Original Article
Bulletin of Pure and Applied Sciences.
Mathematics & Statistics Vol.44 E, No.1.
January-June 2025: P.41-46

Print version ISSN 0970 6577
Online version ISSN 2320 3226
Available online at www.bpasjournals.com

Non-Homogeneous Ternary Cubic Diophantine Equation $5x^2 - 3y^2 = z^3$

Dr. N. Thiruniraiselvi[1], Dr. E. Premalatha[2], and Dr. M.A. Gopalan[3*]

[1, 3]Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
E-mail: drntsmaths@gmail.com, mayilgopalan@gmail.com
[2*]Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India
E-mail: premalathaem@gmail.com

***Corresponding Author:** Dr. M.A. Gopalan, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
E-mail: mayilgopalan@gmail.com

ABSTRACT

An unequal third degree Diophantine equation $5x^2 - 3y^2 = z^3$ has been studied for its solutions in integers. Plenty of integer solutions for the considered equation are obtained by utilizing transformation approach and process of factorization. Also, knowing an integer solution of the title equation, a generating formula to the sequence of solutions is presented.

Keywords: Ternary cubic equation, Non-homogeneous cubic equation, solutions in integers

How to cite this article: Thiruniraiselvi N., Premalatha E., and Gopalan M.A. (2025). Non-Homogeneous Ternary Cubic Diophantine Equation $5x^2 - 3y^2 = z^3$. *Bulletin of Pure and Applied Sciences-Math & Stat.*, 44E (1), 41-46.

Received on 23.02.2025, Revised on 12.05.2025, Accepted on 29.05.2025

1. INTRODUCTION

A significant and important subject area of Theory of Numbers is the theory of Diophantine equations which concentrates on attempting to determine solutions in integers for higher degree and many parameters indeterminate equations. Obviously, polynomial Diophantine equations are many due to definition. Especially, the third degree Diophantine equation in two parameters falls into the theory of elliptic curves which is a developed theory. There are numerous motivating cubic equations with multiple variables which have kindled the interest among Mathematicians. For example, the representation of integers by binary cubic forms is known very little. In this context, for simplicity and brevity, refer various forms of equations of degree three having two, three and four variables in [1-11]. In this paper, an unequal third degree Diophantine equation $5x^2 - 3y^2 = z^3$ has been studied for its solutions in integers. Plenty of integer solutions for the considered equation are obtained by utilizing transformation approach and process of factorization. Also, knowing an integer solution of the title equation, a generating formula to the sequence of solutions is presented.

2. METHOD OF ANALYSIS

The unequal Diophantine equation of degree three having three variables for solving is

$$5x^2 - 3y^2 = z^3 \quad (1)$$

The procedure to obtain various choices of solutions in integers for (1) through different ways is as follows:

Way 2. 1

The choice

$$x = ky, k > 1 \quad (2)$$

in (1) leads to

$$(5k^2 - 3)y^2 = z^3$$

which is satisfied by

$$y = (5k^2 - 3)t^{3s}, z = (5k^2 - 3)t^{2s} \quad (3)$$

In view 0f (2), we have

$$x = k(5k^2 - 3)t^{3s} \quad (4)$$

Thus, (3) and (4) satisfy (1).

Way 2.2

In (1), insertion of

$$y = kx, k > 1 \quad (5)$$

gives

$$(5 - 3k^2)x^2 = z^3$$

Which is satisfied by

$$x = (5 - 3k^2)t^{3s}, z = (5 - 3k^2)t^{2s} \quad (6)$$

In view of (5), we have

$$y = k(5 - 3k^2)t^{3s} \quad (7)$$

Thus, (6) and (7) represent the integer solutions to (1).

Way 2.3

Taking

$$x = 2X + 6kw, y = 2X + 10kw, z = 2w \quad (8)$$

in (1) , we get

$$X^2 = w^2(w + 15k^2) \quad (9)$$

Which is satisfied by

$$w = (s^2 + 6s - 6)k^2, X = (s + 3)(s^2 + 6s - 6)k^3$$

In view of (8), one obtains

$$x = k^3(2s + 12)(s^2 + 6s - 6),$$
$$y = k^3(2s + 16)(s^2 + 6s - 6),$$
$$z = 2(s^2 + 6s - 6)k^2 \quad (10)$$

Which give the integer solutions to (1).

A few solutions are presented in the following Table:

Table- Solutions

k	s	x	y	z
1	1	14	18	2
2	3	3024	3696	168
3	4	18360	22032	612
1	2	160	200	20
2	10	39424	44352	1232
2	12	60480	67200	1680

Fascinating connections among the solutions:

1. $x^3 + 6kxyz + 8k^3z^3 = y^3$
2. The choice

$$s = \alpha^2 + 4\alpha - 3$$

gives

$$y^2 = x^2 + [(2\alpha + 4)kz]^2$$

Which is similar to the well-known Pythagorean equation.

3. $\frac{xy}{k^2z^2}$ is one less than a perfect square.
4. Each of the expressions

$$\frac{(y-x)y}{k^2z^2}, \frac{(yz)}{2k}$$

is a perfect square when $s = 2\alpha^2 + 8\alpha$.

5. Each of the expressions

$$\frac{(y-x)x}{k^2z^2}, \frac{(xz)}{2k}$$

is a perfect square when $s = 2\alpha^2 + 4\alpha - 4$.

6. $(y+x)^2 = 4(y-kz)^2 = 4(x+kz)^2$

Way 2.4

The choice

$$x = X + 48kw, y = X + 80kw, z = 8w \tag{11}$$

in (1) , we get

$$X^2 = 256\, w^2(w + 15k^2) \tag{12}$$

Which is satisfied by

$$w = (s^2 + 6s - 6)\, k^2, X = 16\,(s+3)\,(s^2 + 6s - 6)\, k^3$$

In view of (11) , one obtains

$$x = 16\, k^3\,(s+6)\,(s^2 + 6s - 6),$$

$$y = 16\, k^3\,(s+8)\,(s^2 + 6s - 6),$$

$$z = 8\,(s^2 + 6s - 6)k^2$$

which give the integer solutions to (1).

Way 2.5
Taking

$$x = 2X + 6T, y = 2X + 10T, z = 2w \tag{13}$$

in (1) , we get

$$X^2 - 15T^2 = w^3 \tag{14}$$

Assume

$$w = a^2 - 15b^2 \tag{15}$$

Substituting (15) in (14) and applying factorization, we consider

$$X + \sqrt{15}\,T = (a + \sqrt{15}\,b)^3$$

from which we get

$$X = f(a,b)\ , T = g(a,b)$$

Where

$$f(a,b) = a^3 + 45ab^2\ ,$$

$$g(a,b) = 3a^2 b + 15b^3$$

In view of (13) ,we have

$$x = 2f(a,b) + 6g(a,b)\ ,$$

$$y = 2f(a,b) + 10g(a,b)\ ,$$

$$z = 2(a^2 - 15b^2)$$

Which satisfy (1).
Way 2.6
Rewrite (14) as

$$X^2 - 15T^2 = w^3 * 1 \tag{16}$$

Assume

$$1 = (4 + \sqrt{15})\ (4 - \sqrt{15}) \tag{17}$$

Substituting (15) & (17) in (16) and employing the factorization, one has

$$X + \sqrt{15}\,T = (4 + \sqrt{15})\ (a + \sqrt{15}\,b)^3$$

from which we get

$$X = 4f(a,b) + 15g(a,b)\ , T = f(a,b) + 4g(a,b)$$

In view of (13),

$$x = 14f(a,b) + 54g(a,b)\ ,$$

$$y = 18f(a,b) + 70g(a,b)\ ,$$

$$z = 2(a^2 - 15b^2)\ .$$

satisfy (1).
Note 2. 1
Apart from (17) ,the following representations
for the integer 1 may be considered as shown below:

$$1 = (31 + 8\sqrt{15})\ (31 - 8\sqrt{15})\ ,$$

$$1 = \frac{(8 + \sqrt{15})\ (8 - \sqrt{15})}{49}$$

3. GENERATION OF SOLUTIONS

Let (x_0, y_0, z_0) be any given integer solution to (1).

Assume the second solution to (1) as

$$x_1 = h - x_0, y_1 = h + y_0, z_1 = z_0 \tag{18}$$

Substituting (18) in (1) and simplifying, we have

$$h = 5x_0 + 3y_0$$

Substituting the above value of h in (18), we get

$$x_1 = 4x_0 + 3y_0, y_1 = 5x_0 + 4y_0$$

Following the analysis as above, the nth solution (x_n, y_n, z_n) for (1) is presented below:

$$x_n = \frac{\alpha^n + \beta^n}{2} x_0 + \frac{3(\alpha^n - \beta^n)}{2\sqrt{15}} y_0 ,$$

$$v_n = \frac{5(\alpha^n - \beta^n)}{2\sqrt{15}} x_0 + \frac{\alpha^n + \beta^n}{2} y_0 ,$$

$$z_n = z_0 , n = 1,2,3,...$$

where

$$\alpha = 4 + \sqrt{15} , \beta = 4 - \sqrt{15}$$

4. CONCLUSION

The non-homogeneous cubic equation with three variables given by $5x^2 - 3y^2 = z^3$ is analyzed for patterns of solutions in integers through different techniques. The interested readers may attempt for solving other choices of cubic Diophantine equations having more variables.

REFERENCES

1. Dickson, L. E. (1952). *History of the theory of numbers* (Vol. 2). Chelsea Publishing Company. https://archive.org/details/historyoftheoryo01dick/page/n1/mode/2up
2. Mordell, L. J. (1969). *Diophantine equations*. Academic Press.
3. Vidhyalakshmi, S., Gopalan, M. A., & Thangam, S. A. (2014). On the ternary cubic Diophantine equation. *International Journal of Innovative Research and Review,* *2*(3), 34–39. https://www.cibtech.org/J-Innovative-Research-Review/Publications/2014/Vol-2-No-3/JIRR-02-03-CONTENT.htm
4. Premalatha, E., & Gopalan, M. A. (2020). On the cubic equation with four unknowns. *International Journal of Advances in Engineering and Management,* *2*(2), 31–41. https://ijaem.net/issue_dcp/On%20Homogeneous%20Cubic%20Equation%20with%20Four%20Unknowns.pdf
5. Gopalan, M. A., Thiruniraiselvi, N., & Kiruthika, V. (2015). On the ternary cubic Diophantine equation. *International Journal of Recent Scientific Research,* *6*(9), 6197–6199. https://recentscientific.com/sites/default/files/3379.pdf
6. Gopalan, M. A., Thiruniraiselvi, N., & Sridev. (2015). On the ternary cubic equation. *International Journal of Multidisciplinary Research and Modern Engineering,* *1*(1), 317–319. https://rdmodernresearch.org/wp-content/uploads/2015/10/46.pdf
7. Premalatha, E., Shanthi, J., & Gopalan, M. A. (2021). On non-homogeneous cubic equation with four unknowns. *Biosc.Biotech.Res.Comm.,* *14*(5), 126–129. https://bbrc.in/wp-content/uploads/2021/05/BBRC_Vol_14_No_05_Special-Issue_24.pdf

8. Thiruniraiselvi, N., & Gopalan, M. A. (2021). A search on integer solutions to non-homogeneous ternary equation. *EPRA International Journal of Multidisciplinary Research, 7*(9), 15–18. https://eprajournals.com/IJMR/article/5814/abstract

9. Thiruniraiselvi, N., & Gopalan, M. A. (2024). Binary third degree Diophantine equation. *Oriental Journal of Physical Sciences, 9*(1), 1–4. https://www.orientaljphysicalsciences.org/vol9no1/binary-third-degree-diophantine-equation-5-x-y-3-8xy/

10. Thiruniraiselvi, N., & Gopalan, M. A. (2024). A new class of integer solutions to homogeneous cubic equation with four unknowns. *International Journal of Applied Sciences and Mathematical Theory, 10*(2), 1–7.https://www.iiardjournals.org/journal/?j=IJASMT
https://doi.org/10.56201/ijasmt.v10.no2.2024.pg1.7

11. Thiruniraiselvi, N., Kumar, S., & Gopalan, M. A. (2024). Non-homogeneous binary cubic equation. *Pure and Applied Mathematics Journal, 8*(11), 166–191. https://stm.bookpi.org/RUMCS-V8/article/view/14904 https://doi.org/10.9734/bpi/rumcs/v8/529

****