Bull. Pure Appl. Sci. Sect. E Math. Stat.
39E(1), 165–175 (2020)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2020.00015.6
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387-RPS-DDA Flat, Mansarover Park,
Shahdara, Delhi-110032, India. 2020

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

Trace of the positive integral powers of three and five dimensional rhotrices *

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Abstract The traces of powers of matrices frequently arise in several fields of mathematics. Rhotrices are a new paradigm of matrices which are represented as coupled matrices. The main aim of this paper is to find the trace of positive integral powers of three and five dimensional rhotrices. Further, the trace of positive integral powers of special type of circulant rhotrix is also discussed.

Key words Rhotrix, trace of a rhotrix, determinant, circulant rhotrix.

2020 Mathematics Subject Classification 15A09, 20H30, 11T71.

1 Introduction

The trace of a positive integer power of a matrix needs to be evaluated in various problems in Network Analysis, Number Theory, Matrix Theory, Dynamical Systems and Differential Equations. To study a complex network, we need to compute the total number of triangles of a connected simple graph. The trace of integral powers of square matrices was discussed by Michiel [6] and Pahade and Jha [7]. Cisneros et al. [4] and Cisneros [5] discussed a formula for the trace of symmetric powers of a matrix. Also, Brezinski et al. [3] estimated the trace of powers of self adjoint operators. Zareula [14] discussed the congruences for the trace of powers of some matrices.
Rhotrix is a new paradigm in the study of matrix theory. Rhotrix is represented as coupled matrix. Ajibade [2] defined a 3–dimensional rhotrix, which is, in some way, between 2×2–dimensional and 3×3–dimensional matrices as

$$R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle,$$

where a, b, c, d, e are real numbers and $h(R_3) = c$ is called the heart of rhotrix R_3.

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received April 18, 2019 / Revised March 07, 2020 / Accepted April 03, 2020. Online First Published on June 30, 2020 at https://www.bpasjournals.com/.
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Let $Q_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$, be another 3-dimensional rhotrix, then the addition of two rhotrices is defined as

$$R_3 + Q_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & c+h & d+j \\ & e+k & \end{array} \right\rangle,$$

and for any real number α, the scalar multiplication of a rhotrix R_3 by α is defined as

$$\alpha\, R_3 = \alpha \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha\, a & \\ \alpha\, b & \alpha\, c & \alpha\, d \\ & \alpha\, e & \end{array} \right\rangle.$$

Two types of multiplications of rhotrices are discussed in the literature of rhotrices, namely, the heart oriented multiplication and the row-column multiplication. Ajibade [2] discussed the heart oriented multiplication of 3-dimensional rhotrices as given below:

Let $Q_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$,be another 3-dimensional rhotrix, then the multiplication of two rhotrices is defined as

$$R_3\, o\ Q_3 = \left\langle \begin{array}{ccc} & ah+fc & \\ bh+gc & ch & dh+jc \\ & eh+kc & \end{array} \right\rangle.$$

Further, Absalom et al. [1] generalized the heart oriented multiplication of 3-dimensional rhotrices to n-dimensional rhotrices. The row column multiplication of 3-dimensional rhotrices is defined by Sani [8] as

$$R_3\, o\ Q_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & ch & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

Sani [9] also discussed the row-column multiplication of high dimension rhotrices as follows: Consider an n-dimensional rhotrix

$$P_n = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & \\ & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle,$$

where $t = (n+1)/2$. Sani [10] represented rhotrix as a coupled matrix and denoted it as $P_n = \langle a_{ij},\ c_{lk} \rangle = \langle A, C \rangle$ with $i, j = 1,\ 2,\ ..., t$ and $l, k = 1,\ 2,\ ..., t-1$. The multiplication of two rhotrices P_n and Q_n defined by Sani [9] is as follows:

$$P_n\, o\, Q_n = \langle a_{i_1 j_1},\, c_{l_1 k_1} \rangle\ o\ \langle b_{i_2 j_2},\, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1}\, b_{i_2 j_2})\ ,\ \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1}\, d_{l_2 k_2}) \right\rangle.$$

Sharma and Kanwar [11, 12] defined the trace of a rhotrix and discussed its properties. Sharma et al. [13] introduced circulant rhotrices in the literature of rhotrices.

Definition 1.1. [11] Let

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ a_{t1} & c_{t-11} & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & c_{1t-1} & a_{1t} \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ & & a_{t\,t-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & & \\ & & & a_{t\,t-1} & c_{t-1t-1} & a_{t-1t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

$$= \langle a_{i\,j},\ c_{l\,k} \rangle = \langle A,\ C \rangle\ ,$$

be a rhotrix, where $a_{i\,j},\ c_{l\,k} \in \mathbb{R}$, $1 \le i, j \le t$, $1 \le l, k \le t-1$ and $t = \frac{n+1}{2}$. The trace of R_n is defined as

$$\begin{aligned} Tr\ R_n &= \sum_{i=1}^{t} a_{ii} + \sum_{i=1}^{t-1} c_{ii} \\ &= Tr\ A + Tr\ C \\ &= \sum_{r=1}^{t} x_{rs} \end{aligned}$$

That is, the trace of a rhotrix R_n, which is the sum of all its elements of the vertical diagonal, where $R_n = \langle x_{r\,s} \rangle$ is the general representation to the n–dimensional rhotrix. The elements $x_{r\,s}$ for $r = s\,(r \neq s)$ represent the vertical (non–vertical) diagonal entries of the rhotrix R_n.

2 Trace of positive integral powers of a three dimensional rhotrix

In this section, we put forward a result to evaluate the trace of any positive integral power of a three dimensional rhotrix.

Theorem 2.1. *Let $R_3 =< A, B >$ be a three dimensional rhotrix whose coupled matrices are $A = (a_{ij})_{2\times 2}$ and $B = (c)_{1\times 1}$. Then, the trace of R_3^n, where n is positive even integer, is given by:*

$$TrR_3^n = (TrA)^n - n.|A|.(TrA)^{n-2} + \frac{n\,(n-3)}{2!}.|A|^2.(TrA)^{n-4} - \frac{n\,(n-4)\,(n-5)}{3!}.|A|^3.(TrA)^{n-6} +$$
$$\ldots + (-1)^{\frac{n}{2}} \frac{n\left(n - \frac{n}{2} - 1\right)\left(n - \frac{n}{2} - 2\right) \ldots .up\ to\ \frac{n}{2}\ terms}{\left(\frac{n}{2}\right)!}.|A|^{\frac{n}{2}}(TrA)^{n-2\cdot\frac{n}{2}} + (TrB)^n.$$

Proof. Consider a three dimensional rhotrix

$$R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle,$$

where a, b, c, d, e are real numbers and the coupled matrices are given as $A = \begin{bmatrix} a & d \\ b & e \end{bmatrix}$ and $B = [c]$. Then, $|A| = ae - bd$ and $Tr\ A = a + e$.

$$A^2 = \begin{bmatrix} a^2 + bd & d(a+e) \\ b(a+e) & bd + e^2 \end{bmatrix}$$

and

$$\begin{aligned} Tr\ A^2 = a^2 + 2bd + e^2 &= a^2 + 2ae + e^2 - 2ae + 2bd \\ &= (a+e)^2 - 2(ae - bd) \end{aligned}$$

or,

$$TrA^2 = (TrA)^2 - 2.\,|A|\,. \tag{2.1}$$

Similarly,

$$A^3 = \left[\begin{array}{cc} a^3 + abd + bd(a+e) & a^2 d + d^2 b + ed(a+e) \\ ab\,(a+e) + b^2 d + e^2 b & bd\,(a+e) + dbe + e^3 \end{array} \right]$$

and

$$Tr\ A^3 = a^3 + e^3 + 3bd(a+e) = (a+e)^3 - 3(a+e)(ae-bd)$$

or,

$$TrA^3 = (TrA)^3 - 3.\,|A|\,.\,Tr\ A. \tag{2.2}$$

Replacing A by A^2 in (2.1), we get

$$\begin{aligned} Tr\ \left(A^2\right)^2 &= \left(TrA^2\right)^2 - 2.\,\left|A^2\right| \\ &= \left((TrA)^2 - 2.\,|A|\right)^2 - 2(|A|)^2 \end{aligned}$$

or,

$$TrA^4 = (TrA)^4 - 4.\,|A|\,.(TrA)^2 + \frac{4(4-3)}{2!}.|A|^2.(TrA)^{4-2.\frac{4}{2}}$$

Also, replacing A by A^2 in (2.2), we get

$$Tr\ A^6 = Tr\ \left(A^2\right)^3 = \left(TrA^2\right)^3 - 3.\,\left|A^2\right|.\,TrA^2$$

$$= \left((TrA)^2 - 2.\,|A|\right)^3 - 3.(|A|)^2((Tr\ A)^2 - 2.\,|A|)$$

$$= (TrA)^6 - 8.|A|^3 - 6.\,|A|\,.(TrA)^4 + 12|A|^2.\ (TrA)^2 - 3.(|A|)^2(Tr\ A)^2 + 6.(|A|)^3$$

$$TrA^6 = (TrA)^6 - 6.\,|A|\,.(TrA)^4 + \frac{6\,(6-3)}{2!}.|A|^2.\ (TrA)^2 - \frac{6\,(6-4)\,(6-5)}{3!}.|A|^3.\ (TrA)^{6-2.\frac{6}{2}}.$$

Thus, when n is an even integer, then

$$TrA^n = (TrA)^n - n.|A|.(TrA)^{n-2} + \frac{n\,(n-3)}{2!}.|A|^2.(TrA)^{n-4} - \frac{n\,(n-4)\,(n-5)}{3!}.|A|^3.(TrA)^{n-6} +$$

$$\ldots + (-1)^{\frac{n}{2}} \frac{n\left(n-\frac{n}{2}-1\right)\left(n-\frac{n}{2}-2\right)\ldots \text{up to } \frac{n}{2} \text{ terms}}{\left(\frac{n}{2}\right)!}.|A|^{\frac{n}{2}}(TrA)^{n-2.\frac{n}{2}}.$$

Also,

$$Tr\ B^n = c^n$$

By Definition 1.1 we know that

$$Tr\ R_3 = Tr\ A + Tr\ B.$$

Therefore,

$$TrR_3^n = Tr\ A^n + Tr\ B^n.$$

Hence,

$$TrR_3^n = \sum\nolimits_{k=0}^{n/2} \frac{(-1)^k}{k!} \cdot (n(n-k-1)(n-k-2)\ldots \text{up to } k \text{ terms}) \cdot |A|^k (TrA)^{(n-2k)} + c^n$$

where n is an even integer.

$\square$

Example 2.2. Let

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & -1 \\ & 2 & \end{array} \right\rangle$$

be a three dimensional rhotrix such that its coupled matrices are $A = \left[\begin{array}{cc} 1 & -1 \\ 2 & 2 \end{array} \right]$ and $B = [1]$. Then, $Tr\ A = 3$ and $|A| = 4$ and $Tr\ B = 1$. Now,

$$Tr\ A^{10} = \sum_{k=0}^{5} \frac{(-1)^k}{k!}(n\,(n-k-1)\,(n-k-2)\ldots \text{up to } k \text{ terms}).|A|^k (TrA)^{n-2k}$$

$$= (TrA)^{10} - 10.\,|A|\,.\,(TrA)^8 + \frac{10.7}{2!}.|A|^2.(TrA)^6 - \frac{10.6.5}{3!}.\,|A|^3(TrA)^4 + \frac{10.5.4.3}{4!}.|A|^4(TrA)^2 - \frac{10.4.3.2.1}{5!}.\,|A|^5$$
$$= (3)^{10} - 10.4.\,(3)^8 + 35.\,(4)^2.\,(3)^6 - 50.(4)^3.\,(3)^4 + 25.\,(4)^4.(3)^2 - 2(4)^5 = 1201$$

and $Tr\ B^{10} = 1$. Therefore, ${R_3}^{10} = 1201 + 1 = 1202$.

Theorem 2.3. *Let $R_3 =< A, B >$ be a three dimensional rhotrix whose coupled matrices are $A = (a_{ij})_{2\times 2}$ and $B = (c)_{1\times 1}$ .Then, the trace of R_3^n where, n is a positive odd integer, is given by*

$$TrR_3^n = \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{k!} \cdot (n(n-k-1)(n-k-2)\dots \mathit{up\ to}\ k\ \mathit{terms}) \cdot |A|^k \cdot (TrA)^{n-2k} + (TrB)^n$$

where $|A| = \det A$.

Proof. Consider a three dimensional rhotrix

$$R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

where a, b, c, d, e are real numbers and the coupled matrices are given by $A = \begin{bmatrix} a & d \\ b & e \end{bmatrix}$ and $B = [c]$.

Then, from the Theorem 2.1 TrA^3 is given by

$$TrA^3 = (TrA)^3 - 3.\,|A|\,.\,Tr\ A$$
$$= (TrA)^{3-2.0} + \frac{(-1)^1}{1!}.|A|^1.\ (Tr\ A)^{3-2.1}.$$

Similarly,

$$A^5 = A^3.A^2$$
$$= \begin{bmatrix} a^3 + abd + bd(a+e) & a^2d + d^2b + ed(a+e) \\ ab\,(a+e) + b^2d + e^2b & bd\,(a+e) + dbe + e^3 \end{bmatrix} \begin{bmatrix} a^2 + bd & d(a+e) \\ b(a+e) & bd + e^2 \end{bmatrix}$$

Now,

$$TrA^5 = a^5 + a^3bd + 2a^2bd\,(a+e) + a^3bd + ab^2d^2 + 4b^2d^2\,(a+e) + ebd(a+e)^2$$
$$+ abd(a+e)^2 + 2e^2bd\,(a+e) + eb^2d^2 + 2e^3bd$$
$$= (a+e)^5 - 5.\,(ae-bd)\,.(a+e)^3 + 5(ae-bd)^2.\,(a+e)$$
$$= (TrA)^5 - 5.\,|A|\,.(TrA)^3 + \frac{5\,(5-3)}{2!}.|A|^2.\ (Tr\ A)$$
$$= (TrA)^{5-2.0} + \frac{(-1)^1}{1!}.5.|A|^1.(TrA)^{5-2.1} + \frac{(-1)^2}{2!}5\,(5-3)\,.|A|^2.\ (Tr\ A)^{5-2.2}$$

Thus, when n is an odd integer, then

$$TrA^n = \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{k!} \cdot (n(n-k-1)(n-k-2)\dots \text{ up to } k \text{ terms}) \cdot |A|^k (TrA)^{n-2k}.$$

Also,

$$Tr\ B^n = c^n.$$

Sharma and Kanwar [11] showed that the trace of a rhotrix is given by,

$$Tr\ R_3 = Tr\ A + Tr\ B.$$

Therefore,

$$TrR_3^n = Tr\ A^n + Tr\ B^n.$$

Hence,

$$TrR_3^n = \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\dots \text{up to } k \text{ terms}) \cdot |A|^k (TrA)^{n-2k} + c^n$$

where n is an odd positive integer.

□

Example 2.4. Let

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & -1 \\ & 2 & \end{array} \right\rangle$$

be a three dimensional rhotrix such that its coupled matrices are $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ and $B = [1]$. Then, $Tr\ A = 3$ and $|A| = 4$ and $Tr\ B = 1$. Now,

$$Tr\ A^9 = \sum_{k=0}^{4} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\ldots \text{up to } k \text{ terms}) \cdot |A|^k (TrA)^{n-2k}$$

$$= (TrA)^9 - 9.\,|A|\,.\ (TrA)^7 + \frac{9.7}{2!}.|A|^2.(TrA)^5 - \frac{9.5.4}{3!}.\ |A|^3 (TrA)^3 + \frac{9.4.3}{4!}.|A|^4\ .TrA$$

$$= (3)^9 - 9.4.\ (3)^7 + \frac{63}{2}.\ (4)^2.\ (3)^5 - 30.(4)^3.\ (3)^3 + \frac{9}{2}.\ (4)^4.3 = 15039$$

and $Tr\ B^{10} = 1$. Therefore,

$$Tr\ {R_3}^{10} = 15039 + 1 = 15040.$$

3 Trace of positive integral powers of a five dimensional rhotrix

In this section, we put forward a result to evaluate the trace of any positive power of a five dimensional rhotrix.

Theorem 3.1. *Let $R_5 =< A, B >$ be a five dimensional rhotrix whose coupled matrices are $A = (a_{ij})_{3\times 3}$ and $B = (b_{ij})_{2\times 2}$. Then, the trace of R_5^n where n is a positive even integer is given by*

$$Tr\ R_5^n = \sum_{r_1=0}^{\lfloor \frac{n}{3} \rfloor} \left[\sum_{r_0=0}^{\lfloor \frac{n-3r_1}{2} \rfloor} \frac{(-1)^{r_0}}{r_0!} \left[n\,(n-r_0-2r_1-1)\,(n-r_0-2r_1-2)\ldots(n-2r_0-3r_1+1) \right] \times (TrA)^{n-2r_0-3r_1}.A_2^{r_0} \right] \frac{(|A|)^{r_1}}{r_1!} + \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\ldots(\textit{up to } k \textit{ terms})) \times |B|^k (TrB)^{n-2k}$$

and the trace of R_5^n when n is positive odd integer, is given by

$$Tr\ R_5^n = \sum_{r_1=0}^{\lfloor \frac{n}{3} \rfloor} \left[\sum_{r_0=0}^{\lfloor \frac{n-3r_1}{2} \rfloor} \frac{(-1)^{r_0}}{r_0!} \left[n\,(n-r_0-2r_1-1)\,(n-r_0-2r_1-2)\ldots(n-2r_0-3r_1+1) \right] \times (TrA)^{n-2r_0-3r_1}.A_2^{r_0} \right] \frac{(|A|)^{r_1}}{r_1!} + \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\ldots(\textit{up to } k \textit{ terms})) \times |B|^k (TrB)^{n-2k}$$

where A_2 is the sum of all principal minors of order 2 in A, r_0 is the number of principal minor of order 1, r_1 is the number of principal minors of order 2, $\lfloor x \rfloor$ is the floor function, $|A| = \det A$ and $|B| = \det B$.

Proof. Let $R_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & d & p & b & \\ g & r & e & q & c \\ & h & s & f & \\ & & i & & \end{array} \right\rangle$ be a five dimensional rhotrix whose coupled matrices are $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. As stated earlier, $TrR_5 = TrA + TrB$ and $TrR_5^n = TrA^n + TrB^n$. Since of B is a matrix order 2×2, therefore, the trace of B^n can be written directly from the

Theorem 2.1 and the Theorem 2.3. To get the trace of R_5^n, we need to evaluate the trace of A^n. Now, as

$$|A| = a\,(ei - hf) - b\,(di - gf) + +c\,(dh - eg)$$

and $Tr\ A = a + e + i$, therefore,

$$\begin{aligned} TrA^2 &= (a+e+i)^2 - 2\,(ae - bd + ei - fh + ai - cg) \\ &= (TrA)^2 - 2\sum_{i=1}^{3} |A_{ii}|, \end{aligned}$$

where $|A_{ii}|$ are the principal minors of order 2 given as $|A_{11}| = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$, $|A_{22}| = \begin{vmatrix} e & f \\ g & h \end{vmatrix}$, $|A_{33}| = \begin{vmatrix} a & c \\ g & i \end{vmatrix}$.

Therefore, if we put

$$A_2 = \sum_{i=1}^{3} |A_{ii}|$$

then

$$TrA^2 = (TrA)^2 - 2A_2.$$

Also,

$$\begin{aligned} TrA^4 &= a^4 + 2b^2d^2 + e^4 + 2c^2g^2 + 4a^2\,(bd + cg) + 4cdeh + 4e^2fh + 4cfgh \\ &\quad + 2f^2h^2 + cdhi + 4efhi + 4cgi^2 + 4fhi^2 + i^4 + 4a\,(bde + bfg + cdh + cgi) \\ &\quad + 4b\left(d\left(e^2 + cg + fh\right) + fg(e+i)\right) \\ &= (a+e+i)^4 - 4(a+e+i)^2\,(-bd + ae - cg - fh + ai + ei) \\ &\qquad + 2(-bd + ae - cg - fh + ai + ei)^2 \\ &\qquad + 4(a+e+i)(-ceg + bfg + cdh - afh - bdi + aei) \\ &= (TrA)^4 - 4\sum_{i=1}^{3} |A_{ii}|.TrA + 2(\sum_{i=1}^{3} |A_{ii}|)^2 + 4.TrA.\ |A| \\ &= (TrA)^4 - 4A_2.TrA + 2(A_2)^2 + 4.TrA.\ |A|\,. \end{aligned}$$

Similarly,

$$TrA^5 = (TrA)^5 - 5A_2.(TrA)^3 + 5\ TrA.(A_2)^2 + 5\ (TrA)^2.\ |A| - 5.A_2.|A|\,.$$

Therefore, TrA^n can be generalized as

$$\begin{aligned} &Tr\ A^n \\ &= \sum\nolimits_{r_1=0}^{\lfloor \frac{n}{3} \rfloor} \left[\sum\nolimits_{r_0=0}^{\lfloor \frac{n-3r_1}{2} \rfloor} \frac{(-1)^{r_0}}{r_0!} \left[n\,(n - r_0 - 2r_1 - 1)\,(n - r_0 - 2r_1 - 2)\ldots(n - 2r_0 - 3r_1 + 1)\right] \times \right. \\ &\left. (TrA)^{n-2r_0-3r_1}.A_2^{r_0} \right] \frac{(|A|)^{r_1}}{r_1!} \end{aligned}$$

For even positive integer n and using Theorem 2.1, we have

$$Tr\ B^n = \sum\nolimits_{k=0}^{\frac{n}{2}} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\ldots(\text{up to } k \text{ terms})) \cdot |B|^k (TrB)^{n-2k}$$

Therefore, $TrR_5^n = Tr\ A^n + Tr\ B^n$. Now,

$$\begin{aligned} &Tr\ R_5^n \\ &= \sum\nolimits_{r_1=0}^{\lfloor \frac{n}{3} \rfloor} \left[\sum\nolimits_{r_0=0}^{\lfloor \frac{n-3r_1}{2} \rfloor} \frac{(-1)^{r_0}}{r_0!} \left[n\,(n - r_0 - 2r_1 - 1)\,(n - r_0 - 2r_1 - 2)\ldots(n - 2r_0 - 3r_1 + 1)\right] \times \right. \\ &\left. (TrA)^{n-2r_0-3r_1}.A_2^{r_0} \right] \frac{(|A|)^{r_1}}{r_1!} + \sum\nolimits_{k=0}^{\frac{n}{2}} \frac{(-1)^k}{k!} \cdot (n\,(n-k-1)\,(n-k-2)\ldots(\text{up to } k \text{ terms})) \times \\ &|B|^k (TrB)^{n-2k} \end{aligned}$$

Also, $R_5^n = Tr\ A^n + Tr\ B^n$, for odd n, this gives

$$Tr\ R_5^n$$
$$= \sum_{r_1=0}^{\lfloor \frac{n}{3} \rfloor} \left[\sum_{r_0=0}^{\lfloor \frac{n-3r_1}{2} \rfloor} \frac{(-1)^{r_0}}{r_0!} \left[n\left(n-r_0-2r_1-1\right)\left(n-r_0-2r_1-2\right)\ldots\left(n-2r_0-3r_1+1\right)\right] \times \right.$$
$$\left. (TrA)^{n-2r_0-3r_1}.A_2^{r_0} \right] \frac{(|A|)^{r_1}}{r_1!} + \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{k!} \cdot \left(n\left(n-k-1\right)\left(n-k-2\right)\ldots(\text{up to } k \text{ terms})\right) \times$$
$$|B|^k (TrB)^{n-2k}$$

□

4 Trace of a special type of n-dimensional circulant rhotrix

Now we prove below a theorem for the trace of a special type of n-dimensional circulant rhotrix.

Theorem 4.1. *Let $R_n = \langle A_t, B_{t-1} \rangle$ be an $n-$dimensional special circulant rhotrix whose diagonal entries are zero and all other entries are one. A_t, B_{t-1} are the coupled matrices of R_n, where $t = \frac{n+1}{2}$, $A_t = (a_{ij})$, $1 \le i,j \le t$ and $B_{t-1} = (b_{ij})$, $1 \le i,j \le t-1$ such that $a_{ij} = \begin{cases} 0, & \text{if } i=j \\ 1, & \text{if } i \ne j \end{cases}$ and $b_{ij} = \begin{cases} 0, & \text{if } i=j \\ 1, & \text{if } i \ne j \end{cases}$. Then, the trace of positive powers of R_n, when r is a positive even integer, is given by*

$$Tr\ R_n^r = \sum_{m=1}^{\frac{r}{2}} N(t,m)t(t-1)^m (t-2)^{r-2m} + \sum_{m=1}^{\frac{r}{2}} N(t-1,m)\left(t-1\right)\left(t-2\right)^m (t-3)^{r-2m},$$

and the trace of positive powers of R_n, when r is a positive odd integer, is given by

$$Tr\ R_n^r = \sum_{m=1}^{\frac{r-1}{2}} N(t,m)t(t-1)^m (t-2)^{r-2m} + \sum_{m=1}^{\frac{r-1}{2}} N(t-1,m)\left(t-1\right)\left(t-2\right)^m (t-3)^{r-2m},$$

where, $N\left(k,1\right)=1$, $N\left(k,\frac{k}{2}\right)=1$, $N\left(k,\frac{k-1}{2}\right)=\frac{k-1}{2}$ and $N\left(k,\ r\right)=N\left(k-1,\ r\right)+N\left(k-2,\ r-1\right).$

Proof. The rhotrix R_n is given by

$$R_n = \left\langle \begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & 0 & 1 & & \\ & & & & & \ddots & \\ & & 1 & 0 & 1 & & \\ 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & 1 & 0 & 1 & & \\ & & & 0 & & & \end{array} \right\rangle$$

whose coupled matrices for $t = \frac{n+1}{2}$ are A_t and B_{t-1} given as below:

$$A_t^2 = (a_{ij}) = \begin{cases} t-1, \text{ if } i=j \\ t-2, \text{ if } i \ne j \end{cases} \quad \text{and} \quad B_{t-1}^2 = (b_{ij}) = \begin{cases} t-2, \text{ if } i=j \\ t-3, \text{ if } i \ne j. \end{cases}$$

Therefore,

$$Tr\ A_t^2 = t(t-1) \ \text{ and } \ Tr\ B_{t-1}^2 = \left(t-1\right)\left(t-2\right). \tag{4.1}$$

That is,

$$Tr\ A_t^2 = \sum_{m=1}^{\frac{r}{2}=1} N\left(2,1\right) t(t-1) \ \text{ and } \ B_{t-1}^2 = \sum_{m=1}^{\frac{r}{2}=1} N\left(2,1\right)\left(t-1\right)(t-2).$$

Hence,

$$Tr\ R_n^2 = Tr\ A_t^2 + Tr\ B_{t-1}^2 = \sum_{m=1}^{\frac{r}{2}=1} N\,(2,m)\,t(t-1) + \sum_{m=1}^{\frac{r}{2}=1} N\,(2,m)\,(t-1)(t-2).$$

Similarly,

$$A_t^3 = (a_{ij}) = \begin{cases} (t-1)(t-2), \text{ if } i=j \\ (t-1)+(t-2)^2, \text{ if } i \neq j \end{cases} \quad \text{and} \quad B_{t-1}^3 = (b_{ij}) = \begin{cases} (t-2)(t-3), \text{ if } i=j \\ (t-2)+(t-3)^2, \text{ if } i \neq j, \end{cases}$$

which gives,

$$Tr\ A_t^3 = t(t-1)(t-2) \text{ and } Tr\ B_{t-1}^3 = (t-1)\,(t-2)\,(t-3). \tag{4.2}$$

That is,

$$TrA_t^3 = \sum\nolimits_{m=1}^{\frac{r-1}{2}=1} N\,(3,m)\,t(t-1)(t-2) \quad \text{and} \quad TrB_{t-1}^3 = \sum\nolimits_{m=1}^{\frac{r-1}{2}=1} N\,(3,m)\,(t-1)(t-2)(t-3),$$

which gives,

$$Tr\ R_n^3 = Tr\ A_t^3 + Tr\ B_{t-1}^3 \quad = \sum_{m=1}^{\frac{r-1}{2}=1} N\,(3,m)\,t(t-1)(t-2) + \sum_{m=1}^{\frac{r-1}{2}=1} N\,(3,m)\,(t-1)(t-2)(t-3).$$

Now,

$$A_t^4 = (a_{ij}) = \begin{cases} (t-1)\ \ [(t-1)+(t-2)^2], \text{ if } \ i=j \\ (t-1)\,(t-2)+(t-2)\,[(t-1)+(t-2)^2], \text{ if } \ i \neq j \end{cases}$$

and

$$B_{t-1}^4 = (b_{ij}) = \begin{cases} (t-2)\ \ [(t-2)+(t-3)^2], \text{ if } \ i=j \\ (t-2)\,(t-3)+(t-3)\,[(t-2)+(t-3)^2], \text{ if } \ i \neq j \end{cases}$$

which lends

$$Tr\ A_t^4 = t(t-1)^2 + t(t-1)(t-2)^2 \quad \text{and} \quad Tr\ B_{t-1}^4 = (t{-}1)(t-2)^2 + (t-1)(t-2)(t-3)^2. \tag{4.3}$$

That is,

$$TrA_t^4 = \sum_{m=1}^{\frac{r}{2}=2} N\,(4,m)\,t(t-1)^m(t-2)^{4-2m} \quad \text{and} \quad TrB_t^4 = \sum_{m=1}^{\frac{r}{2}=2} N\,(4,m)\,(t-1)\,(t-2)^m(t-3)^{4-2m}$$

which gives,

$$Tr\ R_n^4 = Tr\ A_t^4 + Tr\ B_{t-1}^4 \quad = \sum_{m=1}^{\frac{r}{2}=2} N\,(4,m)\,t(t-1)^m(t-2)^{4-2m} + \sum_{m=1}^{\frac{r}{2}=2} N\,(4,m)\,(t-1)(t-2)^m(t-3)^{4-2m}.$$

Now,

$$A_t^5 = (a_{ij}) = \begin{cases} 2(t-1)^2\,(t-2)+(t-1)(t-2)^3, \text{ if } \ i=j \\ (t-1)^2+3\,(t-1)\,(t-2)^2+(t-2)^4], \text{ if } \ i \neq j \end{cases}$$

and

$$B_{t-1}^5 = (b_{ij}) = \begin{cases} 2(t-2)^2\,(t-3)+(t-2)(t-3)^3, \text{ if } \ i=j \\ (t-2)^2+3\,(t-2)\,(t-3)^2+(t-3)^4], \text{ if } \ i \neq j, \end{cases}$$

which gives,

$$Tr\ A_t^5 = 2t(t-1)^2(t{-}2) + t(t{-}1)(t-2)^3 \quad \text{and} \quad Tr\ B_{t-1}^5 = 2\,(t-1)\,(t-2)^2\,(t-3) + (t{-}1)(t{-}2)(t-3)^3. \tag{4.4}$$

That is,

$$TrA_t^5 = \sum_{m=1}^{\frac{r}{2}=2} N\,(5,m)\,t(t-1)^m(t-2)^{5-2m} \quad \text{and} \quad TrB_{t-1}^5 = \sum_{m=1}^{\frac{r}{2}=2} N\,(5,m)\,(t-1)(t-2)^m(t-3)^{5-2m}$$

from where follows that

$$Tr\ R_n^5 = Tr\ A_t^5 + Tr\ B_{t-1}^5 = \sum_{m=1}^{\frac{r}{2}=2} N\,(5,m)\,t(t-1)^m(t-2)^{5-2m} + \sum_{m=1}^{\frac{r}{2}=2} N\,(5,m)\,(t-1)(t-2)^m(t-3)^{5-2m}.$$

Thus, by induction, we conclude that

$$Tr\ R_n^r = \sum_{m=1}^{\frac{r}{2}} N(t,m)t(t-1)^m(t-2)^{r-2m} + \sum_{m=1}^{\frac{r}{2}} N(t-1,m)\,(t-1)\,(t-\quad 2)^m(t-3)^{r-2m}$$

where, r is a positive even integer and

$$Tr\ R_n^r = \sum_{m=1}^{\frac{r-1}{2}} N(t,m)t(t-1)^m(t-2)^{r-2m} + \sum_{m=1}^{\frac{r-1}{2}} N(t-1,m)\,(t-1)\,(t-2)^m(t-3)^{r-2m}$$

where, r is a positive odd integer.

□

5 Conclusion

In this paper, we find the trace of positive integral powers of three and five dimensional rhotrices. Also, a result to find the traces of positive integral power of a special type of circulant rhotrix is derived.

Acknowledgments The authors thankfully acknowledge the support of the UGC-SAP and they also sincerely express their gratitude to the Editor-in-Chief of this Journal and the referees for their valuable suggestions for modifying this paper.

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