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Certain properties of modified Hermite-type matrix polynomials *

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Abstract This paper deals with the investigation of an expansion formula, a summation formula, a multiplication theorem, an addition theorem, matrix recurrence relations, fractional integrals, fractional derivatives, Laplace transform, Mellin transform and Fractional Fourier transform for the modified Hermite-type matrix polynomials and some other properties. We also give new definitions for the modified Chebyshev's-type, the modified Legendre's-type and the modified Hermite-Hermite-type matrix polynomials by using these polynomials and we further prove some new results and relations.

Key words Generalized Hermite-type matrix polynomials, Generating matrix functions, modified Chebyshev, modified Legendre, modified Hermite-Hermite.

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1 Introduction and Preliminaries

Hermite polynomials are frequently used in many branches of pure and applied mathematics and physics [1, 15]. An important generalization of special functions are special matrix functions and polynomials. The study of special matrix functions is important due to their applications in certain areas of statistics, physics and engineering. Special matrix functions appear in connection with the matrix analogues of the Hermite and Legendre differential equations and the corresponding polynomials families [17–19, 25, 26, 28, 31]. However, the theory of special functions with matrix parameters is developed and their properties are also studied by some authors, see, for example, Defez et al. [6, 7], Jódar et al. [9, 10], Sayyed et al. [24], Altin and Çekim [2], Metwally et al. [17–19], and Shehata [27]. An important example of orthogonal matrix polynomials are the Hermite matrix polynomials. Motivated by the importance of special matrix polynomials, recently the Hermite matrix polynomials and their extensions and generalizations have been introduced and studied in [14, 29, 30, 35], for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane. The paper is organized as follows. In Section 2, we deal with important properties of the modified Hermite-type matrix polynomials such as an expansion formula, summation formula, multiplication theorem, addition theorem and matrix

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recurrence relations. In Section 3, we obtain some fractional integrals and fractional derivatives for the modified Hermite-type matrix polynomials. We also obtain some results which follow from their generating function. In Section 4, we give the Laplace, Mellin and Fractional Fourier Transforms (see [3, 4]) for these polynomials. In Section 5, the matrix polynomial representations along with the expansions which link the Chebyshev's-type, the modified Legendre's-type matrix polynomials with modified the Hermite-type matrix polynomials and the modified Hermite-Hermite-type matrix polynomials are also deduced. Finally in Section 6, some concluding remarks are given.
If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z (see [6]), then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A and the two-norm denoted by $||A||_2$, is defined by

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^{\mathbb{N}}$, $||y||_2$ denotes the usual Euclidean norm of y, $||y||_2 = (y^T y)^{\frac{1}{2}}$.
If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [5–7], it follows that

$$f(A)g(A) = g(A)f(A).$$

Throughout this study, consider the complex space $\mathbb{C}^{N\times N}$ of all square complex matrices of common order N. If A is a matrix with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A.

Lemma 1.1. *If $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geq 0$, $k \geq 0$, it follows that (Defez and Jódar [6]).*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} A(k,n-mk); m \in \mathbb{N}. \tag{1.1}$$

Similarly to (1.1), we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk); m \in \mathbb{N}. \tag{1.2}$$

Definition 1.2. A matrix A in $\mathbb{C}^{N\times N}$ is said be a positive stable matrix if (see [11])

$$Re(\mu) \nleq 0 \quad \text{for every eigenvalue } \mu \in \sigma(A),\ \sigma(A) := \text{ spectrum of } A. \tag{1.3}$$

Fact 1.3. If B is a matrix in $\mathbb{C}^{N\times N}$ such that (see [13])

$$B + nI \quad \text{is an invertible matrix for all integers } n \geq 0. \tag{1.4}$$

Fact 1.4. For a matrix A in $\mathbb{C}^{N\times N}$ the authors give the following relation (see [12])

$$(1-z)^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_n z^n,\ |z| < 1, \tag{1.5}$$

where

$$(A)_n = A(A+I)\dots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A),\ n \geq 1;\ (A)_0 = I,$$

$\Gamma(A)$ is an invertible matrix in $\mathbb{C}^{N\times N}$ and $\Gamma^{-1}(A)$ is the inverse Gamma matrix function.

If $\Phi(z)$ is a holomorphic function at $z = z_0$ and $\Phi(z_0) \neq 0$, and if $z = z_0 + w\Phi(z)$ and $f(z)$ is an analytic function, we expand a power series in w by the Lagrange expansion formula as (see [22])

$$\frac{f(z)}{1 - w\Phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{d^n}{dz^n} \left[f(z) \Big(\Phi(z) \Big)^n \right] \Big|_{z=z_0}. \tag{1.6}$$

From the definition of the Gamma function, we have (see Srivastava and Karlsson [32, 33])

$$\int_0^\infty e^{-t^2} t^{\frac{2n-mk}{p}} dt = \frac{1}{2}\Gamma\left(\frac{2n-mk}{2p} + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\left(\frac{1}{2}\right)_{\frac{2n-mk}{2p}}, \tag{1.7}$$

and

$$\int_0^\infty e^{-t} t^{\frac{n-(m-p)k}{p}} dt = \Gamma\left(\frac{n-(m-p)k}{p} + 1\right). \tag{1.8}$$

In order to describe more details of our work, we will need some definitions of fractional integrals and fractional derivatives, which are described as below and can be found in standard works in this field, like, [8, 16, 20, 21, 34].

Definition 1.5. Riemann-Liouville fractional integral of order ν is defined as

$$\mathbb{I}_x^\nu\{f(x)\} = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \;\; Re(\nu) > 0. \tag{1.9}$$

Definition 1.6. Let $f(x) \in L(b, c)$, $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$, the left-sided operator of Riemann-Liouville fractional integral of order α is defined as

$${}_b\mathbb{I}_x^\alpha\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_b^x (x-t)^{\alpha-1} f(t)dt, \;\; x > b. \tag{1.10}$$

Definition 1.7. Let $f(x) \in L(b, c)$, $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$, then the right-sided operator of Riemann-Liouville fractional integral of order α is defined as

$${}_x\mathbb{I}_c^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^c (t-x)^{\alpha-1} f(t)dt, \;\; x < c. \tag{1.11}$$

Definition 1.8. The Weyl integral of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$${}_xW_\infty^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \;\; -\infty < x < \infty, \tag{1.12}$$

where $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$.

Definition 1.9. Let $f(x) \in L(b, c)$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$, the left-sided operator of Riemann-Liouville fractional derivative of order α is defined by

$${}_bD_x^\alpha\{f(x)\} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial x}\right)^n \int_b^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \;\; x > b. \tag{1.13}$$

Definition 1.10. Let $f(x) \in L(b, c)$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$, the right-sided operator of Riemann-Liouville fractional derivative of order α is defined by

$${}_xD_c^\alpha\{f(x)\} = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial x}\right)^n \int_x^c \frac{f(t)}{(t-x)^{\alpha-n+1}} dt, \;\; x < c. \tag{1.14}$$

Definition 1.11. Let $f(x) \in L(b, c)$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$, then the Weyl fractional derivative of $f(x)$ of order α, denoted by ${}_xD_\infty^\alpha$, is defined by

$${}_xD_\infty^\alpha\{f(x)\} = \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{\partial}{\partial x}\right)^m \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-m+1}} dt, \tag{1.15}$$

where $-\infty < x < \infty$, $m - 1 \leq \alpha < m$ and $m \in \mathbb{N}$.

Definition 1.12. The Laplace transform of $f(z)$ is defined as (see [4])

$$\pounds\{f(z):s\}=\int_0^\infty e^{-sz}f(z)dz \tag{1.16}$$

It is very pertinent to mention here in this connection that the Upadhyaya transform (see Upadhyaya [36] and Upadhyaya et. al. [37]) till date is the most powerful generalization and unification of all the various types of Laplace transforms introduced into the mathematics research literature by various researchers worldwide during the past thirty years. The interested reader is referred by us to Upadhyaya [36] and Upadhyaya et. al. [37] for further studies in this direction.

Definition 1.13. The Mellin transform of $f(z)$ is defined as (see [3])

$$M\{f(z):s\}=\int_0^\infty z^{s-1}f(z)dz \tag{1.17}$$

Definition 1.14. Let f be a function belonging to $\Phi(R)$, then the fractional Fourier transform of $f(x)$ of order α is defined as (see [23])

$$\Im_\alpha[f(x):\omega]=\int_R e^{i\omega^{\frac{1}{\alpha}}}f(x)dx,\ \omega>0, \tag{1.18}$$

where, $0<\alpha\le 1$.

2 Definition of modified Hermite-type matrix polynomials and some properties

For μ any complex number, if A and B are commutative matrices in $\mathbb{C}^{N\times N}$, μA is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition (1.3) and B is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), we define a matrix version of the modified Hermite-type matrix polynomials by means of the matrix generating function as follows:

$$F^{(\lambda,\zeta)}(x,t;A,B;a,\mu)=\sum_{n=0}^{\infty}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)\frac{t^n}{n!}=a^{\lambda xt^p\sqrt{\mu A}-\zeta Bt^m},\ |t|<\infty, \tag{2.1}$$

where $a>0$ and $a\neq 1$, and m is a positive integer, $\sqrt{\mu A}$ is the square root of the matrix μA in the sense of the functional matrix calculus $\sqrt{\mu A}=\exp(\frac{1}{2}\log(\mu A))$, and log denotes the principal branch of the complex logarithm [5].
Expanding the exponential matrix function, we obtain

$$F^{(\lambda,\zeta)}(x,t;A,B;a,\mu)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\zeta B\ln(a))^k(\lambda x\ln(a)\sqrt{\mu A})^n}{k!\Gamma(n+1)}\,t^{pn+mk}. \tag{2.2}$$

Using (1.1) and (2.2), we write

$$F^{(\lambda,\zeta)}(x,t;A,B;a,\mu)=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\,t^n,$$

comparing the coefficients of t^n, we obtain a matrix version of the modified Hermite-type matrix polynomials for $\Re(\frac{n-mk}{p})>-1$:

$$H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)=\sum_{k=0}^{[\frac{n}{m}]}\frac{n!(-\zeta B\ln(a))^k(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}. \tag{2.3}$$

Remark 2.1. We observe that on taking $a=e$, $\mu=2$, $m=2$, $\lambda=\zeta=1$ and $B=I$ in (2.1), it reduces to $H_{n,2}(x;A,I;e,2)$ the Hermite-type matrix polynomials defined in [29, 30].

Theorem 2.2. *For μ any complex number, let μA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition (1.3), then we have the expansion of*

$$\left(\lambda x\ln(a)\sqrt{\mu\, A}\right)^n = n!\sum_{k=0}^{[\frac{np}{m}]}\frac{(\zeta B\ln(a))^k}{k!\Gamma(np-mk+1)}H^{(\lambda,\zeta)}_{np-mk,m,p}(x,A,B;a,\mu), \Re(np-mk)>-1. \tag{2.4}$$

Proof. By (2.1), we can write

$$a^{\lambda xt^p\sqrt{\mu A}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\zeta B\ln(a))^k}{k!n!}H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)t^{n+mk}, \tag{2.5}$$

and replacing n by $np-mk$ in the right hand side of (2.5), we get

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda x\ln(a)\sqrt{\mu\, A}\right)^n t^{np}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{np}{m}]}\frac{(\zeta B\ln(a))^k}{k!\Gamma(np-mk+1)}H^{(\lambda,\zeta)}_{np-mk,m,p}(x,A,B;a,\mu)t^{np},$$

by comparing the coefficients of t^n in the above equation, we arrive at (2.4). □

Now, we give the summation formulas, multiplication and addition theorems for the modified Hermite-type matrix polynomials in the following results:

Theorem 2.3. *For commutative matrices B, D and $B-D$ in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), the modified Hermite-type matrix polynomials has the finite summation formula:*

$$H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\zeta(D-B)\ln(a))^k}{k!\Gamma\left(n-mk+1\right)}H^{(\lambda,\zeta)}_{n-mk,m,p}(x,A,D;a,\mu), \Re(n-mk)>-1. \tag{2.6}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\frac{t^n}{n!} = a^{(\lambda xt^p\sqrt{\nu A}-\zeta Dt^m)}\cdot a^{(\zeta(D-B)t^m)}$$
$$= a^{(\zeta(D-B)t^m)}\sum_{n=0}^{\infty}H^{(\lambda,\zeta)}_{n,m,p}(x,A,D;a,\mu)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\zeta(D-B)\ln(a))^k}{n!k!}H^{(\lambda,\zeta)}_{n,m,p}(x,A,D;a,\mu)t^{n+mk}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(\zeta(D-B)\ln(a))^k}{k!\Gamma\left(n-mk+1\right)}H^{(\lambda,\zeta)}_{n-mk,m,p}(x,A,D;a,\mu)t^n.$$

Comparing the coefficients of t^n in the above equation leads us to (2.6). □

Theorem 2.4. *The modified Hermite-type matrix polynomials satisfy the multiplication formula:*

$$H^{(\lambda,\zeta)}_{n,m,p}(\alpha x,A,B;a,\mu) = n!\sum_{k=0}^{[\frac{n}{p}]}\frac{\left(\lambda(\alpha-1)x\ln(a)\sqrt{\mu A}\right)^k H^{(\lambda,\zeta)}_{n-kp,m,p}(x,A,B;a,\mu)}{k!\Gamma(n-kp+1)}, \Re(n-kp)>-1, \tag{2.7}$$

where α is constant.

Proof. By (2.1) and (1.1), we have

$$\sum_{n=0}^{\infty} \frac{H_{n,m,p}^{(\lambda,\zeta)}(\alpha x,A,B;a,\mu)t^n}{n!} = a^{\lambda(\alpha-1)xt^p\sqrt{\mu A}} \cdot a^{\lambda x t^p\sqrt{\mu A}-\zeta B t^m}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\lambda(\alpha-1)x\ln(a)\sqrt{\mu A}\right)^k H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)t^{n+kp}}{k!n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{p}]} \frac{\left(\lambda(\alpha-1)x\ln(a)\sqrt{\mu A}\right)^k H_{n-kp,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)t^{n}}{k!\Gamma(n-kp+1)}.$$

On comparing the coefficients of t^n from both sides of the above equation we get (2.7). □

Theorem 2.5. *For commutative matrices B, D and $B+D$ in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), the finite summation formula for the modified Hermite-type matrix polynomials is as below*

$$H_{n,m,p}^{(\lambda,\zeta)}(\alpha x+\beta z,A,B+D;a,\mu)$$
$$= n!\sum_{k=0}^{n} \frac{H_{n-k,m,p}^{(\lambda,\zeta)}(\alpha z,A,B;a,\mu)H_{k,m,p}^{(\lambda,\zeta)}(\beta x,A,D;a,\mu)}{k!(n-k)!}, \tag{2.8}$$

where α and β are constants.

Proof. Using (1.2), we consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{H_{n-k,m,p}^{(\lambda,\zeta)}(\alpha x,A,D;a,\mu)H_{k,m,p}^{(\lambda,\zeta)}(\beta z,A,B;a,\mu)t^n}{k!(n-k)!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_{n,m,p}^{(\lambda,\zeta)}(\alpha x,A,B;a,\mu)H_{k,m,p}^{(\lambda,\zeta)}(\beta z,A,D;a,\mu)t^{n+k}}{k!n!}$$
$$= a^{\lambda(\alpha x+\beta z)t^p\sqrt{\mu A}-\zeta(B+D)t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,p}^{(\lambda,\zeta)}(\alpha x+\beta z,A,B+D;a,\mu)t^n}{n!},$$

in which by comparing the coefficients of t^n, we get (2.8). □

Theorem 2.6. *The modified Hermite-type matrix polynomials satisfy the addition formulas:*

$$H_{n,m,p}^{(\lambda,\zeta)}(x+y,A,B;a,\mu) = n!\sum_{k=0}^{[\frac{n}{p}]} \frac{y^k(\lambda\ln(a)\sqrt{\mu A})^k}{\Gamma(n-pk+1)k!} H_{n-pk,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu), \Re(n-kp)>-1, \tag{2.9}$$

$$H_{n,m,p}^{(\lambda+\nu,\zeta)}(x;A,B;a,\mu) = n!\sum_{k=0}^{[\frac{n}{p}]} \frac{(\nu x\ln(a)\sqrt{\mu A})^k}{\Gamma(n-pk+1)k!} H_{n-pk,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu), \Re(n-kp)>-1, \tag{2.10}$$

and

$$H_{n,m,p}^{(\lambda,\zeta+\eta)}(x;A,B;a,\mu) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(-\eta B\ln(a))^k}{\Gamma(n-mk+1)k!} H_{n-mk,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu), \Re(n-mk)>-1. \tag{2.11}$$

Proof. Rewrite (2.1) in the form

$$a^{-\zeta B t^m} = a^{-\lambda x t^p\sqrt{\mu A}} \sum_{n=0}^{\infty} H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,b,\nu)\frac{t^n}{n!},$$

then replacing x by y in it gives

$$a^{-\zeta B t^m} = a^{-\lambda y t^p\sqrt{\mu A}} \sum_{n=0}^{\infty} H_{n,m,p}^{(\lambda,\zeta)}(y,A,B;a,b,\nu)\frac{t^n}{n!}.$$

On comparing the last two equations, we get

$$a^{\lambda(y-x)t^p\sqrt{\mu A}} \sum_{n=0}^{\infty} H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,b,\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{n,m,p}^{(\lambda,\zeta)}(y,A,B;a,b,\nu)\frac{t^n}{n!}. \tag{2.12}$$

Further, on expanding exponential matrix function in (2.12), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y-x)^k}{n!k!}\left(\lambda\ln(a)\sqrt{\mu\,A}\right)^k H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)t^{n+pk}$$
$$=\sum_{n=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(y,A,B;a,\mu)\frac{t^n}{n!}. \tag{2.13}$$

Replacing n by $n-pk$ and comparing the coefficients of t^n in (2.13), we get

$$n!\sum_{k=0}^{[\frac{n}{p}]}\frac{(y-x)^k}{(n-pk)!k!}\left(\lambda\ln(a)\sqrt{\mu\,A}\right)^k H_{n-pk,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)$$
$$=H_{n,m,p}^{(\lambda,\zeta)}(y,A,B;a,\mu). \tag{2.14}$$

Now by replacing y by $y+x$ in (2.14) we get the addition formula (2.9). Also we have

$$\sum_{n=0}^{\infty}H_{n,m,p}^{(\lambda+\nu,\zeta)}(x;A,B;a,\mu)\frac{t^n}{n!}=a^{(\lambda+\nu)xt^p\sqrt{\mu A}-\zeta Bt^m}$$
$$=a^{\nu xt^p\sqrt{\mu A}}a^{\lambda xt^p\sqrt{\mu A}-\zeta Bt^m}=a^{\nu xt^p\sqrt{\mu A}}\sum_{n=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{t^n}{n!}\frac{(\nu x\ln(a)t^p\sqrt{\mu A})^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{(\nu x\ln(a)\sqrt{\mu A})^k}{n!k!}t^{n+pk}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{p}]}\frac{(\nu x\ln(a)\sqrt{\mu A})^k}{(n-pk)!k!}H_{n-pk,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^n, \tag{2.15}$$

by comparing the coefficients of t^n, we get the required result in (2.10). Similarly we have

$$\sum_{n=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta+\eta)}(x;A,B;a,\mu)\frac{t^n}{n!}=a^{\lambda xt^p\sqrt{\mu A}-(\zeta+\eta)Bt^m}$$
$$=a^{-\eta Bt^m}a^{\lambda xt^p\sqrt{\mu A}-\zeta Bt^m}=a^{-\eta Bt^m}\sum_{n=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{t^n}{n!}\frac{(-\eta Bt^m\ln(a))^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\eta B\ln(a))^k}{n!k!}H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^{n+mk}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\eta B\ln(a))^k}{(n-mk)!k!}H_{n-mk,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^n, \tag{2.16}$$

by comparing the coefficients of t^n, we get the required result of (2.11). □

Theorem 2.7. *The modified Hermite-type matrix polynomials have the following differential representation :*

$$H_{np,mp,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)=\frac{(np)!}{n!}\,a^{\left(\frac{-\zeta B(\lambda\sqrt{\mu A})^{-m}}{(\ln(a))^m}\frac{d^m}{dx^m}\right)}\,(\lambda x\ln(a)\sqrt{\mu A})^n. \tag{2.17}$$

Proof. We observe that

$$a^{\left(\frac{(-\zeta B(\lambda\sqrt{\mu A})^{-m}}{(\ln(a))^m}\frac{d^m}{dx^m}\right)} \cdot a^{(\lambda x t^p\sqrt{\mu A})} = \sum_{n=0}^{\infty}\frac{(-\zeta B\ln(a))^n(\lambda\sqrt{\mu A})^{-mn}}{n!(\ln(a))^{mn}}\frac{d^{mn}}{dx^{mn}}a^{(\lambda x t^p\sqrt{\mu A})}$$

$$= \sum_{n=0}^{\infty}\frac{(-\zeta B\ln(a))^n\, t^{mnp}}{n!}a^{(\lambda x t^p\sqrt{\mu A})} = \sum_{n=0}^{\infty}a^{(\lambda x t^p\sqrt{\mu A}-\zeta B t^{mp})}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}H^{(\lambda,\zeta)}_{n,mp,p}(x;A,B;a,\mu)\,t^n = \sum_{n=0}^{\infty}\frac{1}{(np!)}H^{(\lambda,\zeta)}_{np,mp,p}(x;A,B;a,\mu)\,t^{np}.$$

Thus, the identification of the coefficients of t^{np} on both sides gives (2.17). □

Some recurrence matrix relations have been deduced for the modified Hermite-type matrix polynomials. At first, we record the following theorem.

Theorem 2.8. *The modified Hermite-type matrix polynomials $H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)$ satisfy the following relations:*

$$\frac{\partial^s}{\partial x^s}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = \frac{(\lambda\ln(a)\sqrt{\mu A})^s n!}{\Gamma(n-sp+1)}H^{(\lambda,\zeta)}_{n-sp,m,p}(x;A,B;a,\mu);\ \ 0\le s\le[\frac{n}{p}], \tag{2.18}$$

$$\frac{\partial^s}{\partial \lambda^s}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = \frac{(x\ln(a)\sqrt{\mu A})^s n!}{\Gamma(n-sp+1)}H^{(\lambda,\zeta)}_{n-sp,m,p}(x;A,B;a,\mu);\ \ 0\le s\le[\frac{n}{p}], \tag{2.19}$$

$$\frac{\partial^s}{\partial \zeta^s}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = \frac{(-B\ln(a))^s n!}{\Gamma(n-ms+1)}H^{(\lambda,\zeta)}_{n-ms,m,p}(x;A,B;a,\mu);\ \ 0\le s\le[\frac{n}{m}], \tag{2.20}$$

and

$$\frac{\partial^s}{\partial \mu^s}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = \frac{(\lambda x\ln(a)\sqrt{\mu A})^s n!}{(2\mu)^s\Gamma(n-sp+1)}H^{(\lambda,\zeta)}_{n-sp,m,p}(x;A,B;a,\mu);\ \ 0\le s\le[\frac{n}{p}]. \tag{2.21}$$

Proof. Differentiating the identity (2.1) with respect to x yields

$$(\lambda t^p\ln(a)\sqrt{\mu A})\ a^{\lambda x t^p\sqrt{\mu A}-\zeta B t^m} = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)\frac{t^n}{n!}. \tag{2.22}$$

From (2.1) and (2.22), we have

$$(\lambda\ln(a)\sqrt{\mu A})\sum_{n=0}^{\infty}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)\frac{t^{n+p}}{n!} = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)\frac{t^n}{n!}.$$

Hence, by identifying the coefficients in t^n, it follows that

$$\frac{\partial}{\partial x}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = \frac{(\lambda\ln(a)\sqrt{\mu A})n!}{(n-p)!}H^{(\lambda,\zeta)}_{n-p,m,p}(x;A,B;a,\mu);\ \ n\ge p. \tag{2.23}$$

The iteration of (2.23), for $0\le s\le[\frac{n}{p}]$, implies (2.18). The proofs of (2.19), (2.20) and (2.21) are similar to that of (2.18). □

The following corollary is a consequence of Theorem 2.7.

Corollary 2.9. *The Modified Hermite-type matrix polynomials $H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)$ satisfy the following relations:*

$$(B\ln(a))^p\frac{\partial^m}{\partial x^m}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) - (-1)^p(\lambda\ln(a)\sqrt{\mu A})^m\frac{\partial^p}{\partial\zeta^p}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = 0, \tag{2.24}$$

$$(B\ln(a))^p\frac{\partial^m}{\partial \lambda^m}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) - (-1)^p(x\ln(a)\sqrt{\mu A})^m\frac{\partial^p}{\partial\zeta^p}H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu) = 0, \tag{2.25}$$

$$(B\ln(a))^p \frac{\partial^m}{\partial\mu^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) - (-1)^p(\frac{\lambda x\ln(a)\sqrt{\mu A}}{2\mu})^m \frac{\partial^p}{\partial\zeta^p} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = 0, \quad (2.26)$$

$$x^m \frac{\partial^m}{\partial x^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) - \lambda^m \frac{\partial^m}{\partial\lambda^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = 0, \quad (2.27)$$

$$(\frac{\lambda}{2\mu})^m \frac{\partial^m}{\partial\lambda^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) - \frac{\partial^m}{\partial\mu^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = 0, \quad (2.28)$$

and

$$(\frac{x}{2\mu})^m \frac{\partial^m}{\partial x^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) - \frac{\partial^m}{\partial\mu^m} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = 0. \quad (2.29)$$

Proof. Using (2.18), (2.19), (2.20) and (2.21) the proof follows immediately. □

Theorem 2.10. *Let A and B be commutative matrices in $\mathbb{C}^{N\times N}$. For any complex number μ let μA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3) and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), then we have*

$$\begin{aligned} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) =&(n-1)!\Bigg[\frac{\lambda xp\ln(a)\sqrt{\mu A}}{(n-p)!} H_{n-p,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) \\ &- \frac{m\zeta B\ln(a)}{(n-m)!} H_{n-m,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\Bigg], n\ge m, n\ge p. \end{aligned} \quad (2.30)$$

Proof. Differentiating the identity (2.1) with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial t} F_{n,m,p}^{(\lambda,\zeta)}(x,t;A,B;a,\mu) &= (\lambda xpt^{p-1}\sqrt{\mu A}\ln(a) - m\zeta Bt^{m-1}\ln(a))a^{\lambda xt^p\sqrt{\mu A}-\zeta Bt^m} \\ &= \sum_{n=1}^{\infty} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)\frac{t^{n-1}}{(n-1)!}. \end{aligned}$$

Therefore $F_{n,m,p}^{(\lambda,\zeta)}(x,t;A,B;a,\mu)$ satisfies the matrix partial differential equation

$$(\lambda xpt^{p-1}\sqrt{\mu A}\ln(a) - m\zeta Bt^{m-1}\ln(a))\frac{\partial F^{(\lambda,\zeta)}}{\partial x} - \lambda t^p\sqrt{\mu A}\ln(a)\frac{\partial F^{(\lambda,\zeta)}}{\partial t} = 0.$$

Hence we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda\sqrt{\mu A}\ln(a)}{(n-1)!} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^{n+p-1} &= \lambda xp\sqrt{\mu A}\ln(a)\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^{n+p-1} \\ &-m\zeta B\ln(a)\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu)t^{n+m-1}, \end{aligned}$$

in which by identifying the coefficients of t^{n+p-1}, we obtain

$$\begin{aligned} \frac{\lambda\sqrt{\mu A}\ln(a)}{(n-1)!} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) &= \frac{\lambda xp\sqrt{\mu A}\ln(a)}{n!}\frac{\partial}{\partial x} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) \\ &- \frac{m\zeta B\ln(a)}{(n-m+p)!}\frac{\partial}{\partial x} H_{n-m+p,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu). \end{aligned}$$

Then for $n\ge m$ and $n\ge p$, it follows that

$$\begin{aligned} \frac{1}{(n-1)!} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) &= \frac{xp}{n!}\frac{\partial}{\partial x} H_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) \\ &- \frac{m\zeta B(\lambda\sqrt{\mu A})^{-1}}{(n-m+p)!}\frac{\partial}{\partial x} H_{n-m+p,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu). \end{aligned} \quad (2.31)$$

Using (2.23) and (2.31), we get (2.30). □

3 Fractional integrals and derivatives for the modified Hermite-type matrix polynomials

In this section we obtain the fractional integrals and fractional derivatives for the modified Hermite-type matrix polynomials $H_{n,m,p}^{(\lambda,\xi)}(x,A,B;a,\mu)$.
By using the definitions of fractional integrals and fractional derivatives given above, we now state and prove a number of theorems concerning the Modified Hermite-type matrix polynomials in the succeeding theorems:

Theorem 3.1. *The modified Hermite-type matrix polynomials satisfy the interesting formula:*

$$\mathbb{I}_{\lambda}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\} = \frac{1}{(n+1)_{p\nu}}\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\nu} \times H_{n+p\nu,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),\ n+p\nu\geq 0, \tag{3.1}$$

and

$$\mathbb{I}_{\zeta}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\} = \frac{(-B\ln(a))^{-\nu}}{(n+1)_{m\nu}}\ H_{n+m\nu,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),\ n+m\nu\geq 0. \tag{3.2}$$

Proof. From (2.3) and (1.9), we have

$$\mathbb{I}_{\lambda}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\} = \frac{1}{\Gamma(\nu)}\int_0^{\lambda}(\lambda-t)^{\nu-1}H_{n,m,p}^{(t,\zeta)}(x,A,B;a,\mu)dt$$

$$= \frac{n!}{\Gamma(\nu)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^{\lambda}(\lambda-t)^{\nu-1}t^{\frac{n-mk}{p}}dt.$$

Putting $t=\lambda u$, $dt=\lambda du$, $t=0$, $u=0$ and $t=\lambda$, $u=1$, we get

$$\int_0^{\lambda}(\lambda-t)^{\nu-1}t^{\frac{n-mk}{p}}dt = \lambda^{\nu+\frac{n-mk}{p}}\frac{\Gamma(\nu)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(\nu+\frac{n-mk}{p}+1\right)},$$

and we can write

$$\mathbb{I}_{\lambda}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\} = n!\sum_{k=0}^{[\frac{n+p\nu}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\nu+\frac{n-mk}{p}+1\right)}\lambda^{\nu+\frac{n-mk}{p}}$$

$$= \frac{\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\nu}}{(n+1)_{p\nu}}H_{n+p\nu,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),$$

which gives (3.1). Also we have

$$\mathbb{I}_{\zeta}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\} = \frac{1}{\Gamma(\nu)}\int_0^{\zeta}(\zeta-t)^{\nu-1}H_{n,m,p}^{(\lambda,t)}(x,A,B;a,\mu)dt$$

$$= \frac{n!}{\Gamma(\nu)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^k\left(\lambda x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^{\zeta}(\zeta-t)^{\nu-1}\zeta^k dt.$$

Putting $t=\zeta u$, $dt=\zeta du$, $t=0$, $u=0$ and $t=\zeta$, $u=1$, we get

$$\int_0^{\zeta}(\zeta-t)^{\nu-1}t^k dt=\zeta^{k+\nu}\frac{\Gamma(\nu)\Gamma(k+1)}{\Gamma(k+\nu+1)}.$$

Hence, we have

$$\mathbb{I}_{\zeta}^{\nu}\{H_{n,m,p}^{(\lambda,\zeta)}\}=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^k\left(\lambda x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{\Gamma(k+\nu+1)\Gamma\left(\frac{n-mk}{p}+1\right)}\ \zeta^{\nu+k}$$

$$=(-B\ln(a))^{-\nu}\ n!\sum_{k=0}^{[\frac{n+m\nu}{m}]}\frac{(-\zeta B\ln(a))^k\left(\lambda x\ln(a)\sqrt{\mu A}\right)^{\frac{n+m\nu-mk}{p}}}{k!\Gamma\left(\frac{n+m\nu-mk}{p}+1\right)}$$

$$=\frac{(-B\ln(a))^{-\nu}}{(n+1)_{m\nu}}H_{n+m\nu,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),$$

which gives (3.2). □

Theorem 3.2. *The modified Hermite-type matrix polynomials have the left-sided operator of Riemann-Liouville fractional integral*

$$ {}_b\mathbb{I}_{\lambda}^{\alpha}\{H_{n,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu)\}=\frac{1}{(n+1)_{p\alpha}}\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\alpha} \times H_{n+p\alpha,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu),\ n+p\alpha\geq 0, \tag{3.3}$$

and

$$ {}_b\mathbb{I}_{\zeta}^{\alpha}\{H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)\}=\frac{(-B\ln(a))^{-\alpha}}{(n+1)_{m\mu}}\ H_{n+m\alpha,m,p}^{(\lambda,\zeta-b)}(x,A,B;a,\mu),\ n+m\alpha\geq 0. \tag{3.4}$$

Proof. Using (2.3) in the right hand side of (1.10), we have

$$ {}_b\mathbb{I}_{\lambda}^{\alpha}\{H_{n,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu)\}=\frac{1}{\Gamma(\alpha)}\int_b^{\lambda}(\lambda-t)^{\alpha-1}H_{n,m,p}^{(t-b,\zeta)}(x,A,B;a,\mu)dt$$

$$=\frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_b^{\lambda}(\lambda-t)^{\alpha-1}(t-b)^{\frac{n-mk}{p}}dt.$$

Putting $u=\frac{t-b}{\lambda-b}$, $t-b=(\lambda-b)u$, $dt=(\lambda-b)du$, $t=b$, $u=0$ and $t=\lambda$, $u=1$, we get

$$\int_b^{\lambda}(\lambda-t)^{\alpha-1}(t-b)^{\frac{n-mk}{p}}dt=(\lambda-b)^{\alpha+\frac{n-mk}{p}}\frac{\Gamma(\alpha)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(\alpha+\frac{n-mk}{p}+1\right)},$$

and we can write

$$ {}_b\mathbb{I}_{\lambda}^{\alpha}\{H_{n,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu)\}=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\alpha+\frac{n-mk}{p}+1\right)}$$

$$\times(\lambda-b)^{\alpha+\frac{n-mk}{p}}=\frac{\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\alpha}}{(n+1)_{p\alpha}}H_{n+p\alpha,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu).$$

Thus, we get the desired result (3.3). Also we have

$$
{}_b\mathbb{I}^{\alpha}_{\zeta}\{H^{(\lambda,\zeta-b)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{1}{\Gamma(\alpha)}\int_b^{\zeta}(\zeta-t)^{\alpha-1}H^{(\lambda,t-b)}_{n,m,p}(x,A,B;a,\mu)dt
$$

$$
= \frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^k\left(\lambda x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_b^{\zeta}(\zeta-t)^{\alpha-1}(t-b)^k dt.
$$

Putting $u=\frac{t-b}{\zeta-b}$, $t-b=(\zeta-b)u$, $dt=(\zeta-b)du$ $t=b$, $u=0$ and $t=\zeta$, $u=1$, we get

$$
\int_b^{\zeta}(\zeta-t)^{\alpha-1}(t-b)^{\frac{n-mk}{p}}dt = (\zeta-b)^{\alpha+k}\frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)},
$$

and we can write

$$
{}_b\mathbb{I}^{\alpha}_{\zeta}\{H^{(\lambda,\zeta-b)}_{n,m,p}(x,A,B;a,\mu)\} = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^k\left(\lambda x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{\Gamma(\alpha+k+1)\Gamma\left(\frac{n-mk}{p}+1\right)}\,(\zeta-b)^{\alpha+k}
$$

$$
= \frac{(-B\ln(a))^{-\alpha}}{(n+1)_{m\alpha}}H^{(\lambda,\zeta-b)}_{n+m\alpha,m,p}(x,A,B;a,\mu).
$$

Thus, we get the desired result (3.4). □

Theorem 3.3. *For the modified Hermite-type matrix polynomials we have the formula*

$$
{}_c\mathbb{I}^{\alpha}_{\lambda}\{H^{(c-\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{1}{(n+1)_{p\alpha}}\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\alpha}
\times H^{(c-\lambda,\zeta)}_{n+p\alpha,m,p}(x,A,B;a,\mu),\ n+p\alpha\geq 0, \tag{3.5}
$$

and

$$
{}_c\mathbb{I}^{\alpha}_{\zeta}\{H^{(\lambda,c-\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{(-B\ln(a))^{-\alpha}}{(n+1)_{m\alpha}}\ H^{(\lambda,c-\zeta)}_{n+m\alpha,m,p}(x,A,B;a,\mu),\ n+m\alpha\geq 0. \tag{3.6}
$$

Proof. With the help of (1.11) and (2.3), we obtain (3.5) and (3.6). □

Theorem 3.4. *The Weyl integral of the modified Hermite-type matrix polynomials of order α satisfy the interesting formula*

$$
{}_{\lambda}W^{\alpha}_{\infty}\{H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{(-1)^{\alpha}}{(n+1)_{p\alpha}}\left(x\ln(a)\sqrt{\mu\ A}\right)^{-\alpha}
\times H^{(\lambda,\zeta)}_{n+p\alpha,m,p}(x,A,B;a,\mu),\ n+p\alpha\geq 0, \tag{3.7}
$$

and

$$
{}_{\zeta}W^{\alpha}_{\infty}\{H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{(-1)^{\alpha}(-B\ln(a))^{-\alpha}}{(n+1)_{m\alpha}}H^{(\lambda,\zeta)}_{n+m\alpha,m,p}(x,A,B;a,\mu),\ n+m\alpha\geq 0. \tag{3.8}
$$

Proof. From (2.3) and (1.12), we have

$$
{}_{\lambda}W^{\alpha}_{\infty}\{H^{(\lambda,\zeta)}_{n,m,p}\} = \frac{1}{\Gamma(\alpha)}\int_{\lambda}^{\infty}(t-\lambda)^{\alpha-1}H^{(t,\zeta)}_{n,m,p}(x,A,B;a,\mu)dt
$$

$$
= \frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_{\lambda}^{\infty}(t-\lambda)^{\alpha-1}t^{\frac{n-mk}{p}}dt.
$$

Putting $u=\frac{\lambda}{t}$, $t=\frac{\lambda}{u}$, $dt=-\frac{\lambda}{u^2}du$, $t=\infty$, $u=0$ and $t=\lambda$, $u=1$, we get

$$\int_{\lambda}^{\infty}(t-\lambda)^{\alpha-1}t^{\frac{n-mk}{p}}dt=\lambda^{\alpha+\frac{n-mk}{p}}\frac{\Gamma(\alpha)\Gamma(\frac{mk-n}{p}-\alpha)}{\Gamma(\frac{mk-n}{p})}$$
$$=\frac{(-1)^{\alpha}\lambda^{\alpha+\frac{n-mk}{p}}\Gamma(\alpha)\Gamma(\frac{n-mk}{p}+1)}{\Gamma(\frac{n-mk}{p}+\alpha+1)}$$

and we get

$${}_{\lambda}W_{\infty}^{\alpha}\{H_{n,m,p}\}=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^{k}\Big(x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{k!\Gamma\Big(\alpha+\frac{n-mk}{p}+1\Big)}\lambda^{\alpha+\frac{n-mk}{p}}$$
$$=\frac{(-1)^{\alpha}\Big(x\ln(a)\sqrt{\mu\,A}\Big)^{-\alpha}}{(n+1)_{p\alpha}}H_{n+p\alpha,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu).$$

which gives (3.7). Similarly, we have

$${}_{\zeta}W_{\infty}^{\alpha}\{H_{n,m,p}^{(\lambda,\zeta)}\}=\frac{1}{\Gamma(\alpha)}\int_{\zeta}^{\infty}(t-\zeta)^{\alpha-1}H_{n,m,p}^{(\lambda,t)}(x,A,B;a,\mu)dt$$
$$=\frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^{k}\Big(\lambda x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{k!\Gamma\Big(\frac{n-mk}{p}+1\Big)}\int_{\zeta}^{\infty}(t-\zeta)^{\alpha-1}t^{k}dt.$$

Putting $u=\frac{\zeta}{t}$, $t=\frac{\zeta}{u}$, $dt=-\frac{\zeta}{u^2}du$, $t=\infty$, $u=0$ and $t=\zeta$, $u=1$, we get

$$\int_{\zeta}^{\infty}(t-\zeta)^{\alpha-1}t^{k}dt=\frac{(-1)^{\alpha}\zeta^{\alpha+k}\Gamma(\alpha)\Gamma(k+1)}{\Gamma(k+\alpha+1)}$$

and we get

$${}_{\zeta}W_{\infty}^{\alpha}\{H_{n,m,p}\}=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-B\ln(a))^{k}\Big(\lambda x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{\Gamma\Big(\alpha+k+1\Big)\Gamma\Big(\frac{n-mk}{p}+1\Big)}\zeta^{\alpha+k}$$
$$=\frac{(-1)^{\alpha}(-B\ln(a))^{-\alpha}}{(n+1)_{m\alpha}}H_{n+m\alpha,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu).$$

which gives (3.8). □

Theorem 3.5. *Let $\alpha\in\mathbb{C}$, $Re(\alpha)\geq 0$ and $n=[Re(\alpha)]+1$, ${}_{x}D_{c}^{\alpha}$ be the right sided Riemann-Liouville fractional derivative. Then for the modified Hermite-type matrix polynomials holds the formula*

$${}_{\lambda}D_{c}^{\alpha}\{H_{n,m,p}^{(c-\lambda,\zeta)}(x,A,B;a,\mu)\}=\frac{\Gamma(n+1)}{\Gamma(n+1-p\alpha)}\Big(x\ln(a)\sqrt{\mu\,A}\Big)^{\alpha}$$
$$\times H_{n-p\alpha,m,p}^{(c-\lambda,\zeta)}(x,A,B;a,\mu),\;n-p\alpha\geq 0,\tag{3.9}$$

and

$${}_{\zeta}D_{c}^{\alpha}\{H_{n,m,p}^{(\lambda,c-\zeta)}(x,A,B;a,\mu)\}=\frac{\Gamma(n+1)}{\Gamma(n+1-m\alpha)}(-B\ln(a))^{\alpha}$$
$$\times H_{n-m\alpha,m,p}^{(\lambda,c-\zeta)}(x,A,B;a,\mu),\;n-m\alpha\geq 0.\tag{3.10}$$

Proof. Using (2.3) and (1.14) we have

$$
{}_{\lambda}D_c^{\alpha}\{H_{n,m,p}^{(c-\lambda,\zeta)}(x,A,B;a,\mu)\} = \frac{n!(-1)^n}{\Gamma(n-\alpha)} \sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B\ln(a))^k}{k!\Gamma\Big(\frac{n-mk}{p}+1\Big)}
$$

$$
\times \Big(x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}} \Big(\frac{\partial}{\partial\lambda}\Big)^n \int_{\lambda}^{c} \frac{(c-t)^{\frac{n-mk}{p}}}{(t-\lambda)^{\alpha-n+1}}dt.
$$

Putting $u=\frac{c-t}{c-\lambda}$, $c-t=(c-\lambda)u$, $dt=-(c-\lambda)du$, $t=c$, $u=0$ and $t=\lambda$, $u=1$, we get

$$
\int_{\lambda}^{c} \frac{(c-t)^{\frac{n-mk}{p}}}{(t-\lambda)^{\alpha-n+1}}dt = \int_{\lambda}^{c}(c-t)^{\frac{n-mk}{p}}(c-\lambda-(c-t))^{n-\alpha-1}dt
$$

$$
= (c-\lambda)^{n-\alpha+\frac{n-mk}{p}} \frac{\Gamma(n-\alpha)\Gamma\Big(\frac{n-mk}{p}+1\Big)}{\Gamma\Big(n+\frac{n-mk}{p}-\alpha+1\Big)}
$$

and we can write

$$
{}_{\lambda}D_c^{\alpha}\{H_{n,m,p}^{(c-\lambda,\zeta)}(x,A,B;a,\mu)\}
$$

$$
= n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B\ln(a))^k\Big(x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{k!\Gamma\Big(1-\alpha+\frac{n-mk}{p}\Big)}(c-\lambda)^{\frac{n-mk}{p}-\alpha}
$$

$$
= \frac{\Big(x\ln(a)\sqrt{\mu\, A}\Big)^{\alpha}\Gamma\Big(n+1\Big)}{\Gamma\Big(n-p\alpha+1\Big)} H_{n-p\alpha,m,p}^{(c-\lambda,\zeta)}(x,A,B;a,\mu),
$$

which gives (3.9). Also we have

$$
{}_{\zeta}D_c^{\alpha}\{H_{n,m,p}^{(\lambda,c-\zeta)}(x,A,B;a,\mu)\} = \frac{n!(-1)^n}{\Gamma(n-\alpha)} \sum_{k=0}^{[\frac{n}{m}]} \frac{(-B\ln(a))^k\Big(\lambda x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{k!\Gamma\Big(\frac{n-mk}{p}+1\Big)}
$$

$$
\times \Big(\frac{\partial}{\partial\zeta}\Big)^n \int_{\zeta}^{c} \frac{(c-t)^k}{(t-\zeta)^{\alpha-n+1}}dt.
$$

Putting $u=\frac{c-t}{c-\zeta}$, $c-t=(c-\zeta)u$, $dt=-(c-\zeta)du$, $t=c$, $u=0$ and $t=\zeta$, $u=1$, we get

$$
\int_{\zeta}^{c} \frac{(c-t)^k}{(t-\zeta)^{\alpha-n+1}}dt = \int_{\zeta}^{c}(c-t)^k(c-\zeta-(c-t))^{n-\alpha-1}dt = (c-\zeta)^{n-\alpha+k}\frac{\Gamma(n-\alpha)\Gamma\Big(k+1\Big)}{\Gamma\Big(n-\alpha+k+1\Big)}
$$

Hence, we can write

$$
{}_{\zeta}D_c^{\alpha}\{H_{n,m,p}^{(\lambda,c-\zeta)}(x,A,B;a,\mu)\}
$$

$$
= n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(-B\ln(a))^k\Big(\lambda x\ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{\Gamma\Big(k-\alpha+1\Big)\Gamma\Big(\frac{n-mk}{p}+1\Big)}(c-\zeta)^{k-\alpha}
$$

$$
= \frac{(-B\ln(a))^{\alpha}\Gamma\Big(n+1\Big)}{\Gamma\Big(n+1-m\alpha\Big)} H_{n-m\alpha,m,p}^{(\lambda,c-\zeta)}(x,A,B;a,\mu),
$$

which is the desired result (3.10). □

Theorem 3.6. *The left-sided operator of Riemann-Liouville fractional derivative for Modified Hermite-type matrix polynomials satisfy the interesting formula*

$$\begin{aligned} {}_bD_\lambda^\alpha\{H_{n,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu)\} &= \frac{\Gamma(n+1)}{\Gamma(n+1-p\alpha)}\left(x\ln(a)\sqrt{\mu\ A}\right)^\alpha \\ &\times H_{n-p\alpha,m,p}^{(\lambda-b,\zeta)}(x,A,B;a,\mu),\ n-p\alpha\geq 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} {}_bD_\zeta^\alpha\{H_{n,m,p}^{(\lambda,\zeta-b)}(x,A,B;a,\mu)\} &= \frac{\Gamma(n+1)}{\Gamma(n+1-m\alpha)}(-B\ln(a))^\alpha \\ &\times H_{n-m\alpha,m,p}^{(\lambda,\zeta-b)}(x,A,B;a,\mu), n-m\alpha\geq 0. \end{aligned} \tag{3.12}$$

Proof. With the help of (1.13) and (2.3), one can obtain (3.11) and (3.12). □

Theorem 3.7. *The Weyl fractional derivative of the modified Hermite-type matrix polynomials of order α satisfy the interesting formula*

$${}_\lambda D_\infty^\alpha\{H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)\} = (-n)_{p\alpha}\left(x\ln(a)\sqrt{\mu\ A}\right)^\alpha H_{n-p\alpha,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),\ n-p\alpha\geq 0, \tag{3.13}$$

and

$${}_\zeta D_\infty^\alpha\{H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)\} = (-n)_{m\alpha}\ (-B\ln(a))^\alpha\ H_{n-m\alpha,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu); n-m\alpha\geq 0. \tag{3.14}$$

Proof. Using (2.3) and (1.15) we have

$$\begin{aligned} {}_\lambda D_\infty^\alpha\{H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)\} &= \frac{n!(-1)^m}{\Gamma(m-\alpha)}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\left(x\ln(a)\sqrt{\mu A}\right)^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)} \\ &\times\left(\frac{\partial}{\partial\lambda}\right)^m\int_\lambda^\infty\frac{t^{\frac{n-mk}{p}}}{(t-\lambda)^{\alpha-n+1}}dt. \end{aligned}$$

Putting $u=\frac{\lambda}{t}$, $t=\frac{\lambda}{u}$, $dt=-\frac{\lambda}{u^2}du$, $t=\infty$, $u=0$ and $t=\lambda$, $u=1$, we get

$$\begin{aligned} \int_\lambda^\infty\frac{t^{\frac{n-mk}{p}}}{(t-\lambda)^{\alpha-m+1}}dt &= \frac{\Gamma(m-\alpha)\Gamma\left(\alpha-m+\frac{mk-n}{p}\right)}{\Gamma\left(\frac{mk-n}{p}\right)}\lambda^{m-\alpha+\frac{n-mk}{p}} \\ &= \frac{(-1)^{\alpha-m}\Gamma(m-\alpha)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(\frac{n-mk}{p}+m-\alpha+1\right)}\lambda^{m-\alpha+\frac{n-mk}{p}} \end{aligned}$$

and we can write

$${}_\lambda D_\infty^\alpha\{H_{n,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu)\} = \frac{(-1)^\alpha(x\ln(a)\sqrt{\mu A}\Big)^\alpha\Gamma(n+1)}{\Gamma\Big(n-p\alpha+1\Big)}H_{n-\alpha p,m,p}^{(\lambda,\zeta)}(x,A,B;a,\mu),$$

which gives (3.13). Similarly, we have

$$ {}_{\zeta}D^{\alpha}_{\infty}\{H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{n!(-1)^m}{\Gamma(m-\alpha)} \sum_{k=0}^{[\frac{n}{m}]} \frac{(-B\ln(a))^k \Big(\lambda x \ln(a)\sqrt{\mu A}\Big)^{\frac{n-mk}{p}}}{k!\Gamma\Big(\frac{n-mk}{p}+1\Big)} $$

$$ \times \left(\frac{\partial}{\partial\zeta}\right)^m \int_{\zeta}^{\infty} \frac{t^k}{(t-\zeta)^{\alpha-n+1}} dt. $$

Putting $u=\frac{\zeta}{t}$, $t=\frac{\zeta}{u}$, $dt=-\frac{\zeta}{u^2}du$, $t=\infty$, $u=0$ and $t=\zeta$, $u=1$, we get

$$ \int_{\zeta}^{\infty} \frac{t^k}{(t-\lambda)^{\alpha-m+1}} dt = \frac{(-1)^{\alpha-m}\Gamma(m-\alpha)\Gamma\Big(k+1\Big)}{\Gamma\Big(k+m-\alpha+1\Big)} \zeta^{m-\alpha+k} $$

Hence, we obtain

$$ {}_{\zeta}D^{\alpha}_{\infty}\{H^{(\lambda,\zeta)}_{n,m,p}(x,A,B;a,\mu)\} = \frac{(-1)^{\alpha}\ (-B\ln(a))^{\alpha}\Gamma\big(n+1\big)}{\Gamma\Big(n-m\alpha+1\Big)} H^{(\lambda,\zeta)}_{n-m\alpha,m,p}(x,A,B;a,\mu). $$

which is the desired result of (3.14). □

4 Laplace, Mellin and Fractional Fourier transforms of the Modified Hermite-type matrix polynomials

Theorem 4.1. *The Laplace transform of the modified Hermite-type matrix polynomials is as follows :*

$$ \pounds\{H^{(\lambda,\zeta)}_{n}(x;A,B;a,\mu):s\} = \frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)}{s^{\frac{n}{p}+1}}\ \ exp\Big(\frac{-\zeta B\ s^{\frac{m}{p}}\,(\sqrt{\mu A})^{-\frac{m}{p}}}{\lambda^{\frac{m}{p}}\ (\ln(a))^{\frac{m}{p}-1}}\Big). \tag{4.1} $$

Proof. Taking the Laplace transform of the Modified Hermite-type matrix polynomials, we have

$$ \pounds\{H^{(\lambda,\zeta)}_{n}(x;A,B;a,\mu):s\} = \int_0^{\infty} e^{-sx}\Big[\sum_{k=0}^{[\frac{n}{m}]} \frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\Big]dx. $$

Putting $sx=u$, $dx=\frac{du}{s}$, $x=0$, $u=0$ and $x=\infty$, $u=\infty$, we get

$$ \begin{aligned} \pounds\quad &\{H^{(\lambda,\zeta)}_{n}(x;A,B;a,\mu):s\} = \textstyle\sum_{k=0}^{[\frac{n}{m}]}\Big[\int_0^{\infty} e^{-u}\big(\frac{u}{s}\big)^{\frac{n-mk}{p}} \frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\Big]\frac{du}{s} \\ &= \textstyle\sum_{k=0}^{[\frac{n}{m}]}\Big[\frac{1}{s^{(\frac{n-mk}{p}+1)}}\big(\int_0^{\infty} e^{-u}u^{\frac{n-mk}{p}}du\big)\frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\Big] \\ &= \textstyle\sum_{k=0}^{[\frac{n}{m}]} \frac{1}{s^{(\frac{n-mk}{p}+1)}} \frac{\Gamma(n+1)\Gamma(\frac{n-mk}{p}+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)} \\ &= \frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)}{s^{\frac{n}{p}+1}}\ \ exp\Big(\frac{-\zeta B\ s^{\frac{m}{p}}\,(\sqrt{\mu A})^{-\frac{m}{p}}}{\lambda^{\frac{m}{p}}\ (\ln(a))^{\frac{m}{p}-1}}\Big). \end{aligned} $$

Thus, we get the desired result (4.1). It is worth mentioning here that this result of Theorem 4.1 can be further generalized by employing the properties of the Upadhyaya transform (see, Upadhyaya [36]), which we propose to do in a future communication of ours. □

Theorem 4.2. *The Mellin transform of the modified Hermite-type matrix polynomials is given by :*

$$M\{e^{-x}H_n^{(\lambda,\zeta)}(x;A,B;a,\mu):s\}=\frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)\Gamma(\frac{n}{p}+s)}{\Gamma(\frac{n}{p}+1)\Gamma(k+1)}$$

$$\times\ {}_2F_1[\,1,-\frac{n}{p};-\frac{n}{p}-s+1;(-\zeta B\ln(a))^{\frac{p}{m}}\ (\lambda\ln(a)\sqrt{\mu A})^{-1}\,]. \tag{4.2}$$

Proof. Taking the Mellin transform of the Modified Hermite-type matrix polynomials, we have

$$M\{e^{-x}H_n^{(\lambda,\zeta)}(x;A,B;a,\mu):s\}=\sum_{k=0}^{[\frac{n}{m}]}\frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\int_0^\infty x^{s-1}e^{-x}.x^{\frac{n-mk}{p}}dx$$

$$=\sum_{k=0}^{[\frac{n}{m}]}\frac{\Gamma(n+1)\Gamma(\frac{n-mk}{p}+s)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}.$$

Since we have

$$\Gamma(\frac{n-mk}{p}+1)=\frac{(-1)^{\frac{mk}{p}}\Gamma(\frac{n}{p}+1)}{(-\frac{n}{p})_{\frac{mk}{p}}},\ \Gamma(\frac{n-mk}{p}+s)=\frac{(-1)^{\frac{mk}{p}}\Gamma(\frac{n}{p}+s)}{(-(\frac{n}{p}+s-1))_{\frac{mk}{p}}},\ (1)_{\frac{mk}{p}}=\frac{\Gamma(\frac{mk}{p}+1)}{\Gamma(1)}.$$

Hence we can write

$$M\{e^{-x}H_n^{(\lambda,\zeta)}(x;A,B;a,\mu):s\}$$

$$=\frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)\Gamma(\frac{n}{p}+s)}{\Gamma(\frac{n}{p}+1)\Gamma(k+1)}\sum_{\frac{mk}{p}=0}^{[\frac{n}{p}]}\frac{(1)_{\frac{mk}{p}}\ (-\frac{n}{p})_{\frac{mk}{p}}((-\zeta B\ln(a))^{\frac{p}{m}})^{\frac{mk}{p}}\ (\lambda\ln(a)\sqrt{\mu A})^{\frac{-mk}{p}}}{(\frac{mk}{p})!(-(\frac{n}{p}+s-1))_{\frac{mk}{p}}}.$$

Let $\frac{mk}{p}=r$, then we obtain

$$M\{e^{-x}H_n^{(\lambda,\zeta)}(x;A,B;a,\mu):s\}$$

$$=\frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)\Gamma(\frac{n}{p}+s)}{\Gamma(\frac{n}{p}+1)\Gamma(k+1)}\sum_{r=0}^{[\frac{n}{p}]}\frac{(1)_r\ (-\frac{n}{p})_r((-\zeta B\ln(a))^{\frac{p}{m}})^r(\lambda\ln(a)\sqrt{\mu A})^{-r}}{(r)!(-(\frac{n}{p}+s-1))_r}$$

$$=\frac{(\lambda\ln(a)\sqrt{\mu A})^{\frac{n}{p}}\ \Gamma(n+1)\Gamma(\frac{n}{p}+s)}{\Gamma(\frac{n}{p}+1)\Gamma(k+1)}\ {}_2F_1[\,1,-\frac{n}{p};-\frac{n}{p}-s+1;(-\zeta B\ln(a))^{\frac{p}{m}}\ (\lambda\ln(a)\sqrt{\mu A})^{-1}\,].$$

which is the desired result (4.2). □

Theorem 4.3. *Fractional Fourier transform of the modified Hermite-type matrix polynomials for $x<0$ is given by:*

$$\Im_\alpha[\,H_n^{(\lambda,\zeta)}(x;A,B;a,\mu)\,]=(i\omega^{\frac{1}{\alpha}})^{-1}\ (\ -\lambda(i\omega^{\frac{1}{\alpha}})^{-1}\ \ln(a)\sqrt{\mu A}\ )^{\frac{n}{p}}\ \Gamma(n+1)$$

$$\times exp(\ \frac{-\zeta B(\sqrt{\mu A})^{-\frac{m}{p}}}{(\lambda(i\omega^{\frac{1}{\alpha}})^{-1})^{\frac{m}{p}}\ (\ln(a))^{\frac{m}{p}-1}}\ ). \tag{4.3}$$

Proof.

$$\Im_\alpha[\,H_n^{(\lambda,\zeta)}(x;A,B;a,\mu)\,]=\sum_{k=0}^{[\frac{n}{m}]}\frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\int_{-\infty}^0 e^{i\omega^{\frac{1}{\alpha}}x}\ x^{\frac{n-mk}{p}}dx$$

Putting $t=-i\omega^{\frac{1}{\alpha}}x$, $dt=-i\omega^{\frac{1}{\alpha}}dx$, $x=-\infty$, $t=\infty$ and $x=0$, $t=0$, we get

$$\Im_\alpha[\,H_n^{(\lambda,\zeta)}(x;A,B;a,\mu)\,]=\sum_{k=0}^{[\frac{n}{m}]}\frac{\Gamma(n+1)(-\zeta B\ln(a))^k(\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\int_0^\infty e^{-t}\ (-\frac{t}{i\omega^{\frac{1}{\alpha}}})^{\frac{n-mk}{p}}\ \frac{dt}{i\omega^{\frac{1}{\alpha}}}$$

$$=\sum_{k=0}^{[\frac{n}{m}]}\frac{\Gamma(n+1)(-\zeta B\ln(a))^k(-\lambda\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}\ (i)^{-(\frac{n-mk}{p}+1)}(\omega^{\frac{1}{\alpha}})^{-(\frac{n-mk}{p}+1)}\Gamma(\frac{n-mk}{p}+1)}{k!\Gamma(\frac{n-mk}{p}+1)}$$

$$=(i\omega^{\frac{1}{\alpha}})^{-1}\ (\ -\lambda(i\omega^{\frac{1}{\alpha}})^{-1}\ \ln(a)\sqrt{\mu A}\ )^{\frac{n}{p}}\ \Gamma(n+1)\ \ exp(\ \frac{-\zeta B(\sqrt{\mu A})^{-\frac{m}{p}}}{(\lambda(i\omega^{\frac{1}{\alpha}})^{-1})^{\frac{m}{p}}\ (\ln(a))^{\frac{m}{p}-1}}\ ).$$

Special case: For $\alpha = 1$, the result shows the conventional Fourier transform

$$\Im[\ H_n^{(\lambda,\zeta)}(x;A,B;a,\mu)\ ] = (i\omega)^{-1}\ (\ -\lambda(i\omega)^{-1}\ \ln(a)\sqrt{\mu A}\ )^{\frac{n}{p}}\ \Gamma(n+1)\ \ exp(\ \frac{-\zeta B(\sqrt{\mu A}\ )^{-\frac{m}{p}}}{(\lambda(i\omega)^{-1})^{\frac{m}{p}}\ (\ln(a))^{\frac{m}{p}-1}}).$$

□

5 Definitions of the Modified Chebyshev's-type, the Modified Legendre's-type and the Modified Hermite-Hermite-type matrix polynomials

In this section, we introduce a matrix version of the Modified Chebyshev polynomials and investigate its proof.

Theorem 5.1. *Let A and B be commutative matrices in $\mathbb{C}^{N\times N}$. For any complex number μ and let μA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3) and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), then the Modified Chebyshev-type matrix polynomials of second, third and first kinds are given respectively by modifying the integral transforms involving the Modified Hermite-type matrix polynomials as:*

$$U_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \frac{1}{n!}\int_0^\infty e^{-t}t^{\frac{n}{m}}H_{n,m,p}^{(\lambda,\zeta)}(xt^{\frac{m-p}{m}};A,B;a,\mu))dt, \tag{5.1}$$

$$T_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \frac{(\lambda\ln(a)\sqrt{\nu A})^{-1}}{(n-1)!}\int_0^\infty e^{-t}t^{\frac{n}{m}-1}H_{n,m,p}^{(\lambda,\zeta)}(xt^{\frac{m-p}{m}};A,B;a,\mu))dt,\ n\geq 1, \tag{5.2}$$

where $T_{0,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \mathbf{0}$ *and*

$$W_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \frac{(\lambda\ln(a)\sqrt{\nu A})}{(n+1)!}\int_0^\infty e^{-t}t^{\frac{n}{m}+1}H_{n,m,p}^{(\lambda,\zeta)}(xt^{\frac{m-p}{m}};A,B;a,\mu))dt. \tag{5.3}$$

Proof. Using (1.8), (2.3) and (5.1), we have

$$\begin{aligned} U_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) &= \tfrac{1}{n!}\int_0^\infty e^{-t}t^{\frac{n}{m}}H_{n,m,p}^{(\lambda,\zeta)}(xt^{\frac{m-p}{m}};A,B;a,\mu))dt \\ &= \tfrac{1}{n!}\sum_{k=0}^{[\frac{n}{m}]}\tfrac{n!(-\zeta B\ln(a))^k(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}\int_0^\infty e^{-t}t^{\frac{n-(m-p)k}{p}}dt \\ &= \sum_{k=0}^{[\frac{n}{m}]}\tfrac{(-\zeta B\ln(a))^k\Gamma(\frac{n-(m-p)k}{p}+1)(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}. \end{aligned}$$

Hence, the Modified Chebyshev-type matrix polynomials of the second kind can be defined by

$$U_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\Gamma(\frac{n-(m-p)k}{p}+1)(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}.$$

In a similar way, we define the Modified Chebyshev-type matrix polynomials of the first kind as follows in the form

$$T_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = n(\lambda\ln(a)\sqrt{\nu A})^{-1}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\Gamma(\frac{n-(m-p)k}{p})(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}, n>0,$$

and the Modified Chebyshev-type matrix polynomials of the third kind in the form

$$W_{n,m,p}^{(\lambda,\zeta)}(x;A,B;a,\mu) = \frac{(\lambda\ln(a)\sqrt{\nu A})}{n+1}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-\zeta B\ln(a))^k\Gamma(\frac{n-(m-p)k}{p}+2)(\lambda x\ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}, n>-1.$$

Hence the proof is completed. □

Remark 5.2. On taking $a = e$, $\mu = 2$, $m = 2$, $\lambda = \zeta = 1$ and $B = I$ in (5.1) it reduces to $U^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu)$, the Chebyshev-type matrix polynomials defined in [14].

Here, we introduce a matrix version of the Modified Legendre-type polynomials and investigate its proof.

Theorem 5.3. *Let A and B be commutative matrices in $\mathbb{C}^{N\times N}$. For any complex number μ let μA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3) and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.4), then the Modified Legendre-type matrix polynomials are given by modifying the integral transforms involving modified Hermite-type matrix polynomials as:*

$$P^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{\frac{n}{p}} H^{(\lambda,\zeta)}_{n,m,p}(xt; A, B; a, \mu)dt. \tag{5.4}$$

Proof. From (1.8) and (2.3), we can write

$$\begin{aligned} &\tfrac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{n}{p}} H^{(\lambda,\zeta)}_{n,m,p}(xt; A, B; a, \mu)dt \\ &= \tfrac{2}{\sqrt{\pi}} \sum_{k=0}^{[\frac{n}{m}]} \tfrac{(-\zeta B \ln(a))^k (\lambda x \ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)} \int_0^\infty e^{-t^2} t^{\frac{2n-mk}{p}} dt \\ &= \sum_{k=0}^{[\frac{n}{m}]} \tfrac{(-\zeta B \ln(a))^k \Gamma(\frac{2n-mk}{p}+1)(\lambda x \ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}} k!\Gamma(\frac{n-mk}{p}+1)\Gamma(\frac{2n-mk}{2p})}. \end{aligned}$$

Hence, the Modified Legendre-type matrix polynomials can be defined by

$$P^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B \ln(a))^k \Gamma(\frac{2n-mk}{p}+1)(\lambda x \ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}} k!\Gamma(\frac{n-mk}{p}+1)\Gamma(\frac{2n-mk}{2p})},$$

or,

$$P^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B \ln(a))^k (\frac{1}{2})_{\frac{2n-mk}{2p}} (\lambda x \ln(a)\sqrt{\mu A})^{\frac{n-mk}{p}}}{k!\Gamma(\frac{n-mk}{p}+1)}.$$

□

Remark 5.4. On taking $a = e$, $\mu = 2$, $m = 2$, $\lambda = \zeta = 1$ and $B = I$ in (5.4) it reduces to $P^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu)$, the Legendre-type matrix polynomials defined in [26, 27].

In terms of $H^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu)$, the Modified Hermite-Hermite-type matrix polynomials can be written as

$${}_H H^{(\lambda,\zeta)}_{n,m,p}(x; A, B; a, \mu) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B \ln(a))^k (\sqrt{\mu A})^{n-mk} H^{(\lambda,\zeta)}_{n-mk,m,p}(x; A, B; a, \mu)}{k!(n-mk)!} \tag{5.5}$$

Proof. Using (2.1), we have

$$a^{(\lambda x t^p(\sqrt{\mu A})^{p+1}-\zeta B t^m((\sqrt{\mu A})^m+I)} = a^{\lambda x t^p(\sqrt{\mu A})^{p+1}-\zeta B(t\sqrt{\mu A})^m} \cdot a^{-\zeta B t^m}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\sqrt{\mu A})^{pn+mr}(\lambda x\sqrt{\mu A})^n(-\zeta B)^r(\ln(a))^{n+r}}{n!r!}\, t^{np+mr} \cdot a^{-\zeta B t^m}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{m}]} \frac{(\sqrt{\mu A})^n(-\zeta B\ln(a))^r(\lambda x\sqrt{\mu A}\ln(a))^{\frac{n-mr}{p}}}{\Gamma(\frac{n-mr}{p}+1)r!}\, t^n \cdot a^{-\zeta B t^m}$$
$$= \sum_{n=0}^{\infty} \frac{(\sqrt{\mu A})^n H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)}{n!}\, t^n \sum_{k=0}^{\infty}\frac{(-\zeta B\ln(a))^k t^{mk}}{k!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} \frac{(-\zeta B\ln(a))^k(\sqrt{\mu A})^{n-mk} H^{(\lambda,\zeta)}_{n-mk,m,p}(x;A,B;a,\mu)}{k!(n-mk)!} t^n$$
$$= \sum_{n=0}^{\infty} {}_H H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n, we get (5.5). □

Remark 5.5. On taking $a = e, \mu = 2, m = 2, \lambda = \zeta = 1$ and $B = I$ in (5.5) it reduces to ${}_H H^{(\lambda,\zeta)}_{n,m,p}(x;A,B;a,\mu)$, the Hermite-Hermite-type matrix polynomials defined in [18].

6 Concluding Remarks

In this paper we explored certain properties of the Modified Hermite-type matrix polynomials and showed their relationship to the other similar matrix polynomials existing in the literature.

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