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Odd Harmonious Labeling of Super Subdivision of Star and Comb Graph

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ABSTRACT

An odd harmonious labeling of a graph G with q edges is an injective function $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q - 1)\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \ldots, (2q - 1)\}$ defined by $f^*(xy) = f(x) + f(y)$ is a bijection. A graph that admits odd harmonious labeling is called an odd harmonious graph. In this paper, we prove that the super subdivision of star and comb graphs are odd harmonious.

Keywords: Harmonious Labeling, Comb Graph.

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1. Introduction

An area in Mathematics known as graph theory, which examines how graphs of points are connected by lines, has become a well-known area of study in recent times. Graph theory involves various disciplines to study graphs, one such discipline of graph theory is known as graph labeling. Rosa [5] invented graph labeling in 1967. Graph Labeling is the assignment of integers to the set of vertices or edges or both, subject to certain conditions. Various scholars have defined many graph labeling techniques over the years. More than 3000 studies have examined more than 200 graph labeling strategies in recent years. Gallian [1] provides a thorough analysis of graph labeling. The concept of odd harmonious labeling was initiated by Liang and Bai [4] in 2009. Odd harmonious labeling of a graph G with q edges defined as $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q - 1)\}$ is an injective function and such that the induced function $f^* : E(G) \rightarrow \{1, 3, \ldots, (2q - 1)\}$ defined by $f^* = f(x) + f(y)$ is a bijection. A graph that admits odd harmonious labeling is called an odd harmonious graph. A star is an odd harmonious graph as proved by Liang and Bai [4]. In 2001, Sethuraman and Selvaraju [6] introduced a graph operation called super subdivision of graph, denoted by $SSD(G)$. Jeba Jesintha, Devakirubanithi, and Jeba Sherlin [3] proved the odd graceful labeling of super subdivision of path and star graph. In this paper, we examine odd harmonious labeling on the super subdivision of star and comb graphs.

2. Preliminaries
In this section, a few definitions that will be used to support our theorems is stated.

Definition 2.1
Graham and Sloane [2] defined the *harmonious* graph G.

A graph G with q edges to be *harmonious* if there is an injection from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)(mod\ q)$, the resulting edge labels are distinct.

Definition 2.2
Liang and Bai [4] defined the *odd harmonious labeling* of a graph G

An *odd harmonious labeling* [4] of a graph G with q edges is an injective function $f: V(G) \rightarrow \{0, 1, 2, \ldots, (2q - 1)\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \ldots, (2q - 1)\}$ defined by $f^*(xy) = f(x) + f(y)$ is a bijection. A graph that admits odd harmonious labeling is called an *odd harmonious graph.*

Definition 2.3
A star graph [1] is a tree consisting of one vertex adjacent to all others. A star graph is denoted as $K_{1,n}$ where n number of pendant vertices are connected to one vertex.

Definition 2.4
Sethuraman [6] defined the *super subdivision graph* $SSD(G)$ of a graph G.
The *super subdivision graph* $SSD(G)$ [6] of a graph is obtained from G by replacing every edge uv of G by $K_{2,m}$ and identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members.

3. Main Results

In this section, the odd harmonious labeling has been established on the super subdivision of the star graph and the super subdivision of the comb graph.

Theorem 3.1. *The super subdivision of the star graph $SSD_m(K_{1,n})$ is odd harmonious.*
Proof.
Let $SSD_m(K_{1,n})$ be the super subdivision of a star graph where m denotes the number of times of the super subdivision of the star $K_{1,n}$ with $n + 1$ vertices and n edges. Let a be the central vertex and $a_i: 1 \leq i \leq 2q - 2m$ be the other vertices of the star graph. Let $y_j: 1 \leq j \leq q - 1$ be the vertices of m times the super subdivision of the star. The entire number of vertices and edges of $SSD_m(K_{1,n})$ is given as $V(G) = nm + (n + 1)$, $E(G) = 2nm$ respectively. The general graph of the super subdivision of the star is depicted below in Figure 1.

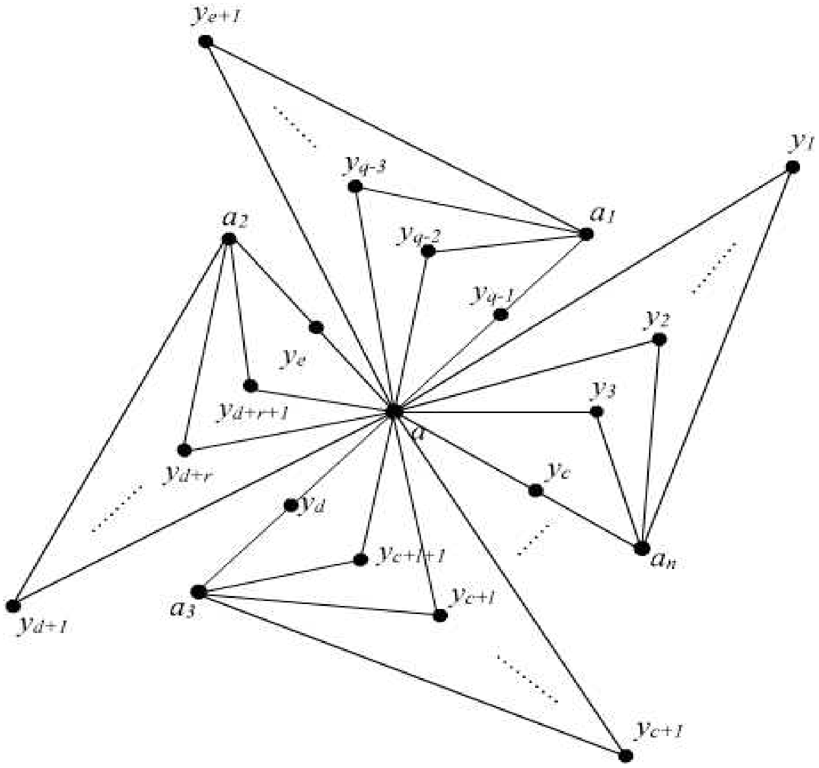

Figure 1: Super subdivision of star $SSD_m(K_{1,n})$

Vertex labeling $f : V(G) \to \{0,1,2, \dots (2q-1)\}$ is defined as
$f(a) = 0$
$f(a_i) = 4mi - 2m$, $1 \le i \le 2q - 2m$
$f(y_j) = 2j - 1$, $1 \le j \le q - 1$,

Edge labeling f^*: $V(G) \to \{0,1,2, \dots (2q-1)\}$ for the graph $SSD_m(K_{1,n})$, as follows:

$f^*(ay_j) = 2j - 1$, $1 \le j \le q - 1$,

$f^*(a_i y_j) = |2j - 1 + 4mi - 2m|$, $1 \le i \le 2q - 2m$, $1 \le j \le q - 1$

which yields the edge labels $E_1 = \{1, 3, 5, \dots, q - 1\}$, $E_2 = \{q + 1, q + 2, \dots, 2q - 1\}$ respectively.

$$E = E_1 \cup E_2 = \{1,3,5, \dots ,2q - 1\}$$

It is observed that the edge labels from sets $\{1, 3, 5, \dots, 2q - 1\}$ are distinct odd numbers based on the computed edge labels. Hence the super subdivision of Star graph $SSD_m(K_{1,n})$ admits odd harmonious labeling. The above theorem is illustrated as follows in Figure 2.

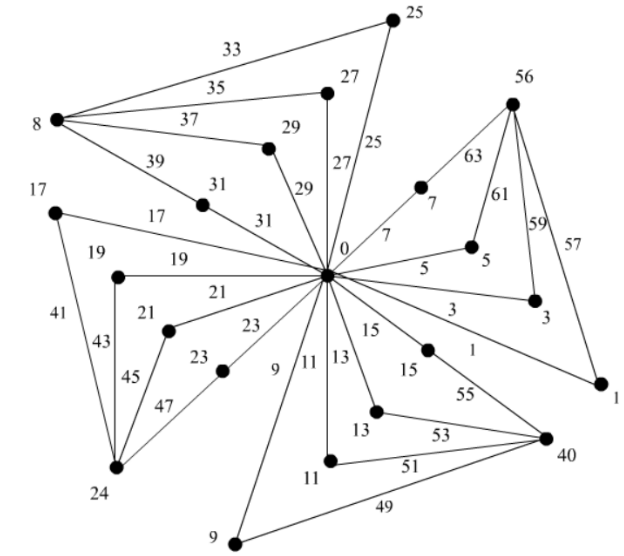

Figure 2: Odd harmonious labeling of Super subdivision of star $SSD_4(K_{1,4})$

Theorem 3.2. *The super subdivision of Comb graph $SSD_m(P_n \odot 1K_1)$ is odd harmonious.*
Proof.
Let $SSD_m(P_n \odot 1K_1)$ be the super subdivision of a comb graph where m denotes the number of times the super subdivision of the comb $SSD_m(P_n \odot 1K_1)$. The description of the graph is given as follows: Let $u_i: 1 \leq i \leq n$ be the vertices of path graph P_n and let $v_i: 1 \leq i \leq n$ be the vertices of $1K_1$ attached with each vertex of the path graph. Let $u_i^j, v_i^j: 1 \leq i \leq n, 1 \leq j \leq m$ be the vertices of m times the super subdivision of comb. The general graph of the super subdivision of the comb is depicted below in Figure 3.

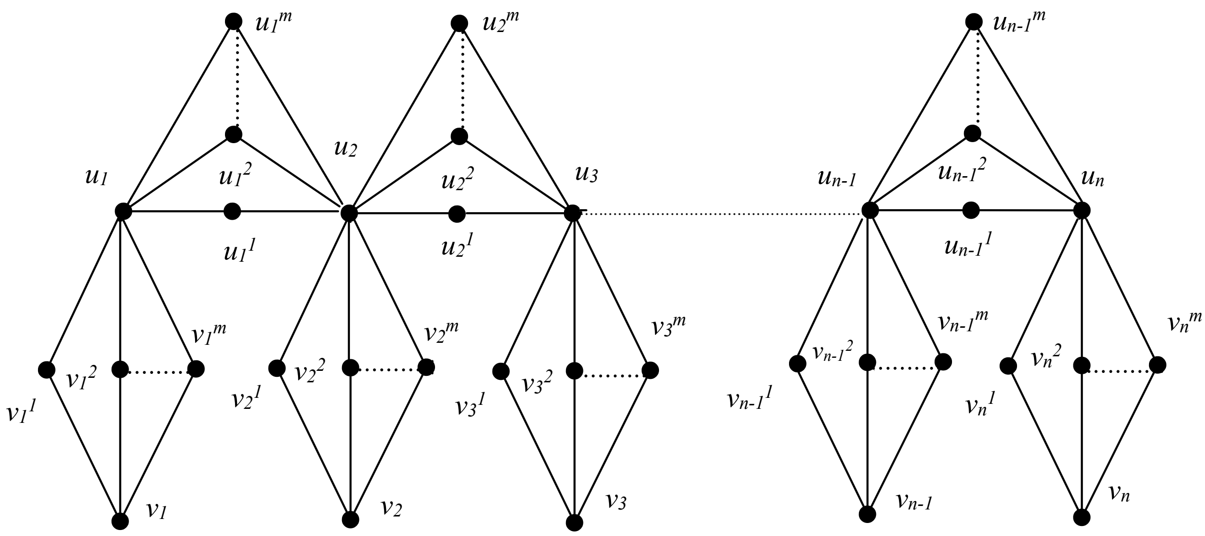

Figure 3: Super subdivision of Comb $SSD_m(P_n \odot 1K_1)$

The number of vertices p and edges q of the graph $SSD_m(P_n \odot 1K_1)$ are defined as:
$p = |V(G)| = m(2n-1)+2n$
$q = |E(G)| = 2(m(2n-1))$

Define vertex labeling $f : V(G) \rightarrow \{0,1,2,\dots(2q-1)\}$ for the graph $SSD_m(P_n \odot 1K_1)$ as follows:
$f(u_1) = 1$
$f(u_i) = \begin{cases} 2m + u_{i-1} & ; i \text{ is odd} \\ 6m + u_{i-1} & ; i \text{ is even} \end{cases}$
$f(v_1) = u_1 + 2m$
$f(v_i) = \begin{cases} 3m + v_{i-1} & ; i \text{ is odd} \\ 2m + v_{i-1} & ; i \text{ is even} \end{cases}$
$f(u_1^j) = u_1^1 + 2j - 2$
$f(u_i^1) = \begin{cases} u_i + 4m - 1; & i \text{ is odd} \\ u_i - 1 \quad ; & i \text{ is even} \end{cases}$
$f(u_i^j) = u_i^1 + (2j-2) \;; \quad 1 \le i \le n$
$f(v_i^1) = \begin{cases} v_1 - 1 & ; \quad 1 \le i \le 2 \\ v_i - 2m - 1 & ; \quad 3 \le i \le n \end{cases}$
$f(v_i^j) = v_i^1 + 2j - 2 \quad ; \quad 3 \le i \le n, 1 \le j \le m$

Define edge labeling $f^* : V(G) \rightarrow \{0,1,2,\dots(2q-1)\}$ for the graph $SSD_m(P_n \odot 1K_1)$, as follows:
$f^*(u_1u_i^1) = \begin{cases} u_i + 4m & ; \quad i \text{ is odd} \\ u_i & ; \quad i = 1, i \text{ is even} \end{cases}$
$f^*(u_1u_i^j) = u_i^1 + 2j - 1 \quad ; 1 \le i \le n$
$f^*(u_iu_i^1) = \begin{cases} 2m + u_{i-1} + u_i + 4m & ; \quad i \text{ is odd} \\ 6m + u_{i-1} + u_i - 1 & ; \quad i \text{ is even} \end{cases}$
$f^*(u_iu_i^j) = \begin{cases} 2m + u_{i-1} + u_1^1 + 2j - 2 & ; \quad i \text{ is odd} \\ 6m + u_{i-1} + u_1^1 + 2j - 2 & ; \quad i \text{ is even} \end{cases}$
$f^*(v_1v_i^1) = \begin{cases} u_1 + 2m + v_1 - 1 & ; \quad 1 \le i \le 2 \\ u_1 + 2m + v_i - (2m+1) & ; \quad 3 \le i \le n \end{cases}$
$f^*(v_1v_i^j) = u_1 + 2m + v_i^1 + (2j-2); \quad 3 \le i \le n, 1 \le j \le m$
$f^*(v_iv_i^1) = \begin{cases} 3m + v_{i-1} + v_1 - 1 & ; \quad i \text{ is odd} \\ 6m + v_{i-1} + v_i - (2m+1) & ; \quad i \text{ is even} \end{cases}$
$f^*(v_iv_i^j) = \begin{cases} 3m + v_{i-1} + v_i^1 + 2j - 2 & ; \quad i \text{ is odd} \\ 6m + v_{i-1} + v_i^1 + 2j - 2 & ; \quad i \text{ is even} \end{cases}$

which yields the edge labels $E_1 = \{1,3,5,\dots,q-1\}$, $E_2 = \{q+1,\dots,2q-1\}$.

$$\therefore E = E_1 \cup E_2 = \{1,3,5,\dots,2q-1\}$$

It is observed that the edge labels from the set $\{1,3,5,\dots,2q-1\}$ are distinct odd numbers based on the computed edge labels. Hence the super subdivision of the comb graph $SSD_m(P_n \odot 1K_1)$ admits odd harmonious labeling. Figure 4 illustrates the above theorem.

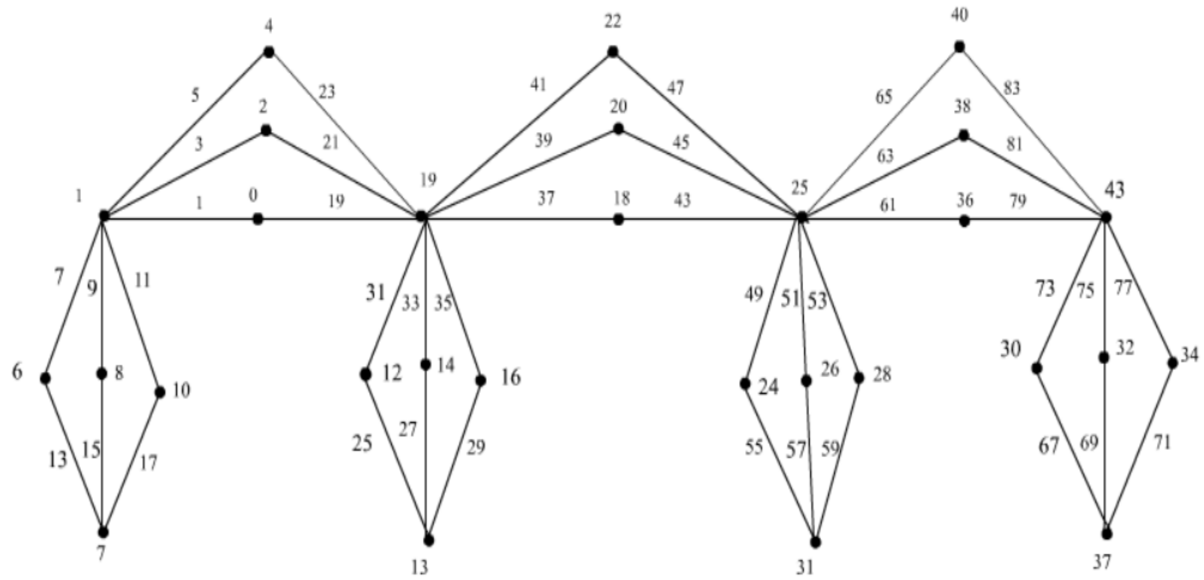

Figure 4: Odd harmonious labeling of Super subdivision of Comb $SSD_3(P_4 \odot 1K_1)$

4. Conclusion

In this paper, we have shown that the super subdivision of star graph and the super subdivision of comb graph admits odd harmonious labeling. Further we intend to prove the odd harmonious labeling for the super subdivision of other graphs.

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