

On the integral representations of some of the Horn's double and Srivastava's triple hypergeometric functions of matrix arguments *

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Abstract We propose to define the Horn's double hypergeometric functions H_3 and H_4 of matrix arguments and deduce some integral representations for these two functions. Utilizing the first author's definitions (Upadhyaya, Lalit Mohan and Dhami, H.S., Matrix generalizations of multiple hypergeometric functions; #1818, Nov.2001, IMA Preprint Series, University of Minnesota, Minneapolis, U.S.A. (Retrieved from the University of Minnesota Digital Conservancy, <http://hdl.handle.net/11299/3706>); Upadhyaya, Lalit Mohan, Matrix Generalizations of Multiple Hypergeometric Functions by Using Mathai's Matrix Transform Techniques (Ph.D. Thesis, Kumaun University, Nainital, Uttarakhand, India), #1943, Nov. 2003, IMA Preprint Series, University of Minnesota, Minneapolis, U.S.A. (<https://www.ima.umn.edu/sites/default/files/1943.pdf> <http://www.ima.umn.edu/preprints/abstracts/1943ab.pdf><http://www.ima.umn.edu/preprints/nov2003/1943.pdf> <http://hdl.handle.net/11299/3955> <https://zbmath.org/?q=an:1254.33008> <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.192.2172&rank=52>). (Retrieved from the University of Minnesota Digital Conservancy, <http://hdl.handle.net/11299/3955>)) of the Srivastava's triple hypergeometric functions H_A and H_B of matrix arguments, we further establish a number of integral representations for these two Srivastava's triple hypergeometric functions, which generalize some of the recent

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results of Choi, Hasanov and Turaev (Choi, Junesang, Hasanov, Anvar and Turaev, Mamasali, Integral representations for Srivastava's hypergeometric function H_B , J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., Vol. 19, No. 2 (May 2012), (2012), 137–145: <http://dx.doi.org/10.7468/jksmeb.2012.19.2.137>; Choi, Junesang, Hasanov, Anvar and Turaev, Mamasali, Integral representations for Srivastava's hypergeometric function H_A , Honam Mathematical J., Vol. 34, No. 1, (2012), 113–124: <http://dx.doi.org/10.5831/HMJ.2012.34.1.113>; Choi, Junesang, Hasanov, Anvar and Turaev, Mamasali, Decomposition formulas and integral representations for some Exton hypergeometric functions, Journal of the Chungcheong Mathematical Society., Vol. 24, No. 4 (December 2011), (2011), 745–758) for these two of the Horn's double and the Srivastava's triple hypergeometric functions. For proving our results for these functions of matrix arguments we invoke the Mathai's matrix transform technique for real symmetric positive definite matrices as arguments. We conclude by stating the corresponding parallel results for these Horn's double and the Srivastava's triple hypergeometric functions, when their argument matrices are complex Hermitian positive definite, with the remark that these parallel results can be easily proved by following our given lines of proofs and by employing the corresponding known results available in the literature.

Key words Hypergeometric functions, Horn's double hypergeometric functions, Srivastava's triple hypergeometric functions, Exton's triple hypergeometric function, matrix argument, matrix transform, real positive definite, Hermitian positive definite.

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1 Introduction

The study of multiple hypergeometric functions of matrix arguments is still a developing field of research. Some initial work in this direction, as far as we are aware of, was initiated by Mathai [4, 6] which led the first author of this paper to investigate the multiple hypergeometric functions to a greater extent in his doctoral dissertation [2] which also contains a detailed study of generalized Horn's functions of matrix arguments [18]. In this paper we continue this study of the first author for some double and triple hypergeometric functions with matrix arguments. In particular, we propose to define the Horn's double hypergeometric functions H_3 and H_4 of matrix arguments with real symmetric positive definite and Hermitian positive definite matrix arguments and establish some integral representations for them. We also aim to deduce some integral representations for the Srivastava's [9, 10] (see also Exton [11] and Srivastava and Karlsson [16]) triple hypergeometric functions H_A and H_B of matrix arguments by recalling the first author's definitions [1, 2] for these functions when the argument matrices are real symmetric positive definite matrices. Based on the results available in the literature (see, Mathai [6] and Mathai and Provost [7]) we also analogously write the definitions of the Srivastava functions H_A and H_B for complex (Hermitian positive definite) matrix arguments and state the complex analogues of all the results proved by us for the case of real symmetric positive matrix arguments in the concluding section of the paper. We remark that all the results that we intend to prove in this paper provide the matrix generalizations of the corresponding results available in the literature due to the recent works of Choi, Hassanov and Turaev [12, 13, 14]. It is pertinent to mention here that the results earlier proved by the first author in his doctoral dissertation [2] contain some double and triple Laplace type integrals for the Srivastava's triple hypergeometric functions H_A, H_B, H_C of matrix arguments and while the first author has very recently introduced a very robust and versatile generalization of the classical Laplace transform, called the Upadhyaya transform [17], for which he has also given the generalizations to the case of real symmetric positive definite matrix arguments and complex (Hermitian) positive definite matrix arguments respectively in the sections 9 and 10 of this paper [17]. It still remains an open problem to explore and develop the theory of these matrix generalizations (see (9.3), p.503 and (10.3), p.504 in [17]) of the Upadhyaya transform and also to investigate the Upadhyaya type integrals (i.e., integrals involving the Upadhyaya transform) for these and other multiple hypergeometric functions of matrix arguments both in the real as well as in the complex cases.

The scheme of the paper now follows. We mention the preliminary results and definitions in the first section of the paper for functions of real symmetric positive definite matrix arguments. The integral representations concerning the Horn's functions H_3 and H_4 of real symmetric positive definite matrices are deduced in the second section and those concerning the Srivastava's triple hypergeometric functions are proved in the third section of the paper. All the matrices appearing in the sections 1 to 3 of this paper are real symmetric positive definite matrices of order $(p \times p)$, while those appearing in the fourth section of the paper are $(p \times p)$ complex (Hermitian) positive definite (see [8]). The Mathai's matrix transform technique [15] is employed by us to deduce our results.

For the notations used in this paper, we mention that $A > 0$ means that the matrix A is positive definite, $A^{1/2}$ represents the symmetric square root of the matrix A , A' denotes the transpose of the matrix A , $\text{Re}(\cdot)$ the real part of $(\cdot)$, while $|A|$ denotes the determinant of the matrix A . $0 < X < I$ means that $X > 0$ and $I - X > 0$, i.e.. all the eigenvalues of X lie between 0 and 1 (see, Mathai [4, p.3]). The matrix transform (M-transform) of a function $f(X)$ of a $(p \times p)$ real symmetric positive definite matrix X was defined by Mathai [15] as follows:

$$M_f(\rho) = \int_{X>0} |X|^{\rho-(p+1)/2} f(X) dX \quad (1.1)$$

for $X > 0$ and $\text{Re}(\rho) > (p-1)/2$, whenever $M_f(s)$ exists. Some preliminary definitions and results that will be used by us in the sequel now follow:

Theorem 1.1. Mathai [3, (2.24), p.23] - Let X and Y be $(p \times p)$ symmetric matrices of functionally independent real variables and A a $(p \times p)$ non singular matrix of constants. Then,

$$Y = AXA' \Rightarrow dY = |A|^{p+1} dX \quad (1.2)$$

and

$$Y = aX \Rightarrow dY = a^{p(p+1)/2} dX \quad (1.3)$$

where a is a scalar quantity.

Theorem 1.2. Gamma integral (Mathai [4, (2.1.3), p.33 and (2.1.2), p. 32]) -

$$\int_{X>0} |X|^{\alpha-(p+1)/2} e^{-\text{tr}(BX)} dX = |B|^{-\alpha} \Gamma_p(\alpha) \quad (1.4)$$

for $\text{Re}(\alpha) > (p-1)/2$, where,

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \cdots \Gamma(\alpha - \frac{p-1}{2}) \quad (1.5)$$

for $\text{Re}(\alpha) > (p-1)/2$ and $\text{tr}(X)$ denotes the trace of the matrix X .

Theorem 1.3. Type-1 Beta Integral (Mathai [4, (2.2.2), p.34]) -

$$B_p(\alpha, \beta) = \int_{0 < X < I} |X|^{\alpha-(p+1)/2} |I - X|^{\beta-(p+1)/2} dX = \frac{\Gamma_p(\alpha) \Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} \quad (1.6)$$

for $\text{Re}(\alpha) > (p-1)/2, \text{Re}(\beta) > (p-1)/2$.

Theorem 1.4. Type-2 Beta Integral (Mathai [4, (2.2.4), p.36]) -

$$B_p(\alpha, \beta) = \int_{Y>0} |Y|^{\alpha-(p+1)/2} |I + Y|^{-(\alpha+\beta)} dY = \frac{\Gamma_p(\alpha) \Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} \quad (1.7)$$

for $\text{Re}(\alpha) > (p-1)/2, \text{Re}(\beta) > (p-1)/2$.

Theorem 1.5. (Mathai [4, (6.13), p. 84]) - For $p = 2$,

$$4^{-p\rho} \frac{\Gamma_p(\frac{a+1}{2} - \rho) \Gamma_p(\frac{a}{2} + \frac{1}{4} - \rho)}{\Gamma_p(\frac{a+1}{2}) \Gamma_p(\frac{a}{2} + \frac{1}{4})} = \frac{\Gamma_p(a - 2\rho)}{\Gamma_p(a)} \quad (1.8)$$

Definition 1.6. (Mathai [4, (6.3), p.76]) The M-transform of Gauss's hypergeometric function ${}_2F_1$ of matrix argument

$${}_2F_1 = {}_2F_1(a, b; c; -X)$$

$$M({}_2F_1) = \int_{X>0} |X|^{\rho-(p+1)/2} {}_2F_1(a, b; c; -X) dX$$

is given by

$$= \frac{\Gamma_p(a-\rho) \Gamma_p(b-\rho) \Gamma_p(c) \Gamma_p(\rho)}{\Gamma_p(a) \Gamma_p(b) \Gamma_p(c-\rho)} \quad (1.9)$$

for $\text{Re}(\rho) > (p-1)/2$.

Definition 1.7. (Mathai [4, c.f. (2.3.5), p.38]) The M-transform of the Lauricella function F_C of n variables

$$F_C = F_C(a, b; c_1, \dots, c_n; -X_1, \dots, -X_n)$$

is given by

$$M(F_C) = \frac{\left\{ \prod_{j=1}^n \{\Gamma_p(c_j) \Gamma_p(\rho_j)\} \right\} \Gamma_p(a - \rho_1 - \dots - \rho_n) \Gamma_p(b - \rho_1 - \dots - \rho_n)}{\Gamma_p(a) \Gamma_p(b) \left\{ \prod_{j=1}^n \Gamma_p(c_j - \rho_j) \right\}} \quad (1.10)$$

for $\text{Re}(\rho_j, c_j - \rho_j, a - \rho_1 - \dots - \rho_n, b - \rho_1 - \dots - \rho_n) > (p-1)/2, j = 1, \dots, n$.

Now we reproduce the definition of the Srivastava functions H_A, H_B of matrix arguments due to the first author (see, [1, 2]).

Definition 1.8. The Srivastava function H_A of matrix arguments

$$H_A = H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z)$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$M(H_A) = \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) dX dY dZ$$

$$= \frac{\Gamma_p(a_1 - \rho_1 - \rho_3) \Gamma_p(a_2 - \rho_1 - \rho_2) \Gamma_p(a_3 - \rho_2 - \rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3)} \times$$

$$\frac{\Gamma_p(c_1) \Gamma_p(c_2) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(c_1 - \rho_1) \Gamma_p(c_2 - \rho_2 - \rho_3)} \quad (1.11)$$

for $\text{Re}(a_1 - \rho_1 - \rho_3, a_2 - \rho_1 - \rho_2, a_3 - \rho_2 - \rho_3, c_1 - \rho_1, c_2 - \rho_2 - \rho_3, \rho_i) > (p-1)/2$, where, $i = 1, 2, 3$.

Definition 1.9. For the Srivastava function H_B of matrix arguments

$$H_B = H_B(a_1, a_2, a_3; c_1, c_2, c_3; -X, -Y, -Z)$$

$$M(H_B) = \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$H_B(a_1, a_2, a_3; c_1, c_2, c_3; -X, -Y, -Z) dX dY dZ$$

$$= \frac{\Gamma_p(a_1 - \rho_1 - \rho_3) \Gamma_p(a_2 - \rho_1 - \rho_2) \Gamma_p(a_3 - \rho_2 - \rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3)} \times$$

$$\frac{\Gamma_p(c_1) \Gamma_p(c_2) \Gamma_p(c_3) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(c_1 - \rho_1) \Gamma_p(c_2 - \rho_2) \Gamma_p(c_3 - \rho_3)} \quad (1.12)$$

for $\text{Re}(a_1 - \rho_1 - \rho_3, a_2 - \rho_1 - \rho_2, a_3 - \rho_2 - \rho_3, c_i - \rho_i, \rho_i) > (p-1)/2$, where, $i = 1, 2, 3$.

2 Definitions of the Horn's functions H_3 and H_4 of matrix arguments and their properties

Now we proceed to define the Horn's functions H_3 and H_4 of matrix arguments when the argument matrices are real symmetric positive definite (see also Upadhyaya and Dhimi [18]).

Definition 2.1. The Horn's function H_3 of matrix arguments

$$H_3 = H_3(a, b; c; -X, -Y)$$

is defined as that class of functions for which the M-transform is the following:

$$\begin{aligned} M(H_3) &= \int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} H_3(a, b; c; -X, -Y) dX dY \\ &= \frac{\Gamma_p(a-2\rho_1-\rho_2) \Gamma_p(b-\rho_2) \Gamma_p(c) \Gamma_p(\rho_1) \Gamma_p(\rho_2)}{\Gamma_p(a) \Gamma_p(b) \Gamma_p(c-\rho_1-\rho_2)} \end{aligned} \quad (2.1)$$

for $\text{Re}(a-2\rho_1-\rho_2, b-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

Definition 2.2. For the Horn's function H_4 of matrix arguments

$$H_4 = H_4(a, b; c; -X, -Y)$$

$$\begin{aligned} M(H_4) &= \int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} H_4(a, b; c, d; -X, -Y) dX dY \\ &= \frac{\Gamma_p(a-2\rho_1-\rho_2) \Gamma_p(b-\rho_2) \Gamma_p(c) \Gamma_p(d) \Gamma_p(\rho_1) \Gamma_p(\rho_2)}{\Gamma_p(a) \Gamma_p(b) \Gamma_p(c-\rho_1) \Gamma_p(d-\rho_2)} \end{aligned} \quad (2.2)$$

where, $\text{Re}(a-2\rho_1-\rho_2, b-\rho_2, c-\rho_1, d-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

The following theorem generalizes the equation (4.1), p.754 of Choi et al. [14] only for the case of (2×2) matrices:

Theorem 2.3. For $p = 2$,

$$\begin{aligned} H_3(a, b; c; -X, -Y) &= \frac{\Gamma_p(c)}{\Gamma_p(b) \Gamma_p(c-b)} \int_0^I |T|^{b-(p+1)/2} |I-T|^{c-b-(p+1)/2} \times \\ &\quad \left| I + T^{1/2} Y T^{1/2} \right|^{-a} {}_2F_1\left(\frac{a+1}{2}, \frac{a}{2} + \frac{1}{4}; c-b; -4\left(I + T^{1/2} Y T^{1/2}\right)^{-1} (I-T)^{1/2} \times \right. \\ &\quad \left. X(I-T)^{1/2} \left(I + T^{1/2} Y T^{1/2}\right)^{-1}\right) dT \end{aligned} \quad (2.3)$$

for $\text{Re}(c-b, b) > (p-1)/2$.

Proof. Taking the M-transform of the right side of (2.3) with respect to the variables X, Y and the parameters ρ_1, ρ_2 respectively, we have,

$$\begin{aligned} &\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{-a} \times \\ &\quad {}_2F_1\left(\frac{a+1}{2}, \frac{a}{2} + \frac{1}{4}; c-b; -4\left(I + T^{1/2} Y T^{1/2}\right)^{-1} (I-T)^{1/2} \times \right. \\ &\quad \left. X(I-T)^{1/2} \left(I + T^{1/2} Y T^{1/2}\right)^{-1}\right) dX dY \end{aligned} \quad (2.4)$$

Applying the transformations

$$\begin{aligned} X_1 &= 4\left(I + T^{1/2} Y T^{1/2}\right)^{-1} (I-T)^{1/2} X (I-T)^{1/2} \left(I + T^{1/2} Y T^{1/2}\right)^{-1} \\ Y_1 &= T^{1/2} Y T^{1/2} \end{aligned}$$

then, from Theorem 1.1 it follows that

$$dX_1 = 4^{p(p+1)/2} |I + Y_1|^{-(p+1)} |I - T|^{(p+1)/2} dX$$

for constant Y , hence, Y_1 . In the light of this (2.4) yields,

$$4^{-p\rho_1} |I - T|^{-\rho_1} \int_{X_1 > 0} \int_{Y > 0} \left| I + T^{1/2} Y T^{1/2} \right|^{2\rho_1 - a} |X_1|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times \\ {}_2F_1 \left(\frac{a+1}{2}, \frac{a}{2} + \frac{1}{4}; c-b; -X_1 \right) dX_1 dY,$$

which, on writing the M-transform of ${}_2F_1$ with the help of (1.9) gives

$$4^{-p\rho_1} |I - T|^{-\rho_1} \frac{\Gamma_p \left(\frac{a+1}{2} - \rho_1 \right) \Gamma_p \left(\frac{a}{2} + \frac{1}{4} - \rho_1 \right) \Gamma_p (c-b) \Gamma_p (\rho_1)}{\Gamma_p \left(\frac{a+1}{2} \right) \Gamma_p \left(\frac{a}{2} + \frac{1}{4} \right) \Gamma_p (c-b-\rho_1)} \times \\ \int_{Y > 0} |Y|^{\rho_2 - (p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{2\rho_1 - a} dY.$$

This expression on simplification with the help of (1.8) produces

$$\frac{\Gamma_p (a-2\rho_1) \Gamma_p (c-b) \Gamma_p (\rho_1)}{\Gamma_p (a) \Gamma_p (c-b-\rho_1)} |I - T|^{-\rho_1} \int_{Y > 0} |Y|^{\rho_2 - (p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{2\rho_1 - a} dY \quad (2.5)$$

Applying the transformation $Y_1 = T^{1/2} Y T^{1/2}$ with $dY_1 = |T|^{(p+1)/2} dY$, $|Y_1| = |T| |Y|$ (from (1.1)) transforms into

$$\frac{\Gamma_p (a-2\rho_1) \Gamma_p (c-b) \Gamma_p (\rho_1)}{\Gamma_p (a) \Gamma_p (c-b-\rho_1)} |T|^{-\rho_2} |I - T|^{-\rho_1} \int_{Y_1 > 0} |Y_1|^{\rho_2 - (p+1)/2} |I + Y_1|^{-((a-2\rho_1-\rho_2)+\rho_2)} dY_1$$

in which Y_1 can now be integrated out by appealing to a type-2 Beta integral ((1.7)) to yield

$$\frac{\Gamma_p (a-2\rho_1-\rho_2) \Gamma_p (c-b) \Gamma_p (\rho_1) \Gamma_p (\rho_2)}{\Gamma_p (a) \Gamma_p (c-b-\rho_1)} |T|^{-\rho_2} |I - T|^{-\rho_1}. \quad (2.6)$$

Substituting (2.6) on the right side of (2.1) yields

$$\frac{\Gamma_p (a-2\rho_1-\rho_2) \Gamma_p (c) \Gamma_p (\rho_1) \Gamma_p (\rho_2)}{\Gamma_p (a) \Gamma_p (b) \Gamma_p (c-b-\rho_1)} \int_0^I |T|^{b-\rho_2-(p+1)/2} |I - T|^{c-b-\rho_1-(p+1)/2} dT$$

in which T can at once be integrated out by using a type-1 Beta integral ((1.6)) to arrive at $M(H_3)$ as given by (2.1), thus proving the theorem. $\square$

On similar lines the following generalization of equation (4.2), p. 755 of Choi et al. [14] can be proved for (2×2) matrices:

Theorem 2.4. For $p = 2$,

$$H_4(a, b; c_1, c_2; -X, -Y) = \frac{\Gamma_p(c_2)}{\Gamma_p(b) \Gamma_p(c_2-b)} \int_0^I |T|^{b-(p+1)/2} |I - T|^{c_2-b-(p+1)/2} \times \\ \left| I + T^{1/2} Y T^{1/2} \right|^{-a} {}_2F_1 \left(\frac{a+1}{2}, \frac{a}{2} + \frac{1}{4}; c_1; -4 \left(I + T^{1/2} Y T^{1/2} \right)^{-1} X \left(I + T^{1/2} Y T^{1/2} \right)^{-1} \right) dT \quad (2.7)$$

for $\text{Re}(c_2 - b, b) > (p-1)/2$.

In the following theorem, we find the M-transform of the Horn's double hypergeometric function $H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z)$ for $(p \times p)$ real symmetric positive definite matrices X, Y, Z :

Theorem 2.5.

$$M(H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z)) \\ = \frac{\Gamma_p(\alpha - 2\rho_1 - 2\rho_2 - \rho_3) \Gamma_p(\beta - \rho_3) \Gamma_p(\gamma) \Gamma_p(\delta) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(\alpha) \Gamma_p(\beta) \Gamma_p(\gamma - \rho_1 - \rho_2) \Gamma_p(\delta - \rho_3)} \quad (2.8)$$

for $\text{Re}(\alpha - 2\rho_1 - 2\rho_2 - \rho_3, \beta - \rho_3, \gamma - \rho_1 - \rho_2, \delta - \rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2$.

Proof. Taking the M-transform of the left side of (2.8) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 we get

$$\begin{aligned} & M(H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z)) \\ &= \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times \\ & \quad H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z) dX dY dZ \end{aligned} \quad (2.9)$$

Put $U = X + Y$, then for fixed Y , $dU = dX$, and $X = U - Y$ may be written as $X = U^{1/2}(I - U^{-1/2}YU^{-1/2})U^{1/2}$, in which on letting $V = U^{-1/2}YU^{-1/2}$, we get $X = U^{1/2}(I - V)U^{1/2}$, from where it follows that $|X| = |U||I - V|$, where, $0 < V < I$. Further, $V = U^{-1/2}YU^{-1/2}$ gives $dV = |U|^{-(p+1)/2}dY$ from (1.1) and $|V| = |U|^{-1}|Y|$. With these substitutions (2.9) takes the form

$$\begin{aligned} & M(H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z)) \\ &= \int_{U>0} \int_{0<V<I} \int_{Z>0} |U|^{\rho_1+\rho_2-(p+1)/2} |I - V|^{\rho_1-(p+1)/2} |V|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times \\ & \quad H_4(\alpha, \beta; \gamma, \delta; -U, -Z) dU dV dZ \end{aligned} \quad (2.10)$$

The above expression can be rewritten immediately as

$$\begin{aligned} & M(H_4(\alpha, \beta; \gamma, \delta; -X - Y, -Z)) \\ &= \left[\int_{U>0} \int_{Z>0} |U|^{\rho_1+\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} H_4(\alpha, \beta; \gamma, \delta; -U, -Z) dU dZ \right] \times \\ & \quad \int_0^I |V|^{\rho_2-(p+1)/2} |I - V|^{\rho_1-(p+1)/2} dV \end{aligned}$$

in which on writing the M-transform of the H_4 function from (2.2) and integrating the V - integral by using a type-1 Beta integral ((1.6)) and canceling out the common factor $\Gamma_p(\rho_1 + \rho_2)$ from the numerator and denominator at once gives the expression on the right hand side of (2.8). $\square$

In a similar manner we may also establish the following result:

Theorem 2.6.

$$\begin{aligned} & M(H_4(\alpha, \beta; \gamma, \delta; -X, -Y - Z)) \\ &= \frac{\Gamma_p(\alpha - 2\rho_1 - \rho_2 - \rho_3) \Gamma_p(\beta - \rho_2 - \rho_3) \Gamma_p(\gamma) \Gamma_p(\delta) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(\alpha) \Gamma_p(\beta) \Gamma_p(\gamma - \rho_1) \Gamma_p(\delta - \rho_2 - \rho_3)} \end{aligned} \quad (2.11)$$

for $\text{Re}(\alpha - 2\rho_1 - \rho_2 - \rho_3, \beta - \rho_2 - \rho_3, \gamma - \rho_1, \delta - \rho_2 - \rho_3, \rho_1, \rho_2, \rho_3) > (p - 1)/2$.

3 Properties of the Srivastava's functions H_A and H_B of matrix arguments

In this section we prove some results concerning the Srivastava's triple hypergeometric functions H_A and H_B of matrix arguments. The following theorem generalizes the result of the equation (2.13) p.142 of Choi et al. [12] for the case of (2×2) real symmetric positive definite matrices only.

Theorem 3.1. For $p = 2$,

$$\begin{aligned} & H_B(a_1, a_2, a_3; c_1, c_2, c_3; -X, -Y, -Z) \\ &= \frac{\Gamma_p(a_1 + a_2 + a_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3)} \int_0^I \int_0^I |T|^{a_1-(p+1)/2} |S|^{a_1+a_2-(p+1)/2} |I - T|^{a_2-(p+1)/2} \times \\ & \quad |I - S|^{a_3-(p+1)/2} F_C^{(3)} \left[\frac{a_1 + a_2 + a_3 + 1}{2}, \frac{a_1 + a_2 + a_3}{2} + \frac{1}{4}; c_1, c_2, c_3; \right. \\ & \quad \left. -4(I - T)^{1/2} T^{1/2} S X S T^{1/2} (I - T)^{1/2}, -4(I - T)^{1/2} (I - S)^{1/2} S^{1/2} Y S^{1/2} (I - S)^{1/2} (I - T)^{1/2}, \right. \\ & \quad \left. -4(I - S)^{1/2} T^{1/2} S^{1/2} Z S^{1/2} T^{1/2} (I - S)^{1/2} \right] dT dS \end{aligned} \quad (3.1)$$

for $\text{Re}(a_1, a_2, a_3) > (p - 1)/2$, where $F_C^{(3)}$ represents the Lauricella function F_C of three variables.

Proof. We take the M-transform of the right side of (3.1) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively to get

$$\int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} F_C^{(3)} \left[\frac{a_1+a_2+a_3+1}{2}, \frac{a_1+a_2+a_3}{2} + \frac{1}{4}; c_1, c_2, c_3; -4(I-T)^{1/2} T^{1/2} S X S T^{1/2} (I-T)^{1/2}, -4(I-T)^{1/2} (I-S)^{1/2} S^{1/2} Y S^{1/2} (I-S)^{1/2} (I-T)^{1/2}, -4(I-S)^{1/2} T^{1/2} S^{1/2} Z S^{1/2} T^{1/2} (I-S)^{1/2} \right] dX dY dZ \quad (3.2)$$

Employing the transformations

$$\begin{aligned} X_1 &= 4(I-T)^{1/2} T^{1/2} S X S T^{1/2} (I-T)^{1/2}, \\ Y_1 &= 4(I-T)^{1/2} (I-S)^{1/2} S^{1/2} Y S^{1/2} (I-S)^{1/2} (I-T)^{1/2}, \\ Z_1 &= 4(I-S)^{1/2} T^{1/2} S^{1/2} Z S^{1/2} T^{1/2} (I-S)^{1/2}, \end{aligned}$$

with

$$\begin{aligned} dX_1 &= 4^{p(p+1)/2} |I-T|^{(p+1)/2} |T|^{(p+1)/2} |S|^{(p+1)} dX, \\ dY_1 &= 4^{p(p+1)/2} |I-T|^{(p+1)/2} |I-S|^{(p+1)/2} |S|^{(p+1)/2} dY, \\ dZ_1 &= 4^{p(p+1)/2} |I-S|^{(p+1)/2} |T|^{(p+1)/2} |S|^{(p+1)/2} dZ, \end{aligned}$$

(from (1.1)) and

$$|X_1| = 4^p |I-T| |T| |S|^2 |X|, |Y_1| = 4^p |I-T| |I-S| |S| |Y|, |Z_1| = 4^p |I-S| |T| |S| |Z|$$

in (3.2) renders it into the following form

$$\begin{aligned} &4^{-p(\rho_1+\rho_2+\rho_3)} |T|^{-\rho_1-\rho_3} |I-T|^{-\rho_1-\rho_2} |S|^{-2\rho_1-\rho_2-\rho_3} |I-S|^{-\rho_2-\rho_3} \times \\ &\int_{X_1>0} \int_{Y_1>0} \int_{Z_1>0} |X_1|^{\rho_1-(p+1)/2} |Y_1|^{\rho_2-(p+1)/2} |Z_1|^{\rho_3-(p+1)/2} \times \\ &F_C^{(3)} \left[\frac{a_1+a_2+a_3+1}{2}, \frac{a_1+a_2+a_3}{2} + \frac{1}{4}; c_1, c_2, c_3; -X_1, -Y_1, -Z_1 \right] dX_1 dY_1 dZ_1, \end{aligned}$$

which, on writing the M-transform of the $F_C^{(3)}$ with the help of (1.10) lends

$$\begin{aligned} &4^{-p(\rho_1+\rho_2+\rho_3)} |T|^{-\rho_1-\rho_3} |I-T|^{-\rho_1-\rho_2} |S|^{-2\rho_1-\rho_2-\rho_3} |I-S|^{-\rho_2-\rho_3} \times \\ &\frac{\left\{ \prod_{j=1}^3 \{ \Gamma_p(c_j) \Gamma_p(\rho_j) \} \right\} \Gamma_p \left(\frac{a_1+a_2+a_3+1}{2} - \rho_1 - \rho_2 - \rho_3 \right) \Gamma_p \left(\frac{a_1+a_2+a_3}{2} + \frac{1}{4} - \rho_1 - \rho_2 - \rho_3 \right)}{\Gamma_p \left(\frac{a_1+a_2+a_3+1}{2} \right) \Gamma_p \left(\frac{a_1+a_2+a_3}{2} + \frac{1}{4} \right) \left\{ \prod_{j=1}^3 \Gamma_p(c_j - \rho_j) \right\}} \end{aligned}$$

An application of (1.8) to the last expression renders it into the form

$$\begin{aligned} &|T|^{-\rho_1-\rho_3} |I-T|^{-\rho_1-\rho_2} |S|^{-2\rho_1-\rho_2-\rho_3} |I-S|^{-\rho_2-\rho_3} \times \\ &\frac{\left\{ \prod_{j=1}^3 \{ \Gamma_p(c_j) \Gamma_p(\rho_j) \} \right\} \Gamma_p(a_1+a_2+a_3-2\rho_1-2\rho_2-2\rho_3)}{\Gamma_p(a_1+a_2+a_3) \left\{ \prod_{j=1}^n \Gamma_p(c_j - \rho_j) \right\}} \end{aligned}$$

Substituting this expression on the right side of (3.1) gives

$$\frac{\left\{ \prod_{j=1}^3 \{ \Gamma_p(c_j) \Gamma_p(\rho_j) \} \right\} \Gamma_p(a_1 + a_2 + a_3 - 2\rho_1 - 2\rho_2 - 2\rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3) \left\{ \prod_{j=1}^n \Gamma_p(c_j - \rho_j) \right\}} \times \\ \int_0^I \int_0^I |T|^{a_1 - \rho_1 - \rho_3 - (p+1)/2} |I - T|^{a_2 - \rho_1 - \rho_2 - (p+1)/2} |S|^{a_1 + a_2 - 2\rho_1 - \rho_2 - \rho_3 - (p+1)/2} \times \\ |I - S|^{a_3 - \rho_2 - \rho_3 - (p+1)/2} dT dS$$

which on integrating T and S by applying a type-1 Beta integral ((1.6)) gives $M(H_B)$ as given by (1.12). $\square$

The following theorem gives the generalization of equation (2.17) p.119 of Choi et al. [13].

Theorem 3.2.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(a_1 + a_2)}{\Gamma_p(a_1) \Gamma_p(a_2)} \int_0^I |T|^{a_1 - (p+1)/2} |I - T|^{a_2 - (p+1)/2} \times \\ & H_4 \left(a_1 + a_2, a_3; c_1, c_2; -(I - T)^{1/2} T^{1/2} X T^{1/2} (I - T)^{1/2}, \right. \\ & \left. -(I - T)^{1/2} Y (I - T)^{1/2} - T^{1/2} Z T^{1/2} \right) dT \end{aligned} \quad (3.3)$$

for $\text{Re}(a_1, a_2) > (p - 1)/2$.

Proof. We take the M-transform of the right side of (3.3) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 to get

$$\begin{aligned} & \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} |Z|^{\rho_3 - (p+1)/2} \times \\ & H_4 \left(a_1 + a_2, a_3; c_1, c_2; -(I - T)^{1/2} T^{1/2} X T^{1/2} (I - T)^{1/2}, \right. \\ & \left. -(I - T)^{1/2} Y (I - T)^{1/2} - T^{1/2} Z T^{1/2} \right) dX dY dZ \end{aligned} \quad (3.4)$$

Appealing to the transformations,

$$X_1 = (I - T)^{1/2} T^{1/2} X T^{1/2} (I - T)^{1/2}, Y_1 = (I - T)^{1/2} Y (I - T)^{1/2}, Z_1 = T^{1/2} Z T^{1/2}$$

with

$$dX_1 = |I - T|^{(p+1)/2} |T|^{(p+1)/2} dX, dY_1 = |I - T|^{(p+1)/2} dY, dZ_1 = |T|^{(p+1)/2} dZ$$

(from (1.1)) and

$$|X_1| = |I - T| |T| |X|, |Y_1| = |I - T| |Y|, |Z_1| = |T| |Z|$$

in (3.4) produces

$$\begin{aligned} & |T|^{-\rho_1 - \rho_3} |I - T|^{-\rho_1 - \rho_2} \int_{X_1>0} \int_{Y_1>0} \int_{Z_1>0} |X_1|^{\rho_1 - (p+1)/2} |Y_1|^{\rho_2 - (p+1)/2} |Z_1|^{\rho_3 - (p+1)/2} \times \\ & H_4(a_1 + a_2, a_3; c_1, c_2; -X_1, -Y_1 - Z_1) dX_1 dY_1 dZ_1, \end{aligned}$$

which on writing the M-transform of the H_4 function by utilizing the Theorem 2.6 generates

$$\begin{aligned} & |T|^{-\rho_1 - \rho_3} |I - T|^{-\rho_1 - \rho_2} \frac{\Gamma_p(a_1 + a_2 - 2\rho_1 - \rho_2 - \rho_3) \Gamma_p(a_3 - \rho_2 - \rho_3)}{\Gamma_p(a_1 + a_2) \Gamma_p(a_3) \Gamma_p(c_1 - \rho_1) \Gamma_p(c_2 - \rho_2 - \rho_3)} \times \\ & \Gamma_p(c_1) \Gamma_p(c_2) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3). \end{aligned}$$

Substituting this expression on the right side of (3.3) and integrating out T by using a type-1 Beta integral ((1.6)) produces $M(H_A)$ as given by (1.11). $\square$

The following theorem gives the generalization of equation (2.1) p.115 of Choi et al. [13].

Theorem 3.3.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(s)}{\Gamma_p(a_1)\Gamma_p(s-a_1)} \int_0^I |T|^{a_1-(p+1)/2} |I-T|^{s-a_1-(p+1)/2} \times \\ & H_A\left(s, a_2, a_3; c_1, c_2; -T^{1/2}XT^{1/2}, -Y, -T^{1/2}ZT^{1/2}\right) dT \end{aligned} \quad (3.5)$$

for $\operatorname{Re}(s-a_1, a_1) > (p-1)/2$.

Proof. This theorem can be proved in a similar manner as the above theorem by first taking the M-transform of (3.5) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 and then applying the transformations $X_1 = T^{1/2}XT^{1/2}$, $Y_1 = Y$, $Z_1 = T^{1/2}ZT^{1/2}$ and then substituting the resulting expression on the right side of (3.5) after writing the M-transform of the involved H_A function in the integrand on the right side of (3.5) and finally integrating out T by applying a type-1 Beta integral to arrive at $M(H_A)$. $\square$

The following Theorems 3.4 to 3.7 respectively generalize the results in equations (2.2), (2.3), (2.4) and (2.5) pp.115–116 of Choi et al. [13].

Theorem 3.4.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(s)}{\Gamma_p(a_2)\Gamma_p(s-a_2)} \int_0^I |T|^{a_2-(p+1)/2} |I-T|^{s-a_2-(p+1)/2} \times \\ & H_A\left(a_1, s, a_3; c_1, c_2; -T^{1/2}XT^{1/2}, -T^{1/2}YT^{1/2}, -Z\right) dT \end{aligned} \quad (3.6)$$

for $\operatorname{Re}(s-a_2, a_2) > (p-1)/2$.

Theorem 3.5.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(s)}{\Gamma_p(a_3)\Gamma_p(s-a_3)} \int_0^I |T|^{a_3-(p+1)/2} |I-T|^{s-a_3-(p+1)/2} \times \\ & H_A\left(a_1, a_2, s; c_1, c_2; -X, -T^{1/2}YT^{1/2}, -T^{1/2}ZT^{1/2}\right) dT \end{aligned} \quad (3.7)$$

for $\operatorname{Re}(s-a_3, a_3) > (p-1)/2$.

Theorem 3.6.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-s)\Gamma_p(s)} \int_0^I |T|^{s-(p+1)/2} |I-T|^{c_1-s-(p+1)/2} \times \\ & H_A\left(a_1, a_2, a_3; s, c_2; -T^{1/2}XT^{1/2}, -Y, -Z\right) dT \end{aligned} \quad (3.8)$$

for $\operatorname{Re}(c_1-s, s) > (p-1)/2$.

Theorem 3.7.

$$\begin{aligned} & H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\ &= \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-s)\Gamma_p(s)} \int_0^I |T|^{s-(p+1)/2} |I-T|^{c_2-s-(p+1)/2} \times \\ & H_A\left(a_1, a_2, a_3; c_1, s; -X, -T^{1/2}YT^{1/2}, -T^{1/2}ZT^{1/2}\right) dT \end{aligned} \quad (3.9)$$

for $\operatorname{Re}(c_2-s, s) > (p-1)/2$.

Now we prove in the following theorem the generalization of equation (2.6) p.116 of Choi et al. [13].

Theorem 3.8.

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(c_2)}{\Gamma_p(a_3)\Gamma_p(c_2 - a_3)} \int_0^I |T|^{a_3 - (p+1)/2} |I - T|^{c_2 - a_3 - (p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{-a_2} \left| I + T^{1/2} Z T^{1/2} \right|^{-a_1} \times \\
&\quad {}_2F_1 \left(a_1, a_2; c_1; - \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} X \times \right. \\
&\quad \left. \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2} \right) dT
\end{aligned} \tag{3.10}$$

for $\operatorname{Re}(c_2 - a_3, a_3) > (p-1)/2$.

Proof. On taking the M-transform of the right side of (3.10) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively we get,

$$\begin{aligned}
& \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} |Z|^{\rho_3 - (p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{-a_2} \left| I + T^{1/2} Z T^{1/2} \right|^{-a_1} \times \\
&\quad {}_2F_1 \left(a_1, a_2; c_1; - \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} X \times \right. \\
&\quad \left. \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2} \right) dX dY dZ
\end{aligned} \tag{3.11}$$

The application of the transformations

$$\begin{aligned}
& Y_1 = T^{1/2} Y T^{1/2}, Z_1 = T^{1/2} Z T^{1/2}, \\
& X_1 = \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} X \left(I + T^{1/2} Y T^{1/2} \right)^{-1/2} \left(I + T^{1/2} Z T^{1/2} \right)^{-1/2}, \\
& \text{i.e., } X_1 = (I + Z_1)^{-1/2} (I + Y_1)^{-1/2} X (I + Y_1)^{-1/2} (I + Z_1)^{-1/2}
\end{aligned}$$

with

$$dX_1 = |I + Z_1|^{-(p+1)/2} |I + Y_1|^{-(p+1)/2} dX, dY_1 = |T|^{(p+1)/2} dY, dZ_1 = |T|^{(p+1)/2} dZ$$

and

$$|X_1| = |I + Z_1|^{-1} |I + Y_1|^{-1} |X|, |Y_1| = |T| |Y|, |Z_1| = |T| |Z|$$

in (3.11) gives

$$\begin{aligned}
& |T|^{-\rho_2 - \rho_3} \int_{X_1>0} \int_{Y_1>0} \int_{Z_1>0} |X_1|^{\rho_1 - (p+1)/2} |Y_1|^{\rho_2 - (p+1)/2} |Z_1|^{\rho_3 - (p+1)/2} |I + Y_1|^{-(a_2 - \rho_1 - \rho_2 + \rho_2)} \times \\
& |I + Z_1|^{-(a_1 - \rho_1 - \rho_3 + \rho_3)} {}_2F_1(a_1, a_2; c_1; -X_1) dX_1 dY_1 dZ_1
\end{aligned}$$

which, on writing the M-transform of the ${}_2F_1$ function by (1.9) and integrating out Y_1 and Z_1 by using a type-2 Beta integral ((1.7)) yields

$$|T|^{-\rho_2 - \rho_3} \frac{\Gamma_p(c_1) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3) \Gamma_p(a_2 - \rho_1 - \rho_2) \Gamma_p(a_1 - \rho_1 - \rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(c_1 - \rho_1)} \tag{3.12}$$

Substituting this expression on the right side of (3.10) and integrating out T by using a type-1 Beta integral ((1.6)) yields $M(H_A)$ as given by (1.11), thus finishing the proof. $\square$

4 Corresponding results for functions of complex matrix argument

We now state the results corresponding to some of the necessary definitions and results mentioned and proven in the sections 2 and 3 above when the argument matrices of the involved hypergeometric functions are complex Hermitian positive definite. This is necessary in view of the remarks made by the first author in the section 4, pp. 213–215 of his paper [8]. It may be mentioned here, as also pointed out earlier that all the matrices appearing in this section of the paper are $(p \times p)$ complex

Hermitian positive definite matrices. The results corresponding to the Jacobians of (complex) matrix transformations and the corresponding definitions of the elementary functions (like the type-1 and type-2 Beta integrals etc.) are available in the monograph of Mathai [6, Chapter 3]. We prefer here to keep the same notation here for complex matrices as we have used in the preceding sections of this paper, unlike that used in the work of Mathai [6], where, the complex matrices are represented by placing a tilde ($\sim$) sign over the symbol of the matrix. It is also pertinent to mention here that the complex analogues of the results of the Theorems 2.3, 2.4 and 3.1 have a different structure (see, Mathai [6, Chapter 6, p.399]) so they will not be mentioned here. A well known fact available in the literature is that for writing the complex analogues of the relevant definitions and results proved above in the sections 2 and 3 is that we have to replace the expression $(p+1)/2$ appearing in the power of the determinants of the various matrices involved in the integrands of the results deduced above by p and the condition of convergence of the integral, which in the case of real symmetric positive definite matrices is $\operatorname{Re}(\cdot) > (p-1)/2$, the same has to be replaced by the expression $\operatorname{Re}(\cdot) > (p-1)$ in the complex case (see Mathai [6, pp. 364-365] and see also Mathai and Provost [7]). The complex analogues of the definitions and results given in section 1 above from (1.1) through (1.10) except the result of Theorem 1.5 ((1.8)) are available in Mathai [6, Chapters 3 and 6]. By using these results and following the parallel steps as detailed above the complex analogues of the above results of sections 2 and 3 can be proved. The interested reader is referred to Mathai [6, Chapters 5 and 6] by us for a detailed discussion about this. Another thing to be noted here is that in this section of the paper $|A|$ stands for the absolute value of the determinant of the matrix A of complex elements. Now we proceed to give below in the Definitions 4.1 and 4.2 the M-transforms of the Srivastava hypergeometric functions H_A and H_B of complex arguments which are respectively the complex analogues of the Definitions 1.8 and 1.9 respectively:

Definition 4.1. The Srivastava function H_A of complex matrix arguments

$$H_A = H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z)$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$\begin{aligned} M(H_A) &= \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} |Z|^{\rho_3-p} \times \\ &\quad H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) dX dY dZ \\ &= \frac{\Gamma_p(a_1 - \rho_1 - \rho_3) \Gamma_p(a_2 - \rho_1 - \rho_2) \Gamma_p(a_3 - \rho_2 - \rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3)} \times \\ &\quad \frac{\Gamma_p(c_1) \Gamma_p(c_2) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(c_1 - \rho_1) \Gamma_p(c_2 - \rho_2 - \rho_3)} \end{aligned} \quad (4.1)$$

for $\operatorname{Re}(a_1 - \rho_1 - \rho_3, a_2 - \rho_1 - \rho_2, a_3 - \rho_2 - \rho_3, c_1 - \rho_1, c_2 - \rho_2 - \rho_3, \rho_i) > (p-1)$, where, $i = 1, 2, 3$.

Definition 4.2. For the Srivastava function H_B of complex matrix arguments

$$\begin{aligned} H_B &= H_B(a_1, a_2, a_3; c_1, c_2, c_3; -X, -Y, -Z) \\ M(H_B) &= \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} |Z|^{\rho_3-p} \times \\ &\quad H_B(a_1, a_2, a_3; c_1, c_2, c_3; -X, -Y, -Z) dX dY dZ \\ &= \frac{\Gamma_p(a_1 - \rho_1 - \rho_3) \Gamma_p(a_2 - \rho_1 - \rho_2) \Gamma_p(a_3 - \rho_2 - \rho_3)}{\Gamma_p(a_1) \Gamma_p(a_2) \Gamma_p(a_3)} \times \\ &\quad \frac{\Gamma_p(c_1) \Gamma_p(c_2) \Gamma_p(c_3) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(c_1 - \rho_1) \Gamma_p(c_2 - \rho_2) \Gamma_p(c_3 - \rho_3)} \end{aligned} \quad (4.2)$$

for $\operatorname{Re}(a_1 - \rho_1 - \rho_3, a_2 - \rho_1 - \rho_2, a_3 - \rho_2 - \rho_3, c_i - \rho_i, \rho_i) > (p-1)$, where, $i = 1, 2, 3$.

Definition 4.3. The Horn's function H_3 of complex matrix arguments

$$H_3 = H_3(a, b; c; -X, -Y)$$

is defined as that class of functions for which the M-transform is the following:

$$\begin{aligned} M(H_3) &= \int_{X>0} \int_{Y>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} H_3(a, b; c; -X, -Y) dX dY \\ &= \frac{\Gamma_p(a-2\rho_1-\rho_2) \Gamma_p(b-\rho_2) \Gamma_p(c) \Gamma_p(\rho_1) \Gamma_p(\rho_2)}{\Gamma_p(a) \Gamma_p(b) \Gamma_p(c-\rho_1-\rho_2)} \end{aligned} \quad (4.3)$$

for $\text{Re}(a-2\rho_1-\rho_2, b-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)$.

Definition 4.4. For the Horn's function H_4 of matrix arguments

$$H_4 = H_4(a, b; c; -X, -Y)$$

$$\begin{aligned} M(H_4) &= \int_{X>0} \int_{Y>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} H_4(a, b; c, d; -X, -Y) dX dY \\ &= \frac{\Gamma_p(a-2\rho_1-\rho_2) \Gamma_p(b-\rho_2) \Gamma_p(c) \Gamma_p(d) \Gamma_p(\rho_1) \Gamma_p(\rho_2)}{\Gamma_p(a) \Gamma_p(b) \Gamma_p(c-\rho_1) \Gamma_p(d-\rho_2)} \end{aligned} \quad (4.4)$$

where, $\text{Re}(a-2\rho_1-\rho_2, b-\rho_2, c-\rho_1, d-\rho_2, \rho_1, \rho_2) > (p-1)$.

The following Theorems 4.5 and 4.6 are respectively the complex analogues of the results of the Theorems 2.5 and 2.6.

Theorem 4.5.

$$\begin{aligned} M(H_4(\alpha, \beta; \gamma, \delta; -X-Y, -Z)) &= \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} |Z|^{\rho_3-p} \times \\ &\quad H_4(\alpha, \beta; \gamma, \delta; -X-Y, -Z) dX dY dZ \\ &= \frac{\Gamma_p(\alpha-2\rho_1-2\rho_2-\rho_3) \Gamma_p(\beta-\rho_3) \Gamma_p(\gamma) \Gamma_p(\delta) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(\alpha) \Gamma_p(\beta) \Gamma_p(\gamma-\rho_1-\rho_2) \Gamma_p(\delta-\rho_3)} \end{aligned} \quad (4.5)$$

for $\text{Re}(\alpha-2\rho_1-2\rho_2-\rho_3, \beta-\rho_3, \gamma-\rho_1-\rho_2, \delta-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)$.

Theorem 4.6.

$$\begin{aligned} M(H_4(\alpha, \beta; \gamma, \delta; -X, -Y-Z)) &= \int_{X>0} \int_{Y>0} \int_{Z>0} |X|^{\rho_1-p} |Y|^{\rho_2-p} |Z|^{\rho_3-p} \times \\ &\quad H_4(\alpha, \beta; \gamma, \delta; -X, -Y-Z) dX dY dZ \\ &= \frac{\Gamma_p(\alpha-2\rho_1-\rho_2-\rho_3) \Gamma_p(\beta-\rho_2-\rho_3) \Gamma_p(\gamma) \Gamma_p(\delta) \Gamma_p(\rho_1) \Gamma_p(\rho_2) \Gamma_p(\rho_3)}{\Gamma_p(\alpha) \Gamma_p(\beta) \Gamma_p(\gamma-\rho_1) \Gamma_p(\delta-\rho_2-\rho_3)} \end{aligned} \quad (4.6)$$

for $\text{Re}(\alpha-2\rho_1-\rho_2-\rho_3, \beta-\rho_2-\rho_3, \gamma-\rho_1, \delta-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)$.

Now we give below in the Theorems 4.7 to 4.13 the complex analogues of the Theorems 3.2 to 3.8 respectively.

Theorem 4.7.

$$\begin{aligned} H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) &= \frac{\Gamma_p(a_1+a_2)}{\Gamma_p(a_1) \Gamma_p(a_2)} \int_0^I |T|^{a_1-p} |I-T|^{a_1-p} \times \\ &\quad H_4\left(a_1+a_2, a_3; c_1, c_2; -(I-T)^{1/2} T^{1/2} X T^{1/2} (I-T)^{1/2}, \right. \\ &\quad \left. -(I-T)^{1/2} Y (I-T)^{1/2} - T^{1/2} Z T^{1/2}\right) dT \end{aligned} \quad (4.7)$$

for $\text{Re}(a_1, a_2) > (p-1)$.

Theorem 4.8.

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(s)}{\Gamma_p(a_1) \Gamma_p(s-a_1)} \int_0^I |T|^{a_1-p} |I-T|^{s-a_1-p} \times \\
& H_A\left(s, a_2, a_3; c_1, c_2; -T^{1/2} X T^{1/2}, -Y, -T^{1/2} Z T^{1/2}\right) dT
\end{aligned} \tag{4.8}$$

for $\operatorname{Re}(s-a_1, a_1) > (p-1)$.**Theorem 4.9.**

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(s)}{\Gamma_p(a_2) \Gamma_p(s-a_2)} \int_0^I |T|^{a_2-p} |I-T|^{s-a_2-p} \times \\
& H_A\left(a_1, s, a_3; c_1, c_2; -T^{1/2} X T^{1/2}, -T^{1/2} Y T^{1/2}, -Z\right) dT
\end{aligned} \tag{4.9}$$

for $\operatorname{Re}(s-a_2, a_2) > (p-1)$.**Theorem 4.10.**

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(s)}{\Gamma_p(a_3) \Gamma_p(s-a_3)} \int_0^I |T|^{a_3-p} |I-T|^{s-a_3-p} \times \\
& H_A\left(a_1, a_2, s; c_1, c_2; -X, -T^{1/2} Y T^{1/2}, -T^{1/2} Z T^{1/2}\right) dT
\end{aligned} \tag{4.10}$$

for $\operatorname{Re}(s-a_3, a_3) > (p-1)$.**Theorem 4.11.**

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-s) \Gamma_p(s)} \int_0^I |T|^{s-p} |I-T|^{c_1-s-p} \times \\
& H_A\left(a_1, a_2, a_3; s, c_2; -T^{1/2} X T^{1/2}, -Y, -Z\right) dT
\end{aligned} \tag{4.11}$$

for $\operatorname{Re}(c_1-s, s) > (p-1)$.**Theorem 4.12.**

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-s) \Gamma_p(s)} \int_0^I |T|^{s-p} |I-T|^{c_2-s-p} \times \\
& H_A\left(a_1, a_2, a_3; c_1, s; -X, -T^{1/2} Y T^{1/2}, -T^{1/2} Z T^{1/2}\right) dT
\end{aligned} \tag{4.12}$$

for $\operatorname{Re}(c_2-s, s) > (p-1)$.**Theorem 4.13.**

$$\begin{aligned}
& H_A(a_1, a_2, a_3; c_1, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(c_2)}{\Gamma_p(a_3) \Gamma_p(c_2-a_3)} \int_0^I |T|^{a_3-p} |I-T|^{c_2-a_3-p} \left| I + T^{1/2} Y T^{1/2} \right|^{-a_2} \left| I + T^{1/2} Z T^{1/2} \right|^{-a_1} \times \\
& {}_2F_1\left(a_1, a_2; c_1; -\left(I + T^{1/2} Z T^{1/2}\right)^{-1/2} \left(I + T^{1/2} Y T^{1/2}\right)^{-1/2} X \times \right. \\
& \left. \left(I + T^{1/2} Y T^{1/2}\right)^{-1/2} \left(I + T^{1/2} Z T^{1/2}\right)^{-1/2}\right) dT
\end{aligned} \tag{4.13}$$

for $\operatorname{Re}(c_2-a_3, a_3) > (p-1)$.

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