\alpha) \Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{p-1}{2}) \tag{1.5}$$

for $\text{Re}(\alpha) > (p-1)/2$ *and $tr(X)$ denotes the trace of the matrix X.*

Theorem 1.3. *Type-1 Beta Integral (Mathai [4, (2.2.2), p.34])-*

$$B_p(\alpha, \beta) = \int_{0<X<I} |X|^{\alpha-(p+1)/2} |I-X|^{\beta-(p+1)/2} dX = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\text{Re}(\alpha) > (p-1)/2, \text{Re}(\beta) > (p-1)/2.$

Theorem 1.4. *Type-2 Beta Integral (Mathai [4, (2.2.4), p.36])-*

$$B_p(\alpha,\beta)=\int_{Y>0}|Y|^{\alpha-(p+1)/2}|I+Y|^{-(\alpha+\beta)}dY=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.7}$$

for $\mathrm{Re}(\alpha)>(p-1)/2, \mathrm{Re}(\beta)>(p-1)/2$.

Theorem 1.5. *(Mathai [4, (6.13), p. 84])- For* $p=2$,

$$4^{-p\rho}\frac{\Gamma_p\left(\frac{a+1}{2}-\rho\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}-\rho\right)}{\Gamma_p\left(\frac{a+1}{2}\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}\right)}=\frac{\Gamma_p\left(a-2\rho\right)}{\Gamma_p\left(a\right)} \tag{1.8}$$

Definition 1.6. (Mathai [4, (2.3.5), p.38]) The M-transform of the multiple hypergeometric function ${}_rF_s$ of matrix argument

$${}_rF_s={}_rF_s\left[a_1,\ldots,a_r;b_1,\ldots,b_s;-X\right]$$

is given by

$$\begin{aligned}M\left({}_rF_s\right)&=\int_{X>0}|X|^{\rho-(p+1)/2}{}_rF_s\left[a_1,\ldots,a_r;b_1,\ldots,b_s;-X\right]dX\\&=\frac{\left\{\prod\limits_{j=1}^{s}\Gamma_p\left(b_j\right)\right\}}{\left\{\prod\limits_{i=1}^{r}\Gamma_p\left(a_i\right)\right\}}\frac{\left\{\prod\limits_{i=1}^{r}\Gamma_p\left(a_i-\rho\right)\right\}}{\left\{\prod\limits_{j=1}^{s}\Gamma_p\left(b_j-\rho\right)\right\}}\Gamma_p\left(\rho\right)\end{aligned} \tag{1.9}$$

for $\mathrm{Re}\left(a_i-\rho,b_j-\rho,\rho\right)>(p-1)/2,\quad i=1,\ldots,r; j=1,\ldots,s.$

Definition 1.7. (Mathai [4, c.f. (2.3.5), p.38]) The M-transform of the Lauricella function F_C of n variables

$$F_C=F_C\left(a,b;c_1,\ldots,c_n;-X_1,\ldots,-X_n\right)$$

is given by

$$M\left(F_C\right)=\frac{\left\{\prod\limits_{j=1}^{n}\left\{\Gamma_p\left(c_j\right)\Gamma_p\left(\rho_j\right)\right\}\right\}\Gamma_p\left(a-\rho_1-\ldots-\rho_n\right)\Gamma_p\left(b-\rho_1-\ldots-\rho_n\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)\left\{\prod\limits_{j=1}^{n}\Gamma_p\left(c_j-\rho_j\right)\right\}} \tag{1.10}$$

for $\mathrm{Re}\left(\rho_j,c_j-\rho_j,a-\rho_1-\ldots-\rho_n,b-\rho_1-\ldots-\rho_n\right)>(p-1)/2, j=1,\ldots,n.$

Theorem 1.8. *(Mathai [4, (2.3.6), p.38]) For a* $(p\times p)$ *real symmetric positive definite matrix* X *such that* $0<X<I$,

$${}_2F_1\left[a,b;c;-X\right]=\frac{\Gamma_p\left(c\right)}{\Gamma_p\left(a\right)\Gamma_p\left(c-a\right)}\int_0^I|Y|^{a-(p+1)/2}|I-Y|^{c-a-(p+1)/2}|I+XY|^{-b}dY \tag{1.11}$$

for $\mathrm{Re}\left(a,c-a\right)>(p-1)/2$.

Definition 1.9. (Mathai [4, (3.2.6), p.55]) For the Appell's function F_4 of matrix arguments $F_4=F_4\left[a,b;c,c';-X,-Y\right]$

$$\begin{aligned}M\left(F_4\right)&=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_4\left[a,b;c,c';-X,-Y\right]dXdY\\&\frac{\Gamma_p\left(c\right)\Gamma_p\left(c'\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)}\frac{\Gamma_p\left(a-\rho_1-\rho_2\right)\Gamma_p\left(b-\rho_1-\rho_2\right)\Gamma_p\left(\rho_1\right)\Gamma_p\left(\rho_2\right)}{\Gamma_p\left(c-\rho_1\right)\Gamma_p\left(c'-\rho_2\right)}\end{aligned} \tag{1.12}$$

for $\mathrm{Re}\left(a-\rho_1-\rho_2,b-\rho_1-\rho_2,c-\rho_1,c'-\rho_2,\rho_1,\rho_2\right)>(p-1)/2.$

2 Definitions of the Exton's function X_2 of matrix arguments

We define in this section the Exton's triple hypergeometric function X_2 of matrix arguments for the case of real symmetric positive definite matrices (see also Upadhyaya [11, 12]).

Definition 2.1. The Exton's function X_2 of matrix arguments

$$X_2 = X_2\left[\begin{array}{ccc} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{array} \mid -X, -Y, -Z\right]$$

is defined as that class of functions which has the following M-transform:

$$\begin{aligned} M(X_2) = & \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times \\ & X_2\left[\begin{array}{ccc} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{array} \mid -X, -Y, -Z\right] dXdYdZ \\ = & \frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3) \end{aligned} \tag{2.1}$$

for $\mathrm{Re}(a_1-2\rho_1-2\rho_2-\rho_3, a_2-\rho_3, c_i-\rho_i, \rho_i) > (p-1)/2, \quad i = 1, 2, 3.$

3 Integral representations of the Exton's function X_2 of matrix arguments

We now, establish a number of integral representations of the Exton's triple hypergeometric function X_2 in this section when the argument matrices are real symmetric positive definite. The following theorem gives the matrix generalization of equation (1.2) p.348 of Choi et al. [14].

Theorem 3.1.

$$\begin{aligned} & X_2\left[\begin{array}{ccc} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{array} \mid -X, -Y, -Z\right] \\ & = \frac{1}{\Gamma_p(a_1)}\int_0^\infty e^{-tr(S)}|S|^{a_1-(p+1)/2}{}_0F_1(\ ; c_1; -SXS')\times \\ & {}_0F_1(\ ; c_2; -SYS')\,{}_1F_1\left(a_2; c_3; -S^{1/2}ZS^{1/2}\right)dS \end{aligned} \tag{3.1}$$

for $\mathrm{Re}(a_1) > (p-1)/2$.

Proof. Taking the M-transform of the right side of (3.1) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively we get

$$\begin{aligned} & \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}{}_0F_1(\ ; c_1; -SXS')\times \\ & {}_0F_1(\ ; c_2; -SYS')\,{}_1F_1\left(a_2; c_3; -S^{1/2}ZS^{1/2}\right)dXdYdZ \end{aligned} \tag{3.2}$$

Invoking the transformations

$$X_1 = SXS, Y_1 = SYS, Z_1 = S^{1/2}ZS^{1/2}$$

with

$$dX_1 = |S|^{p+1}dX, dY_1 = |S|^{p+1}dY, dZ_1 = |S|^{(p+1)/2}dZ$$

(from (1.1)) and

$$|X_1| = |S|^2|X|, |Y_1| = |S|^2|Y|, |Z_1| = |S||Z|$$

in (3.2) yields

$$\begin{aligned} & |S|^{-2\rho_1-2\rho_2-\rho_3}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}\times \\ & {}_0F_1(\ ; c_1; -X_1)\,{}_0F_1(\ ; c_2; -Y_1)\,{}_1F_1(a_2; c_3; -Z_1)\,dX_1dY_1dZ_1 \end{aligned}$$

which, on writing the M-transform of the involved functions with the help of (1.9) gives

$$|S|^{-2\rho_1-2\rho_2-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(\rho_2)}{\Gamma_p(c_2-\rho_2)}\frac{\Gamma_p(c_3)\Gamma_p(\rho_3)}{\Gamma_p(c_3-\rho_3)}\frac{\Gamma_p(a_2-\rho_3)}{\Gamma_p(a_2)} \tag{3.3}$$

Substituting this expression on the right side of (3.1) gives

$$\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(\rho_2)}{\Gamma_p(c_2-\rho_2)}\frac{\Gamma_p(c_3)\Gamma_p(\rho_3)}{\Gamma_p(c_3-\rho_3)}\frac{\Gamma_p(a_2-\rho_3)}{\Gamma_p(a_2)}\times$$
$$\int_0^\infty e^{-tr(S)}|S|^{a_1-2\rho_1-2\rho_2-\rho_3-(p+1)/2}dS$$

which on integrating T by using a Gamma integral ((1.4)) gives $M(X_2)$ as given by (2.1). □

It is worthwhile to mention here that the above theorem presents a Laplace type integral (i.e., an integral involving the Laplace transform) for the function X_2, while it remains an interesting open problem to investigate a more generalized form of this result in terms of an Upadhyaya integral, i.e., an integral involving the Upadhyaya transform of the function X_2 (for more details see, Upadhyaya [17]). The following is the generalization of equation (2.1) p.348 of Choi et al. [14].

Theorem 3.2.

$$\begin{aligned}
&X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3; \end{matrix}\ \middle|\ -X,-Y,-Z\right] \\
&=\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-d)\Gamma_p(d)}\int_0^I |T|^{c_1-d-(p+1)/2}|I-T|^{d-(p+1)/2}\times \\
&X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ d; & c_2; & c_3; \end{matrix}\ \middle|\ -(I-T)^{1/2}X(I-T)^{1/2},-Y,-Z\right]dT
\end{aligned} \tag{3.4}$$

for $\mathrm{Re}(c_1-d,d)>(p-1)/2$.

Proof. On taking the M-transform of the right side of (3.4) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 we have

$$\begin{aligned}
&\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times \\
&X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ d; & c_2; & c_3; \end{matrix}\ \middle|\ -(I-T)^{1/2}X(I-T)^{1/2},-Y,-Z\right]dXdYdZ.
\end{aligned} \tag{3.5}$$

Applying the transformations,

$$X_1=(I-T)^{1/2}X(I-T)^{1/2}, Y_1=Y, Z_1=Z$$

with

$$dX_1=|I-T|^{(p+1)/2}dX, dY_1=dY, dZ_1=dZ$$

(from (1.1)) and

$$|X_1|=|I-T|\,|X|\,,|Y_1|=|Y|\,,|Z_1|=|Z|$$

in (3.5) generates

$$\begin{aligned}
&|I-T|^{-\rho_1}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}\times \\
&X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ d; & c_2; & c_3; \end{matrix}\ \middle|\ -X_1,-Y_1,-Z_1\right]dX_1dY_1dZ_1,
\end{aligned}$$

which on writing the M-transform of the X_2 function ((2.1))gives

$$|I-T|^{-\rho_1}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(d)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(d-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}.$$

Substituting this expression on the right side of (3.4) and integrating out T by using a type-1 Beta integral ((1.6)) produces $M(X_2)$ as given by (2.1). □

The following theorem gives the generalization of equation (2.2) p.348 of Choi et al. [14] and can be proved on the same lines as Theorem 3.2.

Theorem 3.3.

$$X_2\left[\begin{array}{cccc} a_1,a_1; & a_1,a_1; & a_1,a_2; & \mid -X,-Y,-Z \\ c_1; & c_2; & c_3; & \end{array}\right]$$
$$=\frac{\Gamma_p(c_3)}{\Gamma_p(c_3-d)\Gamma_p(d)}\int_0^I |T|^{c_3-d-(p+1)/2}|I-T|^{d-(p+1)/2}\times \tag{3.6}$$
$$X_2\left[\begin{array}{cccc} a_1,a_1; & a_1,a_1; & a_1,a_2; & \mid -X,-Y,-(I-T)^{1/2}Z(I-T)^{1/2} \\ c_1; & c_2; & d; & \end{array}\right]dT$$

for $\mathrm{Re}(c_3-d,d)>(p-1)/2$.

Similarly, the following theorem can also be proved:

Theorem 3.4.

$$X_2\left[\begin{array}{cccc} a_1,a_1; & a_1,a_1; & a_1,a_2; & \mid -X,-Y,-Z \\ c_1; & c_2; & c_3; & \end{array}\right]$$
$$=\frac{\Gamma_p(c_2)}{\Gamma_p(c_2-d)\Gamma_p(d)}\int_0^I |T|^{c_2-d-(p+1)/2}|I-T|^{d-(p+1)/2}\times \tag{3.7}$$
$$X_2\left[\begin{array}{cccc} a_1,a_1; & a_1,a_1; & a_1,a_2; & \mid -X,-(I-T)^{1/2}Y(I-T)^{1/2},-Z \\ c_1; & c_2; & d; & \end{array}\right]dT$$

for $\mathrm{Re}(c_2-d,d)>(p-1)/2$.

The following theorem generalizes equation (2.6) p.349 of Choi et al. [14] for the case of (2×2) matrices:

Theorem 3.5. *For* $p=2$,

$$X_2\left[\begin{array}{cccc} a_1,a_1; & a_1,a_1; & a_1,a_2; & \mid -X,-Y,-Z \\ c_1; & c_2; & c_3; & \end{array}\right]$$
$$=\frac{\Gamma_p(c_3)|I+\Lambda|^{a_3}}{\Gamma_p(a_2)\Gamma_p(c_3-a_2)}\int_0^I |S|^{a_2-(p+1)/2}|I-S|^{c_3-a_2-(p+1)/2}\left|I+S^{1/2}\Lambda S^{1/2}\right|^{-c_3}\times$$
$$\left|I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right|^{-a_1}\times$$
$$F_4\left[\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+\frac{1}{4};c_1,c_2;-4\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.\right.$$
$$\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}X\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.$$
$$\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1},-4\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.$$
$$\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}Y\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.$$
$$\left.\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}\right]dS \tag{3.8}$$

where $\mathrm{Re}(c_3-a_2,a_2)>(p-1)/2$.

Proof. We take the M-transform of the right side of (3.8) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 to obtain

$$\begin{aligned}&\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times\\&F_4\left[\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+\frac{1}{4};c_1,c_2;-4\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.\right.\\&\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}X\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.\\&\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1},-4\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.\\&\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}Y\left\{I+\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}\times\right.\\&\left.\left.(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\right\}^{-1}\right]dXdYdZ\end{aligned}\tag{3.9}$$

For a fixed Z, let us make the transformation,

$$Z_1=\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}(I+\Lambda)^{1/2}S^{1/2}ZS^{1/2}(I+\Lambda)^{1/2}\left(I+S^{1/2}\Lambda S^{1/2}\right)^{-1/2}\tag{3.10}$$

and further we apply the following transformations of the variables X and Y,

$$X_1=4(I+Z_1)^{-1}X(I+Z_1)^{-1}, Y_1=4(I+Z_1)^{-1}Y(I+Z_1)^{-1}$$

with (from (1.1))

$$dX_1=4^{p(p+1)/2}|I+Z_1|^{-(p+1)}dX, dY_1=4^{p(p+1)/2}|I+Z_1|^{-(p+1)}dY$$

and

$$|X_1|=4^p|I+Z_1|^{-2}|X|, |Y_1|=4^p|I+Z_1|^{-2}|Y|,$$

on account of which (3.9) becomes

$$\begin{aligned}&4^{-p(\rho_1+\rho_2)}\int_{Z>0}|Z|^{\rho_3-(p+1)/2}|I+Z_1|^{2\rho_1+2\rho_2-a_1}dZ\left[\int_{X_1>0}\int_{Y_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}\times\right.\\&\left.F_4\left[\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+\frac{1}{4};c_1,c_2;-X_1,-Y_1\right]dX_1dY_1\right].\end{aligned}$$

The above expression on writing the M-transform of the F_4 function gives

$$\begin{aligned}&4^{-p(\rho_1+\rho_2)}\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p\left(\frac{a_1}{2}+\frac{1}{2}-\rho_1-\rho_2\right)\Gamma_p\left(\frac{a_1}{2}+\frac{1}{4}-\rho_1-\rho_2\right)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p\left(\frac{a_1}{2}+\frac{1}{2}\right)\Gamma_p\left(\frac{a_1}{2}+\frac{1}{4}\right)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}\times\\&\int_{Z>0}|Z|^{\rho_3-(p+1)/2}|I+Z_1|^{2\rho_1+2\rho_2-a_1}dZ.\end{aligned}\tag{3.11}$$

An application of (1.8) simplifies (3.11) to the form

$$\frac{\Gamma_p(a_1-2\rho_1-2\rho_2)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}\int_{Z>0}|Z|^{\rho_3-(p+1)/2}|I+Z_1|^{2\rho_1+2\rho_2-a_1}dZ\tag{3.12}$$

Now from (3.10) it follows that

$$dZ_1=\left|I+S^{1/2}\Lambda S^{1/2}\right|^{-(p+1)/2}|I+\Lambda|^{(p+1)/2}|S|^{(p+1)/2}dZ$$

and

$$|Z_1|=\left|I+S^{1/2}\Lambda S^{1/2}\right|^{-1}|I+\Lambda|\,|S|\,|Z|$$

which renders (3.12) as

$$\frac{\Gamma_p(a_1-2\rho_1-2\rho_2)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}|S|^{-\rho_3}|I+\Lambda|^{-\rho_3}\left|I+S^{1/2}\Lambda S^{1/2}\right|^{\rho_3}\times$$
$$\int_{Z_1>0}|Z_1|^{\rho_3-(p+1)/2}|I+Z_1|^{-(a_1-2\rho_1-2\rho_2-\rho_3+\rho_3)}dZ_1. \tag{3.13}$$

Integrating out Z_1 in (3.13) by using a type-2 Beta integral ((1.7)) gives

$$|S|^{-\rho_3}|I+\Lambda|^{-\rho_3}\left|I+S^{1/2}\Lambda S^{1/2}\right|^{\rho_3}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}. \tag{3.14}$$

Substituting this expression on the right side of (3.8) yields

$$|I+\Lambda|^{a_2-\rho_3}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_2)\Gamma_p(c_3-a_2)\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}\times$$
$$\int_0^I|S|^{a_2-\rho_3-(p+1)/2}|I-S|^{c_3-a_2-(p+1)/2}\left|I+S^{1/2}\Lambda S^{1/2}\right|^{\rho_3-c_3}dS \tag{3.15}$$

On comparing the S-integral in (3.15) with (1.11), we can observe that (3.15) can be written as

$$|I+\Lambda|^{a_2-\rho_3}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$
$$ {}_2F_1[a_2-\rho_3,c_3-\rho_3;c_3-\rho_3;-\Lambda]. \tag{3.16}$$

Further, it is obvious that

$$ {}_2F_1[a_2-\rho_3,c_3-\rho_3;c_3-\rho_3;-\Lambda]={}_1F_0[a_2-\rho_3;\quad;-\Lambda]=|I+\Lambda|^{-a_2+\rho_3}. \tag{3.17}$$

With the help of (3.17), (3.16) at once gives $M(X_2)$ in conformity with (2.1), thereby proving the desired result. □

The following theorem generalizes equation (2.4) p.349 of Choi et al. [14] for the case of (2×2) matrices:

Theorem 3.6. *For* $p=2$,

$$X_2\left[\begin{matrix}a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3;\end{matrix}\mid -X,-Y,-Z\right]$$
$$=\frac{\Gamma_p(c_3)}{\Gamma_p(a_2)\Gamma_p(c_3-a_2)}\int_0^I|S|^{a_2-(p+1)/2}|I-S|^{c_3-a_2-(p+1)/2}\left|I+S^{1/2}ZS^{1/2}\right|^{-a_1}\times$$
$$F_4\left[\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+\frac{1}{4};c_1,c_2;-4\left(I+S^{1/2}ZS^{1/2}\right)^{-1}X\left(I+S^{1/2}ZS^{1/2}\right)^{-1},\right.$$
$$\left.-4\left(I+S^{1/2}ZS^{1/2}\right)^{-1}Y\left(I+S^{1/2}ZS^{1/2}\right)^{-1}\right]dS \tag{3.18}$$

where $\mathrm{Re}(s-a_3,a_3)>(p-1)/2$.

Proof. On taking the M-transform of (3.18) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 we get

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\left|I+S^{1/2}ZS^{1/2}\right|^{-a_1}\times$$
$$F_4\left[\tfrac{a_1}{2}+\tfrac{1}{2},\tfrac{a_1}{2}+\tfrac{1}{4};c_1,c_2;-4\left(I+S^{1/2}ZS^{1/2}\right)^{-1}X\left(I+S^{1/2}ZS^{1/2}\right)^{-1},\right.$$
$$\left.-4\left(I+S^{1/2}ZS^{1/2}\right)^{-1}Y\left(I+S^{1/2}ZS^{1/2}\right)^{-1}\right]dXdYdZ. \tag{3.19}$$

Applying the transformations $Z_1 = S^{1/2}ZS^{1/2}$ (which gives, $dZ_1 = |S|^{(p+1)/2}dZ$ and $|Z_1| = |S|\,|Z|$) and

$$X_1 = 4\Big(I + S^{1/2}ZS^{1/2}\Big)^{-1} X\Big(I + S^{1/2}ZS^{1/2}\Big)^{-1},$$
$$Y_1 = 4\Big(I + S^{1/2}ZS^{1/2}\Big)^{-1} Y\Big(I + S^{1/2}ZS^{1/2}\Big)^{-1}$$

along with

$$dX_1 = 4^{p(p+1)/2}|I + Z_1|^{-(p+1)}dX, dY_1 = 4^{p(p+1)/2}|I + Z_1|^{-(p+1)}dY$$

and

$$|X_1| = 4^p|I + Z_1|^{-2}\,|X|\,, |Y_1| = 4^p|I + Z_1|^{-2}\,|Y|$$

in (3.19) leads us to

$$4^{-p(\rho_1+\rho_2)}|S|^{-\rho_3}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}\times$$
$$|I + Z_1|^{-(a_1-2\rho_1-2\rho_2-\rho_3+\rho_3)}F_4\left[\frac{a_1}{2}+\frac{1}{2}, \frac{a_1}{2}+\frac{1}{4}; c_1, c_2;\, X_1, -Y_1\right]dX_1dY_1dZ_1$$

which on writing the M-transform of the F_4 function ((1.12)) and integrating out Z_1 by utilizing a type-2 Beta integral ((1.7)) gives

$$4^{-p(\rho_1+\rho_2)}|S|^{-\rho_3}\frac{\Gamma_p\left(\frac{a_1}{2}+\frac{1}{2}-\rho_1-\rho_2\right)\Gamma_p\left(\frac{a_1}{2}+\frac{1}{4}-\rho_1-\rho_2\right)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p\left(\frac{a_1}{2}+\frac{1}{2}\right)\Gamma_p\left(\frac{a_1}{2}+\frac{1}{4}\right)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}\times$$
$$\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(a_1-2\rho_1-2\rho_2)}.$$

A simple application of (1.8) reduces the last expression into the form

$$|S|^{-\rho_3}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)}$$

which on substitution on the right hand side of (3.18) and integrating out S by using a type-1 Beta integral ((1.6)) gives $M(X_2)$. □

The next theorem generalizes the result of equation (2.3) p.348 of Choi et al. [14]:

Theorem 3.7.

$$X_2\left[\begin{matrix} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{matrix}\;|-X, -Y, -Z\right]$$
$$= \frac{\Gamma_p(a)}{\Gamma_p(a_1)\Gamma_p(a-a_1)}\int_0^I |S|^{a_1-(p+1)/2}|I-S|^{a-a_1-(p+1)/2}\times$$
$$X_2\left[\begin{matrix} a, a; & a, a; & a, a_2; \\ c_1; & c_2; & c_3; \end{matrix}\;|-SXS', -SYS', -S^{1/2}ZS^{1/2}\right]dS \tag{3.20}$$

for $\mathrm{Re}\,(a-a_1, a_1) > (p-1)/2$.

Proof. This theorem can be proved by taking the M-transform of the right side of (3.20) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3, then employing the transformations $X_1 = SXS', Y_1 = SYS', Z_1 = S^{1/2}ZS^{1/2}$ and then writing the M-transform of the X_2 function involved to achieve

$$|S|^{-2\rho_1-2\rho_2-\rho_3}\frac{\Gamma_p(a-2\rho_1-2\rho_2-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)$$

which on substitution on the right side of (3.20) and integrating out S by using a type-1 Beta integral ((1.6)) gives $M(X_2)$. □

In the next theorem we prove the matrix generalization of equation (2.12) p.350 of Choi et al. [14]:

Theorem 3.8. *For $p = 2$,*

$$X_2\left[\begin{array}{ccc} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{array} \mid -X, -Y, -Z\right]$$

$$= \frac{\Gamma_p(a_1+a_2)}{\Gamma_p(a_1)\Gamma_p(a_2)} \int_0^I |S|^{a_1-(p+1)/2}|I-S|^{a_2-(p+1)/2}\times$$

$$F_C^{(3)}\left[\frac{a_1+a_2}{2}+\frac{1}{2}, \frac{a_1+a_2}{2}+\frac{1}{4}; c_1, c_2, c_3; -4SXS', -4SYS',\right.$$

$$\left. -4(I-S)^{1/2}S^{1/2}ZS^{1/2}(I-S)^{1/2}\right] dS \tag{3.21}$$

where $\mathrm{Re}(a_1, a_2) > (p-1)/2$.

Proof. Taking the M-transform of the right side of (3.21) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we have

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$$F_C^{(3)}\left[\frac{a_1+a_2}{2}+\frac{1}{2}, \frac{a_1+a_2}{2}+\frac{1}{4}; c_1, c_2, c_3; -4SXS', -4SYS',\right. \tag{3.22}$$

$$\left. -4(I-S)^{1/2}S^{1/2}ZS^{1/2}(I-S)^{1/2}\right] dXdYdZ$$

which on the application of the transformations

$$X_1 = 4SXS', Y_1 = 4SYS', Z_1 = 4(I-S)^{1/2}S^{1/2}ZS^{1/2}(I-S)^{1/2}$$

along with

$$dX_1 = 4^{p(p+1)/2}|S|^{p+1}dX, dY_1 = 4^{p(p+1)/2}|S|^{p+1}dY,$$
$$dZ_1 = 4^{p(p+1)/2}|I-S|^{(p+1)/2}|S|^{(p+1)/2}dZ$$

and

$$|X_1| = 4^p|S|^2|X|, |Y_1| = 4^p|S|^2|Y|, |Z_1| = 4^p|I-S||S||Z|$$

and then writing the M-transform of the Lauricella function $F_C^{(3)}$ ((1.10)) produces

$$4^{-p(\rho_1+\rho_2+\rho_3)}|S|^{-2\rho_1-2\rho_2-\rho_3}|I-S|^{-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$

$$\frac{\Gamma_p\left(\frac{a_1+a_2}{2}+\frac{1}{2}-\rho_1-\rho_2-\rho_3\right)\Gamma_p\left(\frac{a_1+a_2}{2}+\frac{1}{4}-\rho_1-\rho_2-\rho_3\right)}{\Gamma_p\left(\frac{a_1+a_2}{2}+\frac{1}{2}\right)\Gamma_p\left(\frac{a_1+a_2}{2}+\frac{1}{4}\right)}.$$

The above expression on an application of (1.8) reduces to

$$|S|^{-2\rho_1-2\rho_2-\rho_3}|I-S|^{-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$

$$\frac{\Gamma_p(a_1+a_2-2\rho_1-2\rho_2-2\rho_3)}{\Gamma_p(a_1+a_2)}$$

which on substitution on the right side of (3.21) and then integrating out S by the application of a type-1 Beta integral ((1.6)) gives $M(X_2)$ in agreement with (2.1). □

4 Corresponding results for functions of complex matrix argument

In this section we write the analogous results for definition 2.1 and some of the results proven in section 3, when the argument matrices are Hermitian positive definite. It is our customary to state these parallel results in the light of the remarks made by the first author in section 4 of his paper [12, pp. 213–215]. We emphasize that all the matrices appearing in this section of the paper are $(p \times p)$ complex Hermitian positive definite matrices and for the information of the interested reader we refer to Mathai [7, Chapter 3] where the results corresponding to the Jacobians of (complex) matrix transformations and the corresponding definitions of the type-1 and type-2 Beta integrals of complex matrix arguments can be found. We also mention that we use the same notation here for complex matrices as we have used in the preceding sections of this paper, whereas in the work of Mathai [7], the complex matrices are represented by placing a tilde ($\sim$) sign over the symbol of the matrix. Another important point to be noted in this context is that the complex analogues of the results of the Theorems 3.5, 3.6 and 3.8 have a different structure (see, Mathai [7, Chapter 6, p.399]) so we do not be give them here. It is well established in the literature that the complex analogues of the relevant definition (i.e., def.2.1) and results proved in the preceding sections 2 and 3 can be easily written down by replacing the expression $(p+1)/2$ appearing in the power of the determinants of the various matrices involved in the integrands of the results deduced above by p and the condition of convergence of the integral, which in the case of real symmetric positive definite matrices is $\mathrm{Re}\,(.) > (p-1)/2$, then the same has to be replaced by the expression $\mathrm{Re}\,(.) > (p-1)$ in the complex case (see Mathai [7, pp. 364–365] and see also Mathai and Provost [10]). We also point out that the complex analogues of the definitions and results given in section 1 above from (1.1) through (1.12) except the result of Theorem 1.5 ((1.8)) are available in Mathai [7, Chapters 3 and 6]. Applying the complex analogues of the various preliminary definitions and results mentioned in section 1 of this paper from the monograph of Mathai [7, Chapter3] and following the parallel steps as used by us in deriving the results in section 3 above the complex analogues of the above results, which we list below, can be easily deduced. A detailed and vivid description regarding this can be found in Mathai [7, Chapters 5 and 6]. One thing ought to be remarked here is that in this section of the paper $|A|$ stands for the absolute value of the determinant of the matrix A of complex elements.

The following is the definition of the Exton's triple hypergeometric function X_2 of complex arguments which is simply the complex analogue of the Definition 2.1:

Definition 4.1. The Exton's triple hypergeometric function X_2 of complex matrix arguments

$$X_2 = X_2\left[\begin{matrix} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X, -Y, -Z\right]$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$\begin{aligned} M(X_2) = & \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-p}|Y|^{\rho_2-p}|Z|^{\rho_3-p}\times \\ & X_2\left[\begin{matrix} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X, -Y, -Z\right] dX dY dZ \\ = & \frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\,\Gamma_p(a_2-\rho_3)\,\Gamma_p(c_1)\,\Gamma_p(c_2)\,\Gamma_p(c_3)}{\Gamma_p(a_1)\,\Gamma_p(a_2)\,\Gamma_p(c_1-\rho_1)\,\Gamma_p(c_2-\rho_2)\,\Gamma_p(c_3-\rho_3)}\Gamma_p(\rho_1)\,\Gamma_p(\rho_2)\,\Gamma_p(\rho_3) \end{aligned} \tag{4.1}$$

for $\mathrm{Re}\,(a_1-2\rho_1-2\rho_2-\rho_3, a_2-\rho_3, c_i-\rho_i, \rho_i) > (p-1)\,, \quad i=1,2,3..$

The following Theorems 4.2 – 4.5 are respectively the complex analogues of Theorems 3.1 – 3.4 and Theorem 4.6 is the complex analogue of Theorem 3.7.

Theorem 4.2.

$$\begin{aligned} & X_2\left[\begin{matrix} a_1, a_1; & a_1, a_1; & a_1, a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X, -Y, -Z\right] \\ & = \frac{1}{\Gamma_p(a_1)}\int_0^\infty e^{-tr(S)}|S|^{a_1-p}\,{}_0F_1\left(\ ;c_1;-SXS'\right)\times \\ & {}_0F_1\left(\ ;c_2;-SYS'\right){}_1F_1\left(a_2;c_3;-S^{1/2}ZS^{1/2}\right) dS \end{aligned} \tag{4.2}$$

for $\mathrm{Re}\,(a_1) > (p-1)\,.$

Theorem 4.3.

$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X,-Y,-Z\right]$$
$$= \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-d)\,\Gamma_p(d)} \int_0^I |T|^{c_1-d-p}|I-T|^{d-p} \times \tag{4.3}$$
$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ d; & c_2; & c_3; \end{matrix} \mid -(I-T)^{1/2}X(I-T)^{1/2},-Y,-Z\right] dT$$

for $\operatorname{Re}(c_1-d,d) > (p-1)$.

Theorem 4.4.

$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X,-Y,-Z\right]$$
$$= \frac{\Gamma_p(c_3)}{\Gamma_p(c_3-d)\,\Gamma_p(d)} \int_0^I |T|^{c_3-d-p}|I-T|^{d-p} \times \tag{4.4}$$
$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & d; \end{matrix} \mid -X,-Y,-(I-T)^{1/2}Z(I-T)^{1/2}\right] dT$$

for $\operatorname{Re}(c_3-d,d) > (p-1)$.

Theorem 4.5.

$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X,-Y,-Z\right]$$
$$= \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-d)\,\Gamma_p(d)} \int_0^I |T|^{c_2-d-p}|I-T|^{d-p} \times \tag{4.5}$$
$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & d; \end{matrix} \mid -X,-(I-T)^{1/2}Y(I-T)^{1/2},-Z\right] dT$$

for $\operatorname{Re}(c_2-d,d) > (p-1)$.

Theorem 4.6.

$$X_2\left[\begin{matrix} a_1,a_1; & a_1,a_1; & a_1,a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -X,-Y,-Z\right]$$
$$= \frac{\Gamma_p(a)}{\Gamma_p(a_1)\,\Gamma_p(a-a_1)} \int_0^I |S|^{a_1-p}|I-S|^{a-a_1-p} \times \tag{4.6}$$
$$X_2\left[\begin{matrix} a,a; & a,a; & a,a_2; \\ c_1; & c_2; & c_3; \end{matrix} \mid -SXS',-SYS',-S^{1/2}ZS^{1/2}\right] dS$$

for $\operatorname{Re}(a-a_1,a_1) > (p-1)$.

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