

On Special Spaces of $h(hv)$ –Torsion Tensor C_{jkh} in Generalized Recurrent Finsler Space

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ABSTRACT

The purpose of the present paper is to use the properties of C_2 –like space, C –reducible space, semi- C –reducible space and C_3 –like space in the generalized BP –recurrent space to get new spaces related to generalized recurrent. Further, we find certain identities belong to these spaces.

KEYWORDS: C_2 –like space, C –reducible space, semi- C –reducible space, C_3 –like space.

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1. INTRODUCTION

Various special forms of the $h(hv)$ –torsion tensor C_{jkh} called as C_2 –like space, C –reducible space, semi- C –reducible space and C_3 –like space have been studied by the Finslerian geometers. Matsumoto and Numata [14] and Aveesh et al. [19] introduced definition for C_2 –like space, Singh and Gupta [21] discussed some properties for C_2 –like space

Saxena [16] studied C –reducible Finsler space with Douglas tensor and gave the condition for Finsler space to be C –reducible Finsler space. Matsumoto [12] defined C –reducible Finsler space, Dwivedi [15] obtained every C –reducible Finsler space is P –reducible and converse is not necessarily true.

Tiwari et al. [9] and Heydari [6] introduced definition for semi- C –reducible space and studied some properties for it. Also, Chethana and Narasimhamurthy [7] proved that every semi- C –reducible manifold with C –reducible metric reduces to Landsberg manifold.

Tayebe and Peyghan [5], Pandey and Tripathi [20] and Beizavi [18] introduced definition for C_3 –like space and discussed its relationship with other spaces in Finsler space. In addition, Gangopadhyay and Tiwari [17] obtained that C_3 –like Finsler metric may be consider as a generalization of C –reducible, semi- C –reducible and C_2 –like Finsler metrics. The aim of this paper is to merge the generalized BP –recurrent space with various special forms of the $h(hv)$ –torsion tensor C_{jkh} to get new spaces in Finsler space.

2. PRELIMINARIES

In this section, some definitions will be given for the purpose of this paper. An n –dimensional space X_n equipped with a function $F(x, y)$ which denoted by $F_n = (X_n, F(x, y))$ called a *Finsler space* if the function $F(x, y)$ satisfying the request conditions [10].

The tensor C_{ijk} that is known as $(h)hv$ –torsion tensor defined as [13]

$$C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2.$$

It is positively homogeneous of degree -1 in y^i and symmetric in all its indices. The above tensor C_{ijk} satisfies

$$(2.1) \quad C_{ik}^h = g^{hj} C_{ijk},$$

where C_{jk}^i is called *associate tensor of the tensor* C_{ijk} .

Berwald's covariant derivative $\mathfrak{B}_k T_j^i$ of an arbitrary tensor field T_j^i with respect to x^k is given by [10]

$$\mathfrak{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r.$$

Let Berwald's covariant derivative of first order for the tensors C_{kh}^i and C_{jkh} which are satisfy [3]

$$(2.2) \quad \mathfrak{B}_m C_{kh}^i = \lambda_m C_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k)$$

and

$$(2.3) \quad \mathfrak{B}_m C_{jkh} = \lambda_m C_{jkh} + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Definition 2.1. A Finsler space $F_n (n \geq 2)$ with $C^2 = C_j C^j \neq 0$, it is called a *C2 – like space* if the $(h)hv$ –torsion tensor C_{jkh} can be written in the form [11, 14]

$$(2.4) \quad C_{jkh} = C_j C_k C_h / C^2,$$

where $C_j = g^{kh} C_{jkh}$.

Definition 2.2. A Finsler space F_n is called a *C –reducible space* if the $(h)hv$ –torsion tensor C_{jkh} is characterized by the condition [12, 15, 16]

$$(2.5) \quad C_{jkh} = \frac{1}{(n+1)} (h_{jk} C_h + h_{kh} C_j + h_{hj} C_k),$$

where $h_{jk} = g_{jk} - l_j l_k$ is an angular metric tensor.

Definition 2.3. A Finsler metric F_n is called a *semi-C –reducible* if the $(h)hv$ –torsion tensor C_{jkh} is given by [6, 7, 9]

$$(2.6) \quad C_{jkh} = \left[\frac{p}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|\mathbf{I}\|^2} I_j I_k I_h \right],$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on F_n and $\|\mathbf{I}\|^2 = I^i I_i$.

Definition 2.4. A Finsler metric F_n is called a *C3 –like space* if the $(h)hv$ –torsion tensor C_{jkh} is given by [5, 8, 17, 19]

$$(2.7) \quad C_{jkh} = \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right],$$

where $A_i = A_i(x, y)$ and $B_i(x, y)$ are y –homogeneous scalar functions on F_n of degree -1 and 1 , respectively.

The generalized \mathfrak{BP} –recurrent space has been introduced by Alaa et al. [1, 2, 4]. This space is characterized by the condition

$$(2.8) \quad \mathfrak{B}_m P_{jkh}^i = \lambda_m P_{jkh}^i + \mu_m (\delta_j^i g_{kh} - \delta_k^i g_{jh}), \quad P_{jkh}^i \neq 0.$$

The above space is denoted by $G(\mathfrak{BP}) - RF_n$.

3. A $C2$ –LIKE–GENERALIZED \mathfrak{BP} –RECURRENT SPACE

In this section, we will merge the generalized \mathfrak{BP} –recurrent space (main space) i.e. [it is characterized by the condition (2.8)] with $C2$ –like space in Finser space to get new space contain the same properties of the main space.

Definition 3.1. The generalized \mathfrak{BP} –recurrent space which is $C2$ –like space i.e. satisfies the condition (2.4), will be called a $C2$ –like generalized \mathfrak{BP} –recurrent space and will be denoted briefly by $C2$ –like – $G(\mathfrak{BP})$ – RF_n .

Let us consider a $C2$ – like – $G(\mathfrak{BP})$ – RF_n .

Taking \mathfrak{B} –covariant derivative for the condition (2.4) with respect to x^m and using (2.3), we get

$$\mathfrak{B}_m(C_j C_k C_h / C^2) = \lambda_m C_{jkh} + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Using the condition (2.4) in above equation, we get

$$(3.1) \quad \mathfrak{B}_m(C_j C_k C_h / C^2) = \lambda_m (C_j C_k C_h / C^2) + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Transvecting the condition (2.4) by g^{ij} using (2.1), we get

$$(3.2) \quad C_{kh}^i = C^i C_j C_k / C^2,$$

where $C^i = g^{ij} C_j$.

Taking \mathfrak{B} –covariant derivative for eq. (3.2) with respect to x^m and using (2.2), we get

$$\mathfrak{B}_m(C^i C_j C_k / C^2) = \lambda_m C_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

Using eq. (3.2) in above equation, we get

$$(3.3) \quad \mathfrak{B}_m(C^i C_j C_k / C^2) = \lambda_m (C^i C_j C_k / C^2) + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

From eqs. (3.1) and (3.3), we conclude the following theorem.

Theorem 3.1. In $C2$ –like – $G(\mathfrak{BP})$ – RF_n , Berwald's covariant derivative of first order for the tensors $(C_j C_k C_h / C^2)$ and $(C^i C_j C_k / C^2)$ are given by eqs. (3.1) and (3.3), respectively.

4. A C –REDUCIBLE–GENERALIZED \mathfrak{BP} – RECURRENT SPACE

In this section, we will merge the generalized \mathfrak{BP} –recurrent space (main space) i.e. [it is characterized by the condition (2.8)] with C –reducible space in Finser space to get new space contain the same properties of the main space.

Definition 4.1. The generalized \mathfrak{BP} –recurrent space which is C –reducible space i.e. satisfies the condition (2.5), will be called a C –reducible generalized \mathfrak{BP} –recurrent space and will be denoted briefly by C –reducible – $G(\mathfrak{BP})$ – RF_n .

Let us consider a C –reducible – $G(\mathfrak{BP})$ – RF_n .

Taking \mathfrak{B} –covariant derivative for the condition (2.5) with respect to x^m and using (2.3), we get

$$\mathfrak{B}_m \left[\frac{1}{n+1} (h_{jk} C_h + h_{kh} C_j + h_{hj} C_k) \right] = \lambda_m C_{jkh} + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Using the condition (2.5) in above equation, we get

$$(4.1) \quad \mathfrak{B}_m \left[\frac{1}{n+1} (h_{jk} C_h + h_{kh} C_j + h_{hj} C_k) \right] = \lambda_m \left[\frac{1}{n+1} (h_{jk} C_h + h_{kh} C_j + h_{hj} C_k) \right] + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Transvecting the condition (2.5) by g^{ij} , using (2.1), we get

$$(4.2) \quad C_{kh}^i = \frac{1}{n+1} (h_k^i C_h + h_{kh} C^i + h_h^i C_k),$$

where $C^i = g^{ij} C_j$ and $h_k^i = g^{ij} h_{jk}$.

Taking \mathfrak{B} – covariant derivative for eq. (4.2) with respect to x^m and using eq. (2.2), we get

$$\mathfrak{B}_m \left[\frac{1}{n+1} (h_k^i C_h + h_{kh} C^i + h_h^i C_k) \right] = \lambda_m C_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

Using eq. (4.2) in above equation, we get

$$(4.3) \quad \mathfrak{B}_m \left[\frac{1}{n+1} (h_k^i C_h + h_{kh} C^i + h_h^i C_k) \right] = \lambda_m \left[\frac{1}{n+1} (h_k^i C_h + h_{kh} C^i + h_h^i C_k) \right] + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

From eqs. (4.1) and (4.3), we conclude the following theorem.

Theorem 4.1. In C – reducible – $G(\mathfrak{B}P) - RF_n$, Berwald's covariant derivative of first order for the tensors

$\left[\frac{1}{n+1} (h_{jk} C_h + h_{kh} C_j + h_{hj} C_k) \right]$ and $\left[\frac{1}{n+1} (h_k^i C_h + h_{kh} C^i + h_h^i C_k) \right]$ are given by eqs. (4.1) and (4.3), respectively.

5. A SEMI-C –REDUCIBLE–GENERALIZED $\mathfrak{B}P$ –RECURRENT SPACE

In this section, we will merge the generalized $\mathfrak{B}P$ – recurrent space (main space) i.e. [it is characterized by the condition (2.8)] with semi- C –reducible space in Finsler space to get new space contain the same properties of the main space.

Definition 5.1. The generalized $\mathfrak{B}P$ – recurrent space which is semi- C –reducible space i.e. satisfies the condition (2.6), will be called a *semi- C –reducible generalized $\mathfrak{B}P$ – recurrent space* and will be denoted briefly by *semi- C –reducible – $G(\mathfrak{B}P) - RF_n$* .

Let us consider a *semi- C –reducible – $G(\mathfrak{B}P) - RF_n$* .

Taking \mathfrak{B} – covariant derivative for the condition (2.6) with respect to x^m and using eq. (2.3), we get

$$\mathfrak{B}_m \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|\mathbf{I}\|^2} I_j I_k I_h \right] = \lambda_m C_{jkh} + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Using the condition (2.6) in above equation, we get

$$(5.1) \quad \mathfrak{B}_m \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|\mathbf{I}\|^2} I_j I_k I_h \right] = \lambda_m \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|\mathbf{I}\|^2} I_j I_k I_h \right] + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Transvecting the condition (2.6) by g^{ij} , using (2.1), we get

$$(5.2) \quad C_{kh}^i = g^{ij} \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right].$$

Taking \mathfrak{B} -covariant derivative for eq. (5.2) with respect to x^m and using (2.2), we get

$$\mathfrak{B}_m \left(g^{ij} \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right] \right) = \lambda_m C_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

Using eq. (5.2) in above equation, we get

$$(5.3) \quad \mathfrak{B}_m \left(g^{ij} \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right] \right) \\ = \lambda_m \left(g^{ij} \left[\frac{P}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right] \right) + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

From eqs. (5.1) and (5.3), we conclude the following theorem.

Theorem 5.1. In semi-C-reducible- $G(\mathfrak{B}P) - RF_n$, Berwald's covariant derivative of first order for the tensors $\left[\frac{p}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right]$ and $\left(g^{ij} \left[\frac{p}{1+n} (h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h \right] \right)$ are given by eqs. (5.1) and (5.3), respectively.

6. A C3-LIKE-GENERALIZED $\mathfrak{B}P$ -RECURRENT SPACE

In this section, we will merge the generalized $\mathfrak{B}P$ -recurrent space (main space) i.e. [it is characterized by the condition (2.8)] with C3-like space in Finser space to get new space contain the same properties of the main space.

Definition 6.1. The generalized $\mathfrak{B}P$ -recurrent space which is C3-like space i.e. satisfies the condition (2.7), will be called a C3-like generalized $\mathfrak{B}P$ -recurrent space and will be denoted briefly by C3-like- $G(\mathfrak{B}P) - RF_n$.

Let us consider a C3-like- $G(\mathfrak{B}P) - RF_n$.

Taking \mathfrak{B} -covariant derivative for the condition (2.7) with respect to x^m and using eq. (2.3), we get

$$\mathfrak{B}_m \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] \\ = \lambda_m C_{jkh} + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Using the condition (2.7) in above equation, we get

$$(6.1) \quad \mathfrak{B}_m \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] \\ = \lambda_m \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] + \mu_m (g_{jk} y_h - g_{jh} y_k).$$

Transvecting the condition (2.7) by g^{ij} , using (2.1), we get

$$(6.2) \quad C_{kh}^i = (g^{ij} \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right]).$$

Taking \mathfrak{B} -covariant derivative for eq. (6.2) with respect to x^m and using (2.2), we get

$$\mathfrak{B}_m (g^{ij} \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right]) \\ = \lambda_m C_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

Using eq. (6.2) in above equation, we get

$$(6.3) \quad \mathfrak{B}_m \left(g^{ij} \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] \right) \\ = \lambda_m \left(g^{ij} \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] \right) + \mu_m (\delta_k^i y_h - \delta_h^i y_k).$$

From eqs. (6.1) and (6.3), we conclude the following theorem.

Theorem 6.1. In $C3$ –like– $G(\mathfrak{B}P)$ – RF_n , Berwald's covariant derivative of first order for the tensors $\left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right]$ and $\left(g^{ij} \left[(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h) \right] \right)$ are given by eqs. (6.1) and (6.3), respectively.

7. CONCLUSION

We used some special Finsler spaces in the generalized $\mathfrak{B}P$ –recurrent space to get new spaces which are called $C2$ –like generalized $\mathfrak{B}P$ –recurrent space, C –reducible generalized $\mathfrak{B}P$ –recurrent space, semi– C –reducible generalized $\mathfrak{B}P$ –recurrent space and $C3$ –like generalized $\mathfrak{B}P$ –recurrent space.

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