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Odd gracefulness of cycle with subdivided shell graphs *

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Abstract In 1991 Gnanajothi (Gnanajothi, R.B. (1991). Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, Tamil Nadu, India) introduced a labeling method called odd graceful labeling. A graph G with q edges is said to be odd graceful if there is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$, such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edges labels are distinct and they are members of the set $\{1, 3, 5, \ldots, (2q-1)\}$. In this paper, we prove that the graph obtained by attaching each vertex of C_m with the subdivided shell graph is odd graceful, when $m \equiv 0 \,(\mathrm{mod}\ 4)$.

Key words Odd graceful labeling, Cycle, Shell graph, Subdivided shell graph.

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1 Introduction

Graph labeling methods trace their origin to the graceful labeling introduced by Rosa [5] in 1967. The graceful labeling of a graph G with q edges an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edges are distinct. In 1991, Gnanajothi [3] introduced the odd graceful labeling. An odd-graceful labeling is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, (2q-1)\}$ such that, when each edge xy is assigned the label$|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$.
Gnanajothi [3] proved that every cycle graph is odd graceful if and only if n is even. She also proved that the graph obtained from $P_n \times P_2$ by deleting an edge that joins to end points of the P_n paths is odd graceful. Govindarajan and Srividya [4] proved odd graceful labeling of every odd cycle C_n, $n \geq 7$ with parallel P_k chords for $k = 2, 4$ after the removal of two edges from the cycle C_n. Badr [1] proved that the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ are odd graceful, where k is any positive integer. Joseph Gallian [2] has given a broad and a dynamic survey on various graph labeling methods.

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Graph theory plays a vital role in many fields. Labeled graphs serve as useful mathematical models for a broad range of applications such as the design of good radar type codes, synch-set codes, missile guidance codes and radio astronomy problems. In this paper, we prove that the graph obtained by attaching each vertex of C_m with the subdivided shell graph is odd graceful, when $m \equiv 0 \pmod 4$.

Definition 1.1. The **Shell graph** is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. A shell graph is denoted by $C_{(n,n-3)}$. A shell graph is also called as Fan graph F_{n-1}.

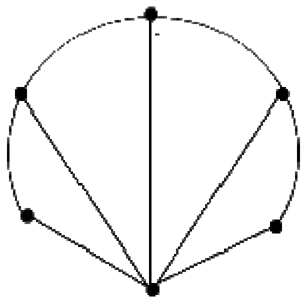

Fig. 1: The Shell Graph $C_{(6,3)}$.

Definition 1.2. The **Subdivided shell graph** is a shell graph in which the edges in the path of the shell are sub divided.

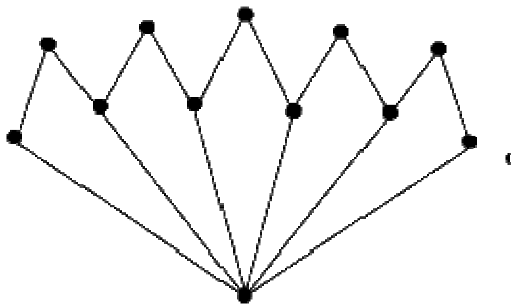

Fig. 2: The Subdivided Shell Graph.

2 Main result

In this section, we prove following main result of this paper:

Theorem 2.1. *The graph obtained by attaching each vertex of cycle C_m with the subdivided shell graph is odd graceful, when $m \equiv 0 \pmod 4$.*

Proof. Let G be the graph obtained by attaching m isomorphic copies of the subdivided shell graph at every vertex of the cycle C_m where $m \equiv 0 \pmod 4$. Let $|V(G)| = p$ and $|E(G)| = q$. The graph G is described as follows: The vertices in the cycle $C_m$in G are denoted by $u_1, u_2, \ldots, u_m$ in the clockwise direction. The middle vertices of the first copy of subdivided shell graph attached at u_1 are denoted by v_{11}, v_{12}, v_{13},..., v_{1j} where $1 \leq j \leq n$. The middle vertices of the second copy of the subdivided shell graph attached at vertex u_2 are denoted by v_{21}, v_{22}, v_{23},..,v_{2j} where, $1 \leq j \leq 2n$. In general, the middle vertices of the i^{th} copy of the subdivided shell graph attached at vertex $u_i$will denoted by $v_{ij}$where $1 \leq i \leq m, 1 \leq j \leq n$. The topmost vertices of the first copy of the subdivided shell graph are denoted by $w_1^1, w_2^1, w_3^1, \ldots, w_m^1$. Similarly, the topmost vertices of the

second copy of the subdivided shell graph are denoted by $w_1^2, w_2^2, w_3^2, \ldots, w_m^2$. In general, the topmost vertices of i^{th} copy of the subdivided shell graph will denoted by w_i^k where $1 \le i \le n$, $1 \le k \le m$. The graph G has $p = (3n+1)\, m$ vertices and $q = k\,(4n+1)$ edges.
The vertex labels for the cycle C_m *are* given below.

$$
\begin{aligned}
f\ (u_{2i-1}) &= (\ 2i-2)\,(n+1)\,, && \text{for} \quad 1 \le i \le \frac{m}{4} \\
f\ (u_{2i-1}) &= (\ 2i-2\ )\,(n+1)+2, && \text{for} \quad \frac{m}{4}+1 \le i \le \frac{m}{2} \\
f\ (u_{2i}) &= (\ 2q-1-2n\ ) - (2i-2)\,(n+1\ )\,, && \text{for} \quad 1 \le i \le \frac{m}{2}
\end{aligned}
\tag{2.1}
$$

The vertex labels for the subdivided shell graph are described as follows:

$$
\begin{aligned}
f\ (v_{2i-1,j}) &= (2q-1) - (n+1)\,(2i-2) - (2j-2)\,, && \text{for} \quad 1 \le i \le \frac{m}{2}\,,\ 1 \le j \le n \\
f\ (v_{2i,j}) &= 2 + (n+1)\,(2i-2) + (2j-2)\,, && \text{for} \quad 1 \le i \le \frac{m}{4}, 1 \le j \le n \\
f\,(v_{2i,j}) &= 4 + (n+1)\,(2i-2) + (2j-2)\,, && \text{for} \quad \frac{m}{4}+1 \le i \le \frac{m}{2}\,,\ 1 \le j \le n
\end{aligned}
$$

$$
f\left(w_l^{(2k-1)}\right) = (2q-4) - (k-1)\,(10n-6) - 6\,(l-1\ )\,, \quad \text{for} \quad 1 \le l \le n-1\,,\ 1 \le k \le \frac{m}{2} \tag{2.2}
$$

$$
f\left(w_l^{(2k)}\right) = \begin{cases} (4n+1) + (k-1)\,(10n-6) + 6\,(l-1)\,, \\ \quad \text{for} \quad 1 \le l \le n \ ,\ 1 \le k \le \frac{m}{4} \\ (4n+3) + (k-1)\,(10n-6\ ) + 6\,(l-1)\,, \\ \quad \text{for} \quad 1 \le l \le n \ ,\ \frac{m}{4}+1 \le k \le \frac{m}{2} \end{cases} \tag{2.3}
$$

From (2.1) to (2.3) we see that the vertex labels for the graph are distinct.
We compute the edge labels for the cycle C_m, for $m \equiv 0\,(\text{mod}4)$ as follows:

$$
|f\ (u_{2i-1}) - f\ (u_{2i})| = \begin{cases} 2\,(n+1)\,(2i-2) - (2q-1-2n), \\ \quad \text{for} \quad 1 \le i \le \frac{m}{4} \\ 2\,(n+1)\,(2i-2) - (2q-1-2n) + 2, \\ \quad \text{for} \quad \frac{m}{4}+1 \le i \le \frac{m}{2} \end{cases}
$$

$$
|f\ (u_{2i+1}) - f\ (u_{2i})| = \begin{cases} (2q-1-2n) - 4i\,(n+1) + 2(n+1), \\ \quad \text{for} \quad 1 \le i \le \frac{m}{4}-1 \\ (2q-1-2n) - 4i\,(n+1) + 2\,(n+1) - 2, \\ \quad \text{for} \quad \frac{m}{4} \le i \le \frac{m}{2}-1 \end{cases}
$$

$$
|\ f\,(u_1) - f\,(u_m)| = 2p - m\,(n+1) + 1 \tag{2.4}
$$

The edge labels for the remaining edges of the subdivided shell graph are computed as below :

$$
|f\,(u_{2i-1}) - f\,(v_{2i-1,j})| = \begin{cases} 4\,(i-1)\,(n+1) + (2j-2) - (2q-1)\,, \\ \quad \text{for} \quad 1 \le i \le \frac{m}{4}\,,\ 1 \le j \le n \\ 4\,(i-1)\,(n+1) + (2j-2) - (2q-1) + 2, \\ \quad \text{for} \quad \frac{m}{4}+1 \le i \le \frac{m}{2}\ ,\ 1 \le j \le n \end{cases}
$$

$$|f(u_{2i}) - f(v_{2i,j})| = \begin{cases} (2q-3-2n) - 4(n+1)(i-1) - (2j-2), \\ \text{for} \quad 1 \le i \le \frac{m}{4},\ 1 \le j \le n \\ (2q-5-2n) - 4(n+1)(i-1) - (2j-2), \\ \text{for} \quad \frac{m}{4}+1 \le i \le \frac{m}{2},\ 1 \le j \le n \end{cases} \tag{2.5}$$

$$\left|f(v_{2i-1,j}) - f\left(w_l^{(2k-1)}\right)\right| = (2i-2)(n+1) + (2j-5) - (k-1)(10n-6) - 6(l-1),$$

$$\text{for} \quad 1 \le i \le \frac{m}{2}, 1 \le j \le n, 1 \le k \le \frac{m}{2}, 1 \le l \le n-1$$

$$\left|f(v_{2i,j}) - f\left(w_l^{(2k)}\right)\right| = n(16-10k-2i) + 6(k+l-2) - 1$$

$$\text{for} \quad 1 \le i \le \frac{m}{2}, 1 \le j \le n, 1 \le k \le \frac{m}{2}, 1 \le l \le n-1 \tag{2.6}$$

From (2.4) to (2.6) we see that the edge labels for the graph G are the distinct odd numbers from the set $\{1, 3, 5, \ldots, (2q-1)\}$. Hence the graph obtained by attaching each vertex of the cycle C_m with the subdivided shell graph is odd graceful, when $m \equiv 0 \pmod 4$.

□

We illustrate the proof of the above theorem as follows:

Illustration 2.2. When $m = 4, n = 4, p = 52, q = 68$ we have the following graph:

3 Conclusion

In this paper we showed that the graph obtained by attaching each vertex of the cycle C_m with the subdivided shell graph is odd graceful, when $m \equiv 0 \pmod 4$.

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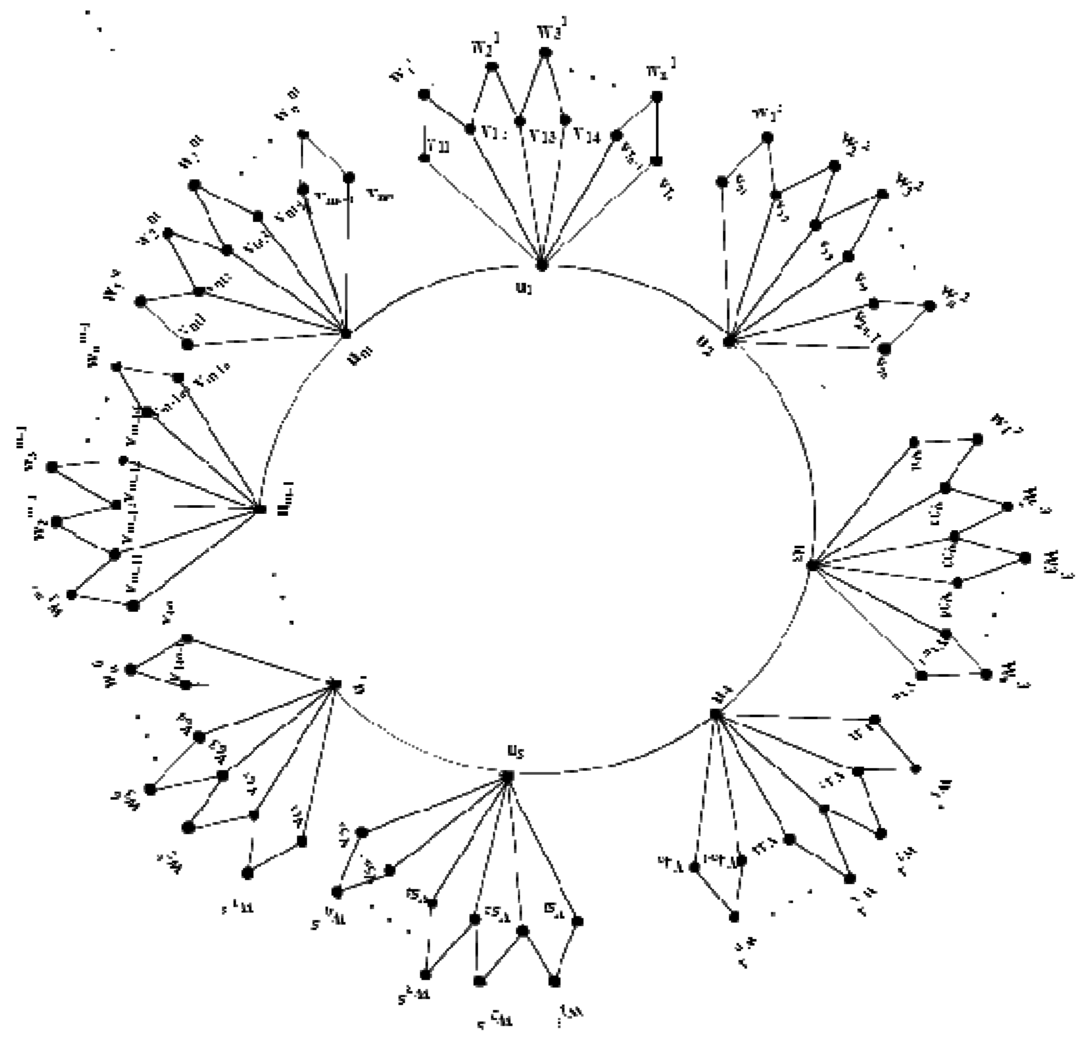

Fig. 3: m- isomorphic copies of Subdivided shell graph attached at each vertex of cycle C_m .

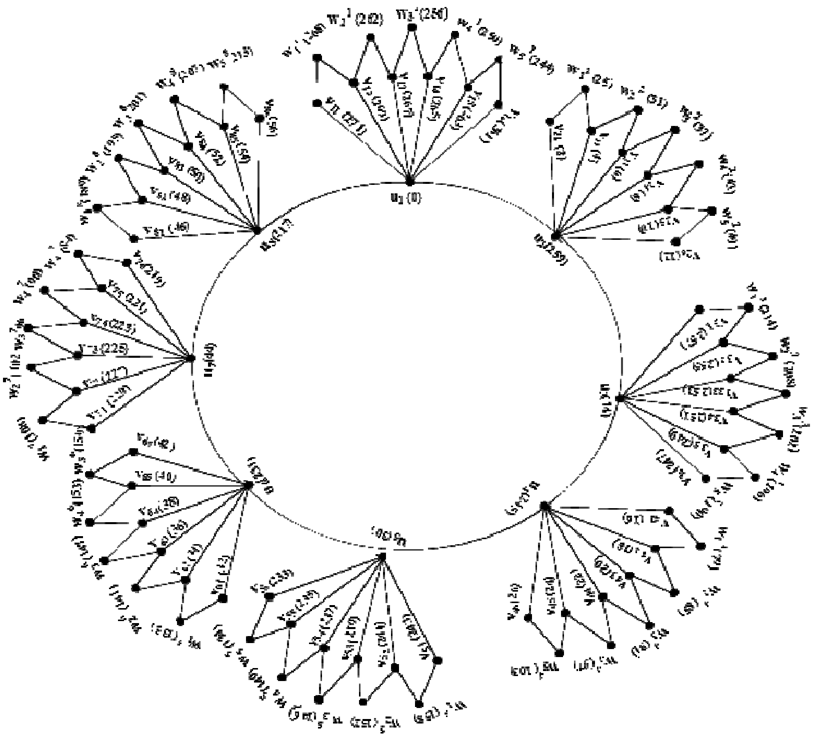

Fig. 4: Isomorphic copies of Dutch Windmill Graphs attached at each vertex of cycle C_4.