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A mixed quadrature rule of modified Birkhoff-Young rule and $SM_2(f)$ rule for the numerical integration of analytic functions *

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Abstract A quadrature rule of higher precision is constructed in this paper by mixing two quadrature rules of lower precision for an approximate evaluation of the integral of an analytic function over a line segment in the complex plane. An asymptotic error estimate of the rule is also determined and the rule is numerically verified.

Key words Quadrature rule, Asymptotic error, Analytic function, Numerical integration, Modified Birkhoff-Young rule, Richardson Extrapolation, $SM_2(f), SM_6(f)$, E_{SM2}, E_{SM6}.

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1 Introduction

There are several rules (see, [4])for the approximate evaluation of real definite integral

$$\int_{-1}^{1} f(z)dz \tag{1.1}$$

However there are only a few quadrature rules for evaluating an integral of the type

$$I(f) = \int_L f(z)\,dz \tag{1.2}$$

where L is a directed line segment from the point $(z_0 - h)$ to $(z_0 + h)$ in the complex plane $\mathbb{C}$ and $f(z)$ is analytic in certain domain $\mathbf{\Omega}$ containing L. Using the transformation $z = z_0 + ht,\ t \in [-1,\ 1]$ (due to Lether [7]), we transform the integral (1.2) to the form

$$h\int_{-1}^{1} f(z_0 + ht)dt \tag{1.3}$$

and make the approximation of the integral by applying the standard quadrature rule meant for an approximate evaluation of the real definite integral (1.1). The rules so formed are termed as the 'transformed rules' for numerical integration of (1.2).
Das and Pradhan [1] have constructed a quadrature rule for the approximate evaluation of (1.1) from two quadrature rules of different types but of equal precision. Such rules are termed as the 'mixed quadrature rules'.

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In this light Mohanty and Dash [5] have constructed a mixed quadrature rule $SM_2(f)$ of precision seven.
Here we mix up the Birkhoff-Young modified rule which is obtained by using the Richardson extrapolation and $SM_2(f)$ quadrature rules each of precision seven. A new rule of precision nine is obtained and this mixed quadrature is used for evaluating an integral of the form (1.2).

2 Birkhoff-Young rule $BY(f)$

The Birkhoff-Young rule due to Birkhoff and Young [3], Acharya and Das [2] and Lether [7] is

$$BY(f) = \frac{h}{15}\left[4\left\{f(z_0+h)+f(z_0-h)\right\}+24f(z_0)-\left\{f(z_0+ih)+f(z_0-ih)\right\}\right]. \tag{2.1}$$

Applying the Taylor's theorem to it we get

$$\begin{aligned} BY\left(f\right) = 2h\Bigg[& f\left(z_0\right)+\frac{h^2}{3!}f^{ii}\left(z_0\right)+\frac{h^4}{5!}f^{iv}\left(z_0\right)+\frac{h^6}{3\times 6!}f^{vi}\left(z_0\right)+ \\ & \frac{h^8}{5\times 8!}f^{viii}\left(z_0\right)+\frac{h^{10}}{3\times 10!}f^{x}\left(z_0\right)+\frac{h^{12}}{5\times 12!}f^{xii}\left(z_0\right)+\ldots\Bigg] \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} I\left(f\right) = 2h\Bigg[& f\left(z_0\right)+\frac{h^2}{3!}f^{ii}\left(z_0\right)+\frac{h^4}{5!}f^{iv}\left(z_0\right)+\frac{h^6}{7!}f^{vi}\left(z_0\right)+ \\ & \frac{h^8}{9!}f^{viii}\left(z_0\right)+\frac{h^{10}}{11!}f^{x}\left(z_0\right)+\frac{h^{12}}{13!}f^{xii}\left(z_0\right)+\ldots\Bigg] \end{aligned} \tag{2.3}$$

The error in this rule is given by

$$E_{BY} = I\left(f\right) - BY\left(f\right).$$

Using (2.2) and (2.3),

$$E_{BY} = \frac{-h^7}{1890}f^{vi}\left(z_0\right)-\frac{h^9}{226800}f^{viii}\left(z_0\right)-\frac{h^{11}}{7484400}f^{x}\left(z_0\right)-\frac{h^{13}}{1945944000}f^{xii}\left(z_0\right)-\ldots. \tag{2.4}$$

Now from (2.1)

$$BY_{h/2} = \frac{2h}{15}\left[24f\left(z_0\right)+4\left\{f\left(z_0+2h\right)+f\left(z_0-2h\right)\right\}-\left\{f\left(z_0+i2h\right)+f\left(z_0-i2h\right)\right\}\right]. \tag{2.5}$$

Using the Taylor's expansion (2.5) converts to

$$\begin{aligned} BY_{h/2}(f) = 4h\Bigg[& f\left(z_0\right)+\frac{2^2h^2}{3!}f^{ii}\left(z_0\right)+\frac{2^4h^4}{5!}f^{iv}\left(z_0\right)+\frac{2^6h^6}{3\times 6!}f^{vi}\left(z_0\right) \\ & +\frac{2^8h^8}{5\times 8!}f^{viii}\left(z_0\right)+\frac{2^{10}h^{10}}{3\times 10!}f^{x}\left(z_0\right)+\frac{2^{12}h^{12}}{5\times 12!}f^{xii}\left(z_0\right)+\ldots\Bigg]. \end{aligned} \tag{2.6}$$

Now the error

$$E_{BY_{h/2}} = I\left(f\right) - BY_{\frac{h}{2}}\left(f\right).$$

Using (2.3) and (2.6),

$$E_{BY_{h/2}} = \frac{-2^7h^7}{1890}f^{vi}\left(z_0\right)-\frac{2^9h^9}{226800}f^{viii}\left(z_0\right)-\frac{2^{11}h^{11}}{7484400}f^{x}\left(z_0\right)-\frac{2^{13}h^{13}}{1945944000}f^{xii}\left(z_0\right)-\ldots. \tag{2.7}$$

3 Modified Birkhoff-Young quadrature rule by the Richardson extrapolation $R_{BY_{\frac{h}{2}}}(f)$

Now

$$I\left(f\right) = BY_h\left(f\right) + E_{BY_h}. \tag{3.1}$$

and

$$I\left(f\right) = BY_{h/2}\left(f\right) + E_{BY_{h/2}}. \tag{3.2}$$

where, $BY_h \equiv BY$ and $E_{BY_h} \equiv E_{BY}$.
Multiplying (3.1) by 2^7 and subtracting (3.2) from the result, we obtain

$$(2^7-1)\, I(f) = [2^7 BY_h(f) - BY_{h/2}(f)] + [2^7 E_{BY_h} - E_{BY_{\frac{h}{2}}}]$$

$$\Rightarrow I(f) = \frac{1}{127}[128\ BY_h(f) - BY_{\frac{h}{2}}(f)] + \frac{1}{127}[128\ E_{BY_h} - E_{BY_{\frac{h}{2}}}]$$

$$= R_{BY_{\frac{h}{2}}}(f) + ER_{BY_{\frac{h}{2}}}$$

where

$$R_{BY_{\frac{h}{2}}}(f) = \frac{1}{127}[128\ BY_h(f) - BY_{\frac{h}{2}}(f)] \tag{3.3}$$

and

$$ER_{BY_{\frac{h}{2}}} = \frac{1}{127}[128\ E_{BY_h} - E_{BY_{\frac{h}{2}}}].$$

Using (2.4) and (2.7)

$$ER_{BY_{\frac{h}{2}}} = \frac{2}{5\times 9!}\left(\frac{128\times 12}{127}\right) h^9 f^{viii}(z_0) + \frac{1920h^{11}}{127\times 7484400} f^{x}(z_0) + \frac{8064\ h^{13}}{127\times 1945944000} f^{xii}(z_0) + \dots \tag{3.4}$$

(3.3) and (3.4) are respectively called the modified Birkhoff-Young extrapolation and the error in the modified Birkhoff-Young extrapolation due to the Richardson extrapolation.
From (3.4) we see that the degree of precision of the rule is 7.

4 The $SM_2(f)$ quadrature rule

From the work of Mohanty and Dash [5], we know that

$$SM_2(f) = \frac{1}{7}[8BL(f) - BY(f)] \tag{4.1}$$

where

$$BL(f) = \frac{h}{45}\left[7\{f(z_0+h) + f(z_0-h)\} + 12f(z_0) + 32\{f(z_0+\frac{h}{2}) + f(z_0-\frac{h}{2})\}\right] \tag{4.2}$$

and from (2.1) and (2.2)

$$BY(f) = \frac{h}{15}[4\{f(z_0+h) + f(z_0-h)\} + 24f(z_0) - \{f(z_0+ih) + f(z_0-ih)\}].$$

The truncation error of the rule $SM_2(f)$ is

$$E_{SM2} = I(f) - SM_2(f)$$
$$= \left[-\frac{78h^9}{105(9!)} f^{viii}(z_0) - \frac{37h^{11}}{84(11!)} f^{x}(z_0) - \frac{3023h^{13}}{240(13!)} f^{xii}(z_0) - \dots\right]$$
$$= -\frac{2\times 13\ h^9}{5\times 7\times 9!} f^{viii}(z_0) - \frac{37\ h^{11}}{84\times 11!} f^{x}(z_0) - \frac{3023\ h^{13}}{240\times 13!} f^{xii}(z_0) - \dots \tag{4.3}$$

This rule is of precision seven.

5 Formulation of the Rule

We formulate the desired Mixed Rule as follows [1,5,6]:
Applying the rules mentioned in (3.3) and (4.1) respectively to the integral in (1.2) we have

$$I(f) = R_{BY_{\frac{h}{2}}}(f) + ER_{BY_{\frac{h}{2}}} \tag{5.1}$$

and

$$I(f) = SM_2 + E_{SM2} \tag{5.2}$$

Now multiplying (5.1) and (5.2) by 1651 and 10752 respectively and then adding the results thus obtained, we get

$$12403\ I\,(f) = 1651\ R_{BY_{\frac{h}{2}}}\,(f) + 10752\ SM_2(f) + 1651\ ER_{BY_{\frac{h}{2}}} + 10752\ E_{SM2}$$

$$\Rightarrow\ I\,(f) = \frac{1}{12403}\left[1651\ R_{BY_{\frac{h}{2}}}\,(f) + 10752\ SM_2(f)\right] + \frac{1}{12403}\left[1651\ ER_{BY_{\frac{h}{2}}} + 10752\ E_{SM2}\right] = SM_6\,(f) + E_{SM6}$$

where,

$$SM_6\,(f) = \frac{1}{12403}\left[1651\ R_{BY_{\frac{h}{2}}}\,(f) + 10752\ SM_2(f)\right] \tag{5.3}$$

and

$$E_{SM6} = \frac{1}{12403}\left[1651\ ER_{BY_{\frac{h}{2}}} + 10752\ E_{SM2}\right]$$

Using (3.4) and (4.3), we have

$$E_{SM6} = \frac{1}{12403\times 11!}\,[133120 - 9472]\,h^{11} f^{x}(z_0) + \frac{1}{12403\times 5\times 13!}\,[1677312 - 1354304)]\,h^{13} f^{xii}(z_0) + \ldots$$

$$= \frac{123648}{12403\times 11!}h^{11} f^{x}\,(z_0) + \frac{323008}{12403\times 5\times 13!}h^{13} f^{xii}(z_0) + \ldots \tag{5.4}$$

From equation (5.4), we see that the degree of precision of $SM_6\,(f)$ rule is nine.
So by combining two quadrature rules of precision 7 (i.e., one is modified Birkhoff-Young rule $R_{BY_{\frac{h}{2}}}\,(f)$ due to Richardson Extrapolation and another one is $SM_2(f)$ quadrature rule), we get the Mixed rule $SM_6\,(f)$ of precision nine for the approximate evaluation of $I(f)$ and truncation error committed in this approximation is given by equation (5.4).

6 Error Analysis

Let $f(z)$ be analytic in the disc $\Omega_R = \{z : |z - z_0| \le R > h\}$, so that the points $z_0, z_0 \pm h, z_0 \pm h\sqrt{\frac{3}{5}}$ are all interior to the disc Ω_R. Now using the Taylor's series expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{n!} f^n(z_0)$$

From (5.4) we have

$$E_{SM6} = \frac{123648}{12403\times 11!}h^{11} f^{x}\,(z_0) + \frac{323008}{12403\times 5\times 13!}h^{13} f^{xii}(z_0) + \ldots$$

From this expression for E_{SM6} we have the following theorem:

Theorem 6.1. *If f is assumed to be analytic in the domain $\Omega \supset L$ then*

$$E_{SM6} = o(h^{11}).$$

Error Comparison
It is shown by Lether [7] that

$$|E_{GL}| \le |E_{BL}|\,.$$

Again from (3.4), (4.3) we have,

$$|ER_{BY_{\frac{h}{2}}}| \le |E_{GL}|$$

Table 7.1: Comparative numerical evaluation of the integrals I_1 (7.1) and I_2 (7.2) .

Sl. No.	Quadrature rules	Approximation value of (I_1, where $i=\sqrt{-1}$.)	|Error| for (I_1)	Approximation value of (I_2, where $i=\sqrt{-1}$.)	|Error|for (I_2)
a	$BL(f)$	$(1.6828781)i$	0.0000638	$(1.0421911)i$	0.0000005
b	$GL(f)$	$(1.6830035)i$	0.0000616	$(1.0421901)i$	0.00000049
c	$SM_2(f)$	$(1.682860071)i$	0.00008	$(1.0421906)i$	0.00000002
d	$R_{BY_{\frac{h}{2}}}(f)$	$(1.6831679)i$	0.00022	$(1.04219066)i$	0.000000068
e	$SM_6(f)$	$(1.682931047)i$	0.0000108	$(1.042190587)i$	0.0000000047
		Exact value of I_1 $(1.6829419239)i$	Least for $SM_6(f)$	Exact value of I_2 $(1.0421905917)i$	Least for $SM_6(f)$

and

$$|E_{SM2}| \leq |E_{GL}|\,.$$

Further from(5.4) and (4.3) we get

$$|E_{SM6}| \leq |ER_{BY_{\frac{h}{2}}}|$$

and

$$|E_{SM6}| \leq |E_{SM2}|\,.$$

Thus, theoretically the mixed rule $SM_6(f)$ developed in this paper is better than the mixed rule $SM_2(f)$ of [5].

7 Numerical Verification

Let us calculate approximate value of the integral I_1 and I_2 by using $BL(f)$, $GL(f)$, $SM_2(f)$, $R_{BY_{\frac{h}{2}}}(f)$ and $SM_6(f)$ for

$$I_1 = \int_{-1}^{1} e^z dz, \tag{7.1}$$

and

$$I_2 = \int_{-i/2}^{i/2} \cos z dz. \tag{7.2}$$

The comparative numerical evaluation of the integrals in (7.1) and (7.2) is shown in the Table 7.1.

8 Conclusion

The new mixed quadrature rule $SM_6(f)$, designed in this paper, gives appreciably better results in comparison to its constituent rules and the mixed rule $SM_2(f)$ as predicted theoretically. This rule can be very much useful for finding the numerical solution of integral equations.

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