Bull. Pure Appl. Sci. Sect. E Math. Stat.
39E(2), 277–288 (2020)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2020.00029.6

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

On the characteristic roots and heart of a class of rhotrices over a finite field *

P.L. Sharma[1,†], Arun Kumar[2] and Arun Kumar Sharma[3]

1. Department of Mathematics and Statistics, Himachal Pradesh University, Shimla, India.
2. Department of Mathematics, Government Post Graduate College, Ghumarwin, Bilaspur, India.
3. Department of Computer Science and Engineering, National Institute of Technology, Hamirpur, Himachal Pradesh, India.

1. E-mail: plsharma1964@gmail.com

2. E-mail: arunch.925@gmail.com , 3. E-mail: arunksharma96@gmail.com

Abstract Matrices play an important role in various branches of mathematics such as coding theory, combinatorics and cryptography. Rhotrices are represented by coupled matrices. The use of rhotrices in cryptography doubles the security of messages which travel over insecure channels. We consider rhotrices of 3-dimension and derive some properties related to their characteristic roots. Further, we take a special class of rhotrices of n-dimension and discuss its properties.

Key words Rhotrix, Finite field, Coupled matrix, Eigen values.

2020 Mathematics Subject Classification 15A09, 20H30, 11T71.

1 Introduction

The concept of rhotrix was introduced by Ajibade [2] in 2003. A rhotrix is defined as a mathematical array, which is in some way, between (2×2)-dimension

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and (3×3)-dimension

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

matrices. The rhotrix of dimension three is defined by

$$R_3 = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a,\ b,\ c,\ d,\ e\ \in \Re \right\} \tag{1.1}$$

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received January 21, 2019 / Revised September 29, 2020 / Accepted October 28, 2020. Online First Published on December 26, 2020 at https://www.bpasjournals.com/.
†Corresponding author P.L. Sharma, E-mail: plsharma1964@gmail.com

Here, $h(R_3)= c$ is the heart of rhotrix and all the entries of rhotrix are real numbers. Ajibade [2] also discussed the following operations of addition and scalar multiplication.

Let $R_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & h(R_5) & h & i \\ & j & k & l & \\ & & m & & \end{array} \right\rangle$ and $Q_5 = \left\langle \begin{array}{ccccc} & & n & & \\ & o & p & q & \\ r & s & h(Q_5) & u & v \\ & w & x & y & \\ & & z & & \end{array} \right\rangle$ be 5-dimensional rhotrices, then the addition of these rhotrices is defined as

$$R_5 + Q_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & h(R_5) & h & i \\ & j & k & l & \\ & & m & & \end{array} \right\rangle + \left\langle \begin{array}{ccccc} & & n & & \\ & o & p & q & \\ r & s & h(Q_5) & u & v \\ & w & x & y & \\ & & z & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & a+n & & \\ & b+o & c+p & d+q & \\ e+r & f+s & h(R_5)+h(Q_5) & h+u & i+v \\ & j+w & k+x & l+y & \\ & & m+z & & \end{array} \right\rangle. \tag{1.2}$$

The scalar multiplication $\alpha\, R_5$ for the real scalar α is defined as

$$\alpha\, R_5 = \alpha \left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & h(R_5) & h & i \\ & j & k & l & \\ & & m & & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & \alpha a & & \\ & \alpha b & \alpha c & \alpha d & \\ \alpha e & \alpha f & \alpha h(R_5) & \alpha h & \alpha i \\ & \alpha j & \alpha k & \alpha l & \\ & & \alpha m & & \end{array} \right\rangle. \tag{1.3}$$

The row-column multiplication of rhotrices is discussed in [6] as follows:

For $R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $Q_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$,

$$R_3\, o\, Q_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & ch & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

A method of converting a rhotrix to a coupled matrix is discussed in [8]. Various problems involving the $m \times m$ and the $(m-1) \times (m-1)$ matrices can be easily solved by coupled matrix simultaneously, see [4, 15]. The heart-oriented multiplication is discussed in [1, 5].

The row-column multiplication of n-dimensional rhotrices is discussed by Sani [7]. The rhotrix of n-dimension is given by

$$R_n = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ & .. & .. & .. & .. & .. & \\ a_{n1} & .. & .. & .. & .. & .. & a_{1n} \\ & .. & .. & .. & .. & .. & \\ & a_{n\,n-2} & c_{n-1n-2} & a_{n-1n-1} & c_{n-2n-1} & a_{n-2\,n} & \\ & & a_{n\,n-1} & c_{n-1n-1} & a_{n-1\,n} & & \\ & & & a_{nn} & & & \end{array} \right\rangle = \langle a_{ij},\, c_{lk} \rangle,$$

The total number of elements in R_n (n is always an odd number ≥ 3) is given by $|R_n| = \frac{n^2+1}{2}$, where $R_n = \langle a_{ij},\, c_{lk} \rangle$is the coupled matrix with $i,\, j = 1,\ 2,\ ...,t;\ l,\ k = 1,\ 2,\ ...,t-1$ for $t = \frac{n+1}{2}$.

Further, for two coupled rhotrices $R_n{=}\langle a_{i_1j_1},\ c_{l_1k_1}\rangle$ and $Q_n{=}\langle b_{i_2j_2},\ d_{l_2k_2}\rangle$, the multiplication is defined as

$$R_n\, o\, Q_n = \langle a_{i_1j_1},\ c_{l_1k_1}\rangle\ o\ \langle b_{i_2j_2},\ d_{l_2k_2}\rangle = \left\langle \sum_{i_2j_1=1}^{t} (a_{i_1j_1}\, b_{i_2j_2})\ ,\ \sum_{l_2k_1=1}^{t-1} (c_{l_1k_1}\, d_{l_2k_2}) \right\rangle. \tag{1.4}$$

The inverse of the rhotrix

$$R_3 = \left\langle \begin{matrix} & a & \\ b & e & d \\ & c & \end{matrix} \right\rangle$$

is defined as

$$R_3^{-1} = \left\langle \begin{matrix} & \frac{c}{ac-bd} & \\ -\frac{b}{bd-ac} & \frac{1}{e} & -\frac{d}{bd-ac} \\ & \frac{a}{ac-bd} & \end{matrix} \right\rangle. \tag{1.5}$$

Matrices are important in cryptography. Some cryptosystems are based on matrices, such as Hill cipher, see [3]. The rhotrices are represented as coupled matrices. Therefore, the use of rhotrix in cryptography means the use of the double matrices which increase the security of the cipher. The properties of rhotrices are discussed in [9–14, 16].
We discuss some properties of rhotrices $R_n = \langle a_{ij},\ c_{lk}\rangle$ over a finite field $\mathbb{F}_2$ which satisfies the following condition:

$$\mathbf{P}\ : \alpha_{ij} = \begin{cases} 1 & \text{if } i+j = \text{even}, \\ 0 & \text{if } i+j = \text{odd}. \end{cases} \tag{1.6}$$

2 The main results

Theorem 2.1. *Let R be a rhotrix of 3-dimension which satisfies the condition* **P** *of (1.6) and $\lambda_1,\ \lambda_2,\ \lambda_3$ be the characteristic roots of R. Then $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2$ are the characteristic roots of R^2 over the finite field $\mathbb{F}_2$.*

Proof. Let

$$R = \left\langle \begin{matrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{matrix} \right\rangle \tag{2.1}$$

be the rhotrix satisfying the condition **P** of (1.6). Now, the characteristic roots of R are given by

$$|R - \lambda I| = 0,$$

That is,

$$\left| \begin{matrix} & a_{11}-\lambda & \\ 0 & c_{11}-\lambda & 0 \\ & a_{22}-\lambda & \end{matrix} \right| = 0.$$

This gives

$$\lambda^3 + (a_{11} + a_{22} + c_{11})\,\lambda^2 + (a_{11}c_{11} + a_{22}c_{11} + a_{22}a_{11})\,\lambda + (a_{11}a_{22}c_{11}) = 0. \tag{2.2}$$

Since, $\lambda_1, \lambda_2, \lambda_3$ are the characteristic roots of R, therefore, there exists a non-zero vector X such that

$$RX = \lambda X,$$

where, $X = (x_1, x_2, x_3)$ is a non-zero vector corresponding to the characteristic roots $\lambda_1, \lambda_2, \lambda_3$.
Thus, we get

$$\left\langle \begin{matrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{matrix} \right\rangle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.3}$$

Multiplying both sides by R, we get

$$\left\langle \begin{matrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{matrix} \right\rangle \left\langle \begin{matrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{matrix} \right\rangle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\langle \begin{matrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{matrix} \right\rangle \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

that is,

$$\left\langle \begin{array}{ccc} & a_{11}a_{11} & \\ 0 & c_{11}c_{11} & 0 \\ & a_{22}a_{22} & \end{array} \right\rangle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{array} \right\rangle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \tag{2.4}$$

Using (2.3) in (2.4), we get

$$\begin{aligned} \left\langle \begin{array}{ccc} & a_{11}a_{11} & \\ 0 & c_{11}c_{11} & 0 \\ & a_{22}a_{22} & \end{array} \right\rangle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \left(\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right), \\ &= \left(\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \right)^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\ &= \left(\begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned} \tag{2.5}$$

Since, $x = (x_1, x_2, x_3)$ is a non-zero vector, therefore, it is clear from the above equation that $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the characteristic roots of the rhotrix R^2. □

Theorem 2.2. *Let R be a rhotrix of 3-dimension satisfying the condition* **P** *of (1.6). Then there exists an invertible rhotrix S, such that $S^{-1}RS$ and R have the same characteristic roots (eigenvalues).*

Proof. Let the rhotrix be

$$R = \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{array} \right\rangle .$$

Let $\lambda_1, \ \lambda_2, \ \lambda_3$ be its characteristic roots. Therefore,

$$\lambda^3 + (a_{11} + a_{22} + c_{11})\,\lambda^2 + (a_{11}c_{11} + a_{22}c_{11} + a_{22}a_{11})\,\lambda + (a_{11}a_{22}c_{11}) = 0.$$

Let the rhotrix

$$S = \left\langle \begin{array}{ccc} & b_{11} & \\ 0 & d_{11} & 0 \\ & b_{22} & \end{array} \right\rangle$$

be non-singular.
Therefore, the inverse of S rhotrix is given by

$$S^{-1} = \left\langle \begin{array}{ccc} & \frac{1}{b_{11}} & \\ 0 & \frac{1}{d_{11}} & 0 \\ & \frac{1}{b_{22}} & \end{array} \right\rangle . \tag{2.6}$$

Now,

$$\begin{aligned} S^{-1}RS &= \left\langle \begin{array}{ccc} & \frac{1}{b_{11}} & \\ 0 & \frac{1}{d_{11}} & 0 \\ & \frac{1}{b_{22}} & \end{array} \right\rangle \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{array} \right\rangle \left\langle \begin{array}{ccc} & b_{11} & \\ 0 & d_{11} & 0 \\ & b_{22} & \end{array} \right\rangle , \\ &= \left\langle \begin{array}{ccc} & \frac{1}{b_{11}} & \\ 0 & \frac{1}{d_{11}} & 0 \\ & \frac{1}{b_{22}} & \end{array} \right\rangle \left\langle \begin{array}{ccc} & a_{11}b_{11} & \\ 0 & c_{11}d_{11} & 0 \\ & a_{22}b_{22} & \end{array} \right\rangle , \\ &= \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{array} \right\rangle . \end{aligned} \tag{2.7}$$

Therefore, the characteristic equation of $S^{-1}RS$ is given by

$$\left| S^{-1}RS - \lambda \right| = 0,$$

i.e.,

$$\begin{vmatrix} & a_{11}-\lambda & \\ 0 & c_{11}-\lambda & 0 \\ & a_{22}-\lambda & \end{vmatrix} = 0,$$

which gives

$$\lambda^3 + (a_{11}+a_{22}+c_{11})\,\lambda^2 + (a_{11}c_{11}+a_{22}c_{11}+a_{22}a_{11})\,\lambda + (a_{11}a_{22}c_{11}) = 0. \tag{2.8}$$

It is clear from (2.2) and (2.8) that the characteristic roots of R and $S^{-1}RS$ are the same. □

Theorem 2.3. *Let R be a rhotrix of 3-dimension satisfying the condition* **P** *of* (1.6). *Also $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the set of characteristic roots of the rhotrix R over the finite field $\mathbb{F}_2$. Then $\frac{|R|}{\lambda_i}$ (for some $i = 1, 2, 3$) is the characteristic root of the rhotrix R.*

Proof. Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the set of characteristic roots of rhotrix

$$R = \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{array} \right\rangle.$$

The determinant of the rhotrix R is given by

$$|R| = \begin{vmatrix} & a_{11} & \\ 0 & c_{11} & 0 \\ & a_{22} & \end{vmatrix}$$
$$= (a_{11}a_{22}c_{11}) \tag{2.9}$$

Therefore,

$$\frac{|R|}{\lambda_i} = \frac{(a_{11}a_{22}c_{11})}{\lambda_i}. \tag{2.10}$$

Since c_{11} is one of the characteristic roots of R among $\{\lambda_1, \lambda_2, \lambda_3\}$, thus taking $\lambda_i = c_{11}$ and putting in (2.10), we get

$$\frac{|R|}{\lambda_i} = \frac{(a_{11}a_{22}c_{11})}{c_{11}}$$
$$= a_{11}a_{22}. \tag{2.11}$$

Putting the value of $\frac{|R|}{\lambda_i}$ from (2.11) in the characteristic equation of R given in (2.2), we get

$$(a_{11}a_{22})^3 + (a_{11}+a_{22}+c_{11})\,(a_{11}a_{22})^2 + (a_{11}c_{11}+a_{22}c_{11}+a_{22}a_{11})\,(a_{11}a_{22}) + (a_{11}a_{22}c_{11}) = 0. \tag{2.12}$$

Therefore, it is clear from (2.12) that $\frac{|R|}{\lambda_i}$ satisfies the equation (2.2). This implies that $\frac{|R|}{\lambda_i}$ is the characteristic root of R. □

Theorem 2.4. *Let R_n be a rhotrix over finite field $\mathbb{F}_2$, satisfying the condition* **P** *of* (1.6). *Then,* $\mathrm{H}(R_n^2) = 1$.

Proof. Let the rhotrix R_n be given by

$$R_n = \left\langle \begin{array}{cccccc}
 & & & & & a_{11} \\
 & & & & a_{21} & c_{11} \\
 & & & a_{31} & c_{21} & a_{22} \\
 & & \cdots & \cdots & \cdots & \cdots \\
 & a_{(n-1)1} & \cdots & \cdots & \cdots & \sum a_{\frac{(n-1)}{2}\frac{(n-1)}{2}} \\
a_{n1} & c_{(n-1)1} & \cdots & \cdots & \sum a_{\frac{(n+1)}{2}\frac{(n-1)}{2}} & \sum c_{\frac{(n-1)}{2}\frac{(n-1)}{2}} \\
 & a_{n2} & \cdots & \cdots & \cdots & \sum a_{\frac{(n+1)}{2}\frac{(n+1)}{2}} \\
 & & \cdots & \cdots & \cdots & \cdots \\
 & & & a_{n(n-2)} & c_{n(n-1)} & a_{(n-1)(n-1)} \\
 & & & & a_{n(n-1)} & c_{(n-1)(n-1)} \\
 & & & & & a_{nn}
\end{array} \right.$$

$$
\begin{array}{ccccc}
a_{12} & & & & \\
c_{12} & a_{13} & & & \\
\cdots & \cdots & \cdots & & \\
\cdots & \cdots & \cdots & a_{1(n-1)} & \\
\sum a_{\frac{(n-1)}{2}\ \frac{(n+1)}{2}} & \cdots & \cdots & c_{1(n-1)} & a_{1n} \\
\cdots & \cdots & \cdots & a_{2n} & \\
\cdots & \cdots & \cdots & & \\
c_{(n-1)n} & a_{(n-2)n} & & & \\
a_{(n-1)n} & & & &
\end{array}
\Bigg\rangle .
$$

Since the rhotrix R_n satisfies the condition **P** of (1.6), therefore R_n becomes

$$
R_n = \left\langle
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & 1 & 0 & & & & \\
 & & & 1 & 0 & 1 & 0 & 1 & & & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & 0 & \cdots & \cdots & \cdots & \sum a_{\frac{(n-1)}{2}\ \frac{(n-1)}{2}} & \cdots & \cdots & \cdots & 0 & \\
a_{n1} & 0 & \cdots & \cdots & 0 & \sum c_{\frac{(n-1)}{2}\ \frac{(n-1)}{2}} & 0 & \cdots & \cdots & 0 & a_{1n} \\
 & 0 & \cdots & \cdots & \cdots & \sum a_{\frac{(n+1)}{2}\ \frac{(n+1)}{2}} & \cdots & \cdots & \cdots & 0 & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & & a_{n(n-2)} & 0 & a_{(n-1)(n-1)} & 0 & a_{(n-2)n} & & & \\
 & & & & 0 & c_{(n-1)(n-1)} & 0 & & & & \\
 & & & & & a_{nn} & & & & &
\end{array}
\right\rangle , \qquad (2.13)
$$

where $a_{11}, c_{11}, a_{21}, a_{12}, c_{21}, c_{21}, \ldots, c_{(n-1)(n-1)}, a_{nn} \in \mathbb{F}_2$ and $c_{\frac{(n-1)}{2}\ \frac{(n-1)}{2}}$ is the heart of the rhotrix R_n, that is, $H(R_n)$.
Since, $R_n^2 = R_n R_n$, therefore,

$$
R_n^2 = \left\langle
\begin{array}{cccccccccc}
 & & & & & & & & & \sum\limits_{i,j=1}^{n} a_{ij}a_{ji} \\
 & & & & & & & & 0 & \sum\limits_{i,j=1}^{n} c_{ij}c_{ji} \\
 & & & & & & & \sum\limits_{i=3,j=1}^{n} a_{ij}a_{ji} & 0 & \sum\limits_{i,j=2}^{n} a_{ij}a_{ji} \\
 & & & & & & 0 & \sum\limits_{i=3,j=1}^{n} c_{ij}c_{ji} & 0 & \sum\limits_{i,j=2}^{n} c_{ij}c_{ji} \\
 & & & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \sum\limits_{i,j=1}^{n} a_{\frac{(n-1)}{2}\, i}\, a_{j\, \frac{(n-1)}{2}} \\
\sum\limits_{i=n} \sum\limits_{j=1}^{n} a_{ij}a_{ji} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sum\limits_{i,j=1}^{n} c_{\frac{(n-1)}{2}\, i}\, c_{j\, \frac{(n-1)}{2}} \\
 & 0 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \sum\limits_{i,j=1}^{n} a_{\frac{(n+1)}{2}\, i}\, a_{j\, \frac{(n+1)}{2}} \\
 & & & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & 0 & \cdots & \cdots & \cdots \\
 & & & & & & & \cdots & \cdots & \cdots \\
 & & & & & & & \sum\limits_{i=n} \sum\limits_{j=3}^{n} a_{ij}a_{(j-2)i} & 0 & \sum\limits_{i,j=3}^{n} a_{(i-1)j}a_{(j-1)i} \\
 & & & & & & & & 0 & \sum\limits_{i,j=2}^{n} c_{(i-1)j}c_{(j-1)i} \\
 & & & & & & & & & \sum\limits_{i=1}^{n} a_{ii}a_{ii}
\end{array}
\right.
$$

$$
\left.
\begin{array}{lllllllllll}
0 \\
0 & \sum\limits_{i=1,j=3}^{n} a_{ij}a_{ji} \\
0 & \sum\limits_{i=1,j=3}^{n} c_{ij}c_{ji} & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sum\limits_{j=n}\sum\limits_{i=1}^{n} a_{ij}a_{ji} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & 0 \\
0 & \sum\limits_{j=n}\sum\limits_{i=3}^{n} a_{(i-2)j}a_{ji} \\
0
\end{array}
\right\rangle \tag{2.14}
$$

Obviously, the heart of the rhotrix R_n^2 is

$$
H\left(R_n^2\right) = \sum_{i,j=1}^{n} c_{\frac{(n-1)}{2}\, i}\, c_{j\, \frac{(n-1)}{2}}\,, \tag{2.15}
$$

where $\frac{(n-1)}{2}$ is always even.
Therefore,

$$
\begin{aligned}
\sum_{i,j=1}^{n} c_{\frac{(n-1)}{2}\, i}\, c_{j\, \frac{(n-1)}{2}} &= c_{1i}c_{j1} + c_{2i}c_{j2} + c_{3i}c_{j3} + \ . \ . \ . \ + c_{\frac{(n-1)}{2}\, i}\, c_{j\, \frac{(n-1)}{2}}\,, \\
&= c_{11}c_{11} + c_{13}c_{31} + c_{15}c_{51} + c_{21}c_{12} + \ . \ . \ . \ + c_{\frac{(n-1)}{2}\, n}\, c_{n\, \frac{(n-1)}{2}}\,.
\end{aligned} \tag{2.16}
$$

By using the condition **P** of (1.6) in (2.16) the terms like $c_{12}c_{21}$ become zero. Similarly, all the terms with $i+j =$ an odd number are zero and only the terms for which $i+j =$ an even number survive. Since these terms are always odd in number, therefore, it is clear form (2.16) that sum of all these terms is equal to 1 over the finite field $\mathbb{F}_2$. Hence, we conclude that heart of rhotrix R_n^2 is always equal to 1. □

Theorem 2.5. *Let R_n be a rhotrix satisfying the condition* **P** *of* (1.6). *Then, there exist a sub-rhotrix*

$$
\begin{cases}
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, & \text{for } n = 4k_1 \pm 1, \text{ where } k_1 \text{ is an even number}, \\
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle, & \text{for } n = 4k_2 \pm 1, \text{ where } k_2 \text{ is an odd number},
\end{cases} \tag{2.17}
$$

of R_n^2 over the finite field $\mathbb{F}_2$.

Proof. Let the rhotrix R_n be

$$R_n = \left\langle \begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & 1 & 0 & & & & \\
 & & & 1 & 0 & 1 & 0 & 1 & & & \\
 & & .. & .. & .. & .. & .. & .. & .. & & \\
 & 0 & .. & .. & .. & .. & .. & .. & .. & 0 & \\
a_{n1} & 0 & .. & .. & .. & .. & .. & .. & .. & 0 & a_{1n} \\
 & 0 & .. & .. & .. & .. & .. & .. & .. & 0 & \\
 & & .. & .. & .. & .. & .. & .. & & & \\
 & & & a_{n(n-2)} & 0 & a_{(n-1)(n-1)} & 0 & a_{(n-2)n} & & & \\
 & & & & 0 & c_{(n-1)(n-1)} & 0 & & & & \\
 & & & & & a_{nn} & & & & &
\end{array} \right\rangle,$$

Since, $R_n^2 = R_n R_n$, therefore,

$$R_n^2 = \left\langle \begin{array}{cccccccccc}
 & & & & & & & & & \sum\limits_{i,j=1}^{n} a_{ij}a_{ji} \\
 & & & & & & & & 0 & \sum\limits_{i,j=1}^{n} c_{ij}c_{ji} \\
 & & & & & & & \sum\limits_{i=3,j=1}^{n} a_{ij}a_{ji} & 0 & \sum\limits_{i,j=2}^{n} a_{ij}a_{ji} \\
 & & & & & & 0 & \sum\limits_{i=3,j=1}^{n} c_{ij}c_{ji} & 0 & \sum\limits_{i,j=2}^{n} c_{ij}c_{ji} \\
 & & & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \sum\limits_{i,j=1}^{n} c_{\frac{(n-3)}{2} i} c_{j \frac{(n-3)}{2}} \\
 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sum\limits_{i,j=1}^{n} a_{\frac{(n-1)}{2} i} a_{j \frac{(n-1)}{2}} \\
\sum\limits_{i=n} \sum\limits_{j=1}^{n} a_{ij}a_{ji} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \sum\limits_{i,j=1}^{n} c_{\frac{(n+1)}{2} i} c_{j \frac{(n-3)}{2}} & 0 & \sum\limits_{i,j=1}^{n} c_{\frac{(n-1)}{2} i} c_{j \frac{(n-1)}{2}} \\
 & 0 & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sum\limits_{i,j=1}^{n} a_{\frac{(n+1)}{2} i} a_{j \frac{(n+1)}{2}} \\
 & & & & & \cdots & \cdots & \cdots & \cdots & \sum\limits_{i,j=1}^{n} c_{\frac{(n+1)}{2} i} c_{j \frac{(n+1)}{2}} \\
 & & & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & \cdots & \cdots & \cdots \\
 & & & & & & & \sum\limits_{i=n} \sum\limits_{j=3}^{n} a_{ij}a_{(j-2)i} & 0 & \sum\limits_{i,j=2}^{n} a_{(i-1)j}a_{(j-1)i} \\
 & & & & & & & & 0 & \sum\limits_{i,j=2}^{n} c_{(i-1)j}c_{(j-1)i} \\
 & & & & & & & & & \sum\limits_{i=1}^{n} a_{ii}a_{ii}
\end{array}\right.$$

$$
\begin{array}{lllllllllll}
0 & & & & & & & & & & \\
0 & \sum_{i=1,j=3}^{n} a_{ij}a_{ji} & & & & & & & & & \\
0 & \sum_{i=1,j=3}^{n} a_{ij}a_{ji} & 0 & & & & & & & & \\
\cdots & \cdots & \cdots & \cdots & & & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & \\
0 & \sum_{i,j=1}^{n} a_{\frac{(n-3)}{2}i}\, a_{j\frac{(n+1)}{2}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sum_{j=n}\sum_{i=1}^{n} a_{ij}a_{ji} & \Big\rangle \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & & & & & \\
\cdots & \cdots & \cdots & \cdots & & & & & & & \\
\cdots & \cdots & 0 & & & & & & & & \\
0 & \sum_{j=n}\sum_{i=3}^{n} a_{(i-2)j}a_{ji} & & & & & & & & & \\
0 & & & & & & & & & &
\end{array}
\qquad (2.18)
$$

For $n = 5$, the entries of R_n^2 are as follows:

$$
\begin{aligned}
\sum_{n\,=\,5}\sum_{i,j=1}^{3} a_{\frac{(n-1)}{2}\,i}\, a_{j\,\frac{(n-1)}{2}} &= a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} \\
&= 0 + 1 + 0 \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n\,=\,5}\sum_{i,j=1}^{2} c_{\frac{(n-1)}{2}\,i}\, c_{j\,\frac{(n-1)}{2}} &= c_{21}c_{12} + c_{22}c_{22} \\
&= 1 + 0 \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n\,=\,5}\sum_{i,j=1}^{2} c_{\frac{(n-3)}{2}\,i}\, c_{j\,\frac{(n-3)}{2}} &= c_{11}c_{11} + c_{12}c_{21} \\
&= 0 + 1 \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n\,=\,5}\sum_{i,j=1}^{2} c_{\frac{(n-1)}{2}\,i}\, c_{j\,\frac{(n-3)}{2}} &= c_{11}c_{12} + c_{12}c_{22} \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

and $\sum_{n\,=\,5}\ \sum_{i,j=1}^{2} c_{\frac{(n-3)}{2}\,i}\, c_{j\,\frac{(n-1)}{2}} = c_{21}c_{11} + c_{22}c_{21}$

$$
\begin{aligned}
&= 0 + 0 \\
&= 0.
\end{aligned}
$$

Clearly, $\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$ is a sub-rhotrix of R_n^2 over the finite field $\mathbb{F}_2$.

Similarly, for $n = 7$, the entries of R_n^2 are as follows:

$$
\sum_{n\,=\,7}\sum_{i,j=1}^{3} c_{\frac{(n-3)}{2}\,i}\, c_{j\,\frac{(n-3)}{2}} = c_{21}c_{12} + c_{22}c_{22} + c_{23}c_{32}
$$

$$= 0 + 1 + 0$$
$$= 1,$$

$$\sum_{n\,=\,7}\ \sum_{i\,,j=1}^{4} a_{\frac{(n-1)}{2}\ i}\, a_{j\ \frac{(n-1)}{2}} = a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} + a_{34}a_{43}$$
$$= 1 + 0 + 1 + 0$$
$$= 0,$$

$$\sum_{n\,=\,7}\ \sum_{i\,,j=1}^{4} a_{\frac{(n-3)}{2}\ i}\, a_{j\ \frac{(n-3)}{2}} = a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} + a_{24}a_{42}$$
$$= 0 + 1 + 0 + 1$$
$$= 0,$$

$$\sum_{n\,=\,7}\ \sum_{i\,,j=1}^{4} a_{\frac{(n-1)}{2}\ i}\, a_{j\ \frac{(n-3)}{2}} = a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} + a_{34}a_{42}$$
$$= 0 + 0 + 0 + 0$$
$$= 0,$$

and $\sum_{n\,=\,7}\ \sum_{i\,,j=1}^{4} a_{\frac{(n-3)}{2}\ i}\, a_{j\ \frac{(n-1)}{2}} = a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} + a_{24}a_{43}$

$$= 0 + 0 + 0 + 0$$
$$= 0,$$

Clearly, $\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$ is a sub-rhotrix of R_n^2 over the finite field $\mathbb{F}_2$.

Therefore, in the same way for other values of n, we conclude the desired result of (2.17) for R_n^2 rhotrix. □

3 Illustrations

Here, we illustrate some results given in section 2 with the help of examples.

Example 3.1. Let R_5 be a rhotrix satisfying the condition **P** of (1.6) and let it given by

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle.$$

Now, $R_5^2 = R_5 R_5$

$$= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{ccccc} & & 2 & & \\ & 0 & 1 & 0 & \\ 2 & 0 & 1 & 0 & 2 \\ & 0 & 1 & 0 & \\ & & 2 & & \end{array} \right\rangle.$$

Since, all the entries of above rhotrix belong to the finite field $\mathbb{F}_2$, therefore, we get

$$R_5^2 = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array} \right\rangle. \tag{3.1}$$

Clearly, the heart of matrix R_5^2 over the finite field $\mathbb{F}_2$ is 1. Further, from (3.1) it is concluded that, there is a sub-rhotrix of R_5^2 of the form $\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$.

Example 3.2. Let R be a rhotrix of 7-dimensions satisfying the condition **P** of (1.6) and suppose it is given by

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

Now,

$$R_7^2 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{ccccccc} & & & 2 & & & \\ & & 0 & 2 & 0 & & \\ & 2 & 0 & 2 & 0 & 2 & \\ 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ & 2 & 0 & 2 & 0 & 2 & \\ & & 0 & 2 & 0 & & \\ & & & 2 & & & \end{array} \right\rangle.$$

After reducing all the entries over the finite field $\mathbb{F}_2$, we get

$$R_7^2 = \left\langle \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & & \\ & & & 0 & & & \end{array} \right\rangle. \tag{3.2}$$

Therefore, it is clear from (3.2) that there exist a sub-rhotrix of R_7^2 which is of the form $\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$.

4 Conclusion

We considered the rhotrices of 3-dimensions and derived some properties related to their characteristic roots. Further, we discussed results related to the heart and sub-rhotrix of n-dimensional rhotrices.

Acknowledgments The first author thankfully acknowledges the support of the UGC-SAP and the second author is grateful to the UGC for granting the Rajeev Gandhi National Fellowship (RGNF). The authors also express their gratitude to the Editor-in-Chief for his painstaking efforts for most accurately typesetting and editing this manuscript in latex and to the anonymous reviewers for providing their valuable feedback on the original version of this manuscript.

References

[1] Absalom, E.E., Sani, B. and Sahalu, J.B. (2011). The concept of heart-oriented rhotrix multiplication, *Global J. Sci. Fro. Research*, 11(2), 35–42.

[2] Ajibade, A.O. (2003). The concept of rhotrices in mathematical enrichment, *Int. J. Math. Educ. Sci. Technol.*, 34(2), 175–179.

[3] Menezes, Alfred, Oorschot, Paul and Vanstone, Scott (2001). Handbook of Applied Cryptography, CRC Press.

[4] Aminu, A. (2012). A note on the rhotrix system of equation, *Journal of the Nigerian Association of Mathematical Physics*, 21, 289–296.

[5] Mohammed, A., Ezugwu, E.A. and Sani, B. (2011). On generalization and algorithmatization of heart-based method for multiplication of rhotrices, *Int. J. Comp. Inform. Syst.*, 2, 46–49.

[6] Sani, B. (2004). An alternative method for multiplication of rhotrices, *Int. J. Math. Educ. Sci. Technol.*, 35(5), 777–781.

[7] Sani, B. (2007). The row-column multiplication for high dimensional rhotrices, *Int. J. Math. Educ. Sci. Technol.*, 38(5), 657–662.

[8] Sani, B. (2008). Conversion of a rhotrix to a coupled matrix, *Int. J. Math. Educ. Sci. Technol.*, 39(2), 244–249.

[9] Sharma, P.L. and Kanwar, R.K. (2011). A note on relationship between invertible rhotrices and associated invertible matrices, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 30E(2), 333–339.

[10] Sharma, P.L. and Kanwar, R.K. (2012). Adjoint of a rhotrix and its basic properties, *Int. J. Math. Sci.*, 11(3-4), 337–343.

[11] Sharma, P.L., Kumar, S. and Rehan, M. (2014). On construction of Hadamard codes using Hadamard rhotrices, *Int. J. Theo. and Appl. Sci.*, 6(1), 102–111.

[12] Sharma, P.L., Kumar, S. and Rehan, M. (2013). On Hadamard rhotrix over finite field, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 32E(2), 181–190.

[13] Sharma, P.L., Gupta, S. and Dhiman, N. (2017). Sylvester rhotrices and their properties over finite field, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 36E(1), 70–80.

[14] Sharma, P.L., Gupta, S. and Rehan, M. (2017). On circulant like rhotrices over finite fields, *Applications and Applied Mathematics (AAM)*, 12(1), 506–520.

[15] Tudunkaya, S.M. (2013). Rhotrix polynomial and polynomial rhotrix, *Pure Appl. Math. J.*, 2(1), 38–41.

[16] Tudunkaya, S.M. and Makanjuola, S.O. (2010). Rhotrices and the construction of finite fields, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 29E(2), 225–229.