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ON THE SRIVASTAVA'S FUNCTION H_B OF MATRIX ARGUMENTS

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Abstract
We establish some integral representations for the Srivastava's triple hypergeometric function H_B of matrix arguments which generalize some of the recent results of Choi, Hasanov and Turaev [12] for this function. We prove our results by employing the Mathai's matrix transform technique for real symmetric positive definite matrices as arguments. Towards the end of the paper we also give the corresponding results when the argument matrices are complex Hermitian positive definite.

Keywords: hypergeometric functions, Srivastava's triple hypergeometric functions, Exton's triple hypergeometric function, matrix argument, matrix transform, real positive definite, Hermitian positive definite.

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1. INTRODUCTION

Among the multiple hypergeometric functions, the triple hypergeometric functions H_A, H_B and H_C introduced by Srivastava [9,10] deserve an important place. Much work has been done on these functions ever since then. Without going into any more details of this (because the interested reader can easily find the vast literature existing in this field), we only mention here that the first author had first of all defined these three functions for the case of matrix arguments with real symmetric positive definite matrices as the arguments in 2001 [1] (see also [2]). A number of integral representations for these functions are available in the literature, some of the very earlier ones can be found in [9,10], besides [11, chapters fourth, fifth and sixth] and many others. Choi, Hasanov and Turaev [12] have also given a number of integral representations for the Srivastava function H_B, which will be the focus of our study here. Our object here is to give the generalizations of some of these integral representations for the function H_B when the arguments of this function are real symmetric positive definite matrices. We prove our results by appealing to the Mathai's matrix transform technique[13]. We also give the corresponding parallel results when the argument matrices are complex Hermitian positive definite in the concluding section of the paper.

The organization of the paper is as follows: the first section contains the necessary background material; in the second section we reproduce the first author's definition [1] of the Srivastava function H_B for the real symmetric positive definite matrix arguments which we utilize to prove our results in the third section of the paper. Finally, in the fourth section of the paper we state the corresponding parallel results for the Srivastava function H_B of complex matrix arguments (see also [8]) which can be proved in a similar manner as the results in the third section by utilizing the corresponding tools available in the literature pertinent references to which are also pointed out there.

We mention that all the matrices appearing in the first, second and the third sections of this paper are real symmetric positive matrices of order $(p\times p)$ while those appearing in the fourth section of this paper are Hermitian positive definite of order $(p\times p)$. $A>0$ denotes that the matrix A is positive definite, $A^{1/2}$ represents the symmetric square root of the matrix A and A' denotes the transpose of the matrix A. $\mathrm{Re}(.)$ denotes the real part of $(.)$, while $|A|$ denotes the determinant of the matrix A. $0<X<I$ means that $X>0$ and $I-X>0$, i.e. all the eigenvalues of X lie between 0 and 1 (see, Mathai [4, p.3]). The matrix transform (M- transform) of a function $f(X)$ of a $(p\times p)$ real symmetric positive definite matrix X was defined by Mathai [13] as follows:

$$M_f(\rho)=\int_{X>0}|X|^{\rho-(p+1)/2}f(X)dX \tag{1.1}$$

for $X>0$ and $\mathrm{Re}(\rho)>(p-1)/2$, whenever $M_f(s)$ exists.

The following results and definition will be used by us at various places in this paper.

Theorem 1.1: Mathai [3, (2.24), p.23] - Let X and Y be $(p\times p)$ symmetric matrices of functionally independent real variables and A a $(p\times p)$ non singular matrix of constants. Then,

$$Y=AXA' \Rightarrow dY=|A|^{p+1}dX \tag{1.2}$$

and

$$Y=aX \Rightarrow dY=a^{p(p+1)/2}dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2: Gamma integral (Mathai [4, (2.1.3), p.33 and (2.1.2), p. 32]) -

$$\int_{X>0}|X|^{\alpha-(p+1)/2}e^{-tr(BX)}dX=|B|^{-\alpha}\Gamma_p(\alpha) \tag{1.4}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$, where,

$$\Gamma_p(\alpha)=\pi^{p(p-1)/4}\Gamma(\alpha)\Gamma(\alpha-\tfrac{1}{2})\cdots\Gamma(\alpha-\tfrac{p-1}{2}) \tag{1.5}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$ and $tr(X)$ denotes the trace of the matrix X.

Theorem 1.3: Type-1 Beta Integral (Mathai [4, (2.2.2), p.34])-

$$B_p(\alpha,\beta)=\int_{0<X<I}|X|^{\alpha-(p+1)/2}|I-X|^{\beta-(p+1)/2}dX=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\mathrm{Re}(\alpha)>(p-1)/2, \mathrm{Re}(\beta)>(p-1)/2$.

Definition 1.4: We reproduce below the definition of the Exton's X_4 function of matrix argument from an earlier paper of the first author (see (2.1), p.209, [8]) which we utilize in proving a result in the third section of this paper.

The Exton's function $X_4 = X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X, -Y, -Z\right]$ of matrix arguments is defined as that class of functions which has the following matrix transform (M-transform):

$$M\left(X_4\right) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X, -Y, -Z\right] dXdYdZ$$

$$= \frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \qquad (1.7)$$

for $\mathrm{Re}(a_1-2\rho_1-\rho_2-\rho_3, a_2-\rho_2-\rho_3, c_i-\rho_i, \rho_i) > (p-1)/2, i = 1,2,3.$

Theorem 1.5: (Mathai [4, (6.13), p. 84])- For $p = 2$,

$$4^{-p\rho}\frac{\Gamma_p\left(\frac{a+1}{2}-\rho\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}-\rho\right)}{\Gamma_p\left(\frac{a+1}{2}\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}\right)} = \frac{\Gamma_p\left(a-2\rho\right)}{\Gamma_p\left(a\right)} \qquad (1.8)$$

Theorem 1.6: (Mathai [4, (2.3.6), p.38]) - For a $\left(p\times p\right)$ real symmetric positive definite matrix X such that $0 < X < I$,

$${}_2F_1\left(\alpha,\beta;\gamma;-X\right) = \frac{\Gamma_p\left(\gamma\right)}{\Gamma_p\left(\alpha\right)\Gamma_p\left(\gamma-\alpha\right)}\int_0^I |S|^{\alpha-(p+1)/2} |I-S|^{\gamma-\alpha-(p+1)/2} |I+XS|^{-\beta} dS \qquad (1.9)$$

for $\mathrm{Re}\left(\alpha,\gamma-\alpha\right) > \left(p-1\right)/2$. For the corresponding result of scalar arguments for this well known integral see Slater [5].

2. THE SRIVASTAVA FUNCTION H_B OF MATRIX ARGUMENTS

Now we reproduce the definition of the Srivastava function H_B of matrix arguments due to the first author [1] (see also [2]).

Definition 2.1: The Srivastava function H_B of matrix arguments

$$H_B = H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$M\left(H_B\right) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right) dXdYdZ$$

$$= \frac{\Gamma_p\left(a_1-\rho_1-\rho_3\right)\Gamma_p\left(a_2-\rho_1-\rho_2\right)\Gamma_p\left(a_3-\rho_2-\rho_3\right)}{\Gamma_p\left(a_1\right)\Gamma_p\left(a_2\right)\Gamma_p\left(a_3\right)} \times$$

$$\frac{\Gamma_p\left(c_1\right)\Gamma_p\left(c_2\right)\Gamma_p\left(c_3\right)\Gamma_p\left(\rho_1\right)\Gamma_p\left(\rho_2\right)\Gamma_p\left(\rho_3\right)}{\Gamma_p\left(c_1-\rho_1\right)\Gamma_p\left(c_2-\rho_2\right)\Gamma_p\left(c_3-\rho_3\right)} \qquad (2.1)$$

for $\mathrm{Re}\left(a_1-\rho_1-\rho_3, a_2-\rho_1-\rho_2, a_3-\rho_2-\rho_3, c_i-\rho_i, \rho_i\right) > \left(p-1\right)/2$, where, $i = 1,2,3.$

3. INTEGRAL REPRESENTATIONS FOR THE SRIVASTAVA'S FUNCTION H_B OF MATRIX ARGUMENTS

We now establish some integral representations for the Srivastava's function H_B of matrix when the argument matrices are real symmetric positive definite. We give the proofs of only the representative results, other results can be proved on similar lines.

Theorem 3.1:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p(s)}{\Gamma_p(a_1)\Gamma_p(s-a_1)}\int_0^I |T|^{a_1-(p+1)/2}|I-T|^{s-a_1-(p+1)/2}\times$$
$$H_B\left(s,a_2,a_3;c_1,c_2,c_3;-T^{1/2}XT^{1/2},-Y,-T^{1/2}ZT^{1/2}\right)dT \qquad (3.1)$$

for $\operatorname{Re}(s-a_1,a_1)>(p-1)/2$.

Proof: Taking the matrix transform (M-transform) of the right side of (3.1) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we get

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$
$$H_B\left(s,a_2,a_3;c_1,c_2,c_3;-T^{1/2}XT^{1/2},-Y,-T^{1/2}ZT^{1/2}\right)dXdYdZ \qquad (3.2)$$

Applying the transformations $X_1=T^{1/2}XT^{1/2},Y_1=Y,Z_1=T^{1/2}ZT^{1/2}$ to the above expression with $dX_1=|T|^{(p+1)/2}dX,dY_1=dY,dZ_1=|T|^{(p+1)/2}dZ$ (from (1.2)) and $|X_1|=|T||X|,|Y_1|=|Y|,|Z_1|=|T||Z|$ we get

$$|T|^{-\rho_1-\rho_3}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}\times$$
$$H_B\left(s,a_2,a_3;c_1,c_2,c_3;-X_1,-Y_1,-Z_1\right)dX_1dY_1dZ_1$$

which, on writing the M-transform of the H_B function with the help of (2.1), yields

$$|T|^{-\rho_1-\rho_3}\frac{\Gamma_p(s-\rho_1-\rho_3)\Gamma_p(a_2-\rho_1-\rho_2)\Gamma_p(a_3-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(s)\Gamma_p(a_2)\Gamma_p(a_3)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3) \qquad (3.3)$$

Substituting this expression on the right side of (3.1) gives

$$\frac{\Gamma_p(s-\rho_1-\rho_3)\Gamma_p(a_2-\rho_1-\rho_2)\Gamma_p(a_3-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(s-a_1)\Gamma_p(a_2)\Gamma_p(a_3)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\int_0^I|T|^{a_1-\rho_1-\rho_3-(p+1)/2}|I-T|^{s-a_1-(p+1)/2}dT$$

in which the variable T can be integrated out with the help of (1.6) to give $M\left(H_B\right)$ as given by (2.1).

■

The above theorem generalizes the integral given in (2.1) p.139 of Choi et al. [12] for the case of H_B with real symmetric positive definite matrix arguments. Similarly the following theorems give the generalizations of the results in (2.2), (2.3) and (2.4) on p.139 and (2.5) and (2.6) on p. 140 of Choi et al. [12] respectively.

Theorem 3.2:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p(s)}{\Gamma_p(a_2)\Gamma_p(s-a_2)}\int_0^I|T|^{a_2-(p+1)/2}|I-T|^{s-a_2-(p+1)/2}\times$$
$$H_B\left(a_1,s,a_3;c_1,c_2,c_3;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2},-Z\right)dT \qquad (3.4)$$

for $\operatorname{Re}(s-a_2,a_2)>(p-1)/2$.

Theorem 3.3:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z)=\frac{\Gamma_p(s)}{\Gamma_p(a_3)\Gamma_p(s-a_3)}\int_0^I|T|^{a_3-(p+1)/2}|I-T|^{s-a_3-(p+1)/2}\times$$
$$H_B\left(a_1,a_2,s;c_1,c_2,c_3;-X,-T^{1/2}YT^{1/2},-T^{1/2}ZT^{1/2}\right)dT \qquad (3.5)$$

for $\operatorname{Re}(s-a_3,a_3)>(p-1)/2$.

Theorem 3.4:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z)=\frac{\Gamma_p(c_1)}{\Gamma_p(s)\Gamma_p(c_1-s)}\int_0^I|T|^{s-(p+1)/2}|I-T|^{c_1-s-(p+1)/2}\times$$
$$H_B\left(a_1,a_2,a_3;s,c_2,c_3;-T^{1/2}XT^{1/2},-Y,-Z\right)dT \qquad (3.6)$$

for $\operatorname{Re}(c_1-s,s)>(p-1)/2$.

Theorem 3.5:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z)=\frac{\Gamma_p(c_2)}{\Gamma_p(s)\Gamma_p(c_2-s)}\int_0^I|T|^{s-(p+1)/2}|I-T|^{c_2-s-(p+1)/2}\times$$
$$H_B\left(a_1,a_2,a_3;c_1,s,c_3;-X,-T^{1/2}YT^{1/2},-Z\right)dT \qquad (3.7)$$

for $\operatorname{Re}(c_2-s,s)>(p-1)/2$.

Theorem 3.6:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z)=\frac{\Gamma_p(c_3)}{\Gamma_p(s)\Gamma_p(c_3-s)}\int_0^I|T|^{s-(p+1)/2}|I-T|^{c_3-s-(p+1)/2}\times$$
$$H_B\left(a_1,a_2,a_3;c_1,c_2,s;-X,-Y,-T^{1/2}ZT^{1/2}\right)dT \qquad (3.8)$$

for $\operatorname{Re}(c_3-s,s)>(p-1)/2$.

Now we prove the following generalization of (2.7) p.140 of Choi et al. [12]:

Theorem 3.7:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z)=\frac{\Gamma_p(a_1+a_2)}{\Gamma_p(a_1)\Gamma_p(a_2)}\int_0^I|T|^{a_1-(p+1)/2}|I-T|^{a_2-(p+1)/2}\times$$
$$X_4\left[\begin{matrix}a_1+a_2,a_1+a_2;a_1+a_2,a_3;a_1+a_2,a_3\\ c_1;\qquad c_2;\qquad c_3\end{matrix}\right.$$
$$\left.\Big|-(I-T)^{1/2}T^{1/2}XT^{1/2}(I-T)^{1/2},-(I-T)^{1/2}Y(I-T)^{1/2},-T^{1/2}ZT^{1/2}\right]dT \qquad (3.9)$$

for $\operatorname{Re}(a_1,a_2)>(p-1)/2$.

Proof: On taking the M-transform of the right side of (3.9) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we obtain

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times X_4\left[\begin{matrix}a_1+a_2,a_1+a_2;a_1+a_2,a_3;a_1+a_2,a_3\\ c_1;\qquad c_2;\qquad c_3\end{matrix}\right.$$
$$\left.\Big|-(I-T)^{1/2}T^{1/2}XT^{1/2}(I-T)^{1/2},-(I-T)^{1/2}Y(I-T)^{1/2},-T^{1/2}ZT^{1/2}\right]dXdYdZ \qquad (3.10)$$

Using the transformations

$$X_1=(I-T)^{1/2}T^{1/2}XT^{1/2}(I-T)^{1/2},Y_1=(I-T)^{1/2}Y(I-T)^{1/2},Z_1=T^{1/2}ZT^{1/2}$$

with

$$dX_1 = |I-T|^{(p+1)2} |T|^{(p+1)2} dX, dY_1 = |I-T|^{(p+1)2} dY, dZ_1 = |T|^{(p+1)2} dZ$$

and

$$|X_1| = |I-T||T||X|, |Y_1| = |I-T||Y|, |Z_1| = |T||Z|$$

in (3.10) and then writing the M-transform of the Exton's triple hypergeometric function X_4 by using (1.7) yields

$$|T|^{-\rho_1-\rho_3} |I-T|^{-\rho_1-\rho_2} \frac{\Gamma_p(a_1+a_2-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_3-\rho_2-\rho_3)}{\Gamma_p(a_1+a_2)\Gamma_p(a_3)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \times$$
$$\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3) \tag{3.11}$$

Now substituting this expression on the right side of (3.10) and integrating out T in the consequent expression by utilizing (1.6) produces $M(H_B)$ in conformity with (2.1). ■

We proceed now to give below a result that holds good only for the case of (2×2) real symmetric positive definite matrices.

Theorem 3.8: Let $0 < (M-\Lambda)(\Lambda-N)^{-1} < I$ and

$$G = \frac{\Gamma_p(a_1+a_2+a_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(a_3)} |M-N|^{a_1+a_2} |\Lambda-N|^{-a_1-a_2} \int_0^I \int_0^I |T|^{a_1-(p+1)/2} |S|^{a_1+a_2-(p+1)/2} |I-T|^{a_2-(p+1)/2} \times$$
$$|I-S|^{a_3-(p+1)/2} \left|I+(M-\Lambda)(\Lambda-N)^{-1} S\right|^{-a_1-a_2-a_3} F_C^{(3)}\Bigg[\frac{a_1+a_2+a_3+1}{2}, \frac{a_1+a_2+a_3}{2}+\frac{1}{4}; c_1, c_2, c_3;$$
$$-4\left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1} (M-N)(\Lambda-N)^{-1} (I-T)^{1/2} T^{1/2} SXST^{1/2} (I-T)^{1/2} (\Lambda-N)^{-1} \times$$
$$(M-N)\left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1}, -4\left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1} (\Lambda-N)^{-1/2} (M-N)^{1/2} \times$$
$$(I-T)^{1/2} (I-S)^{1/2} S^{1/2} YS^{1/2} (I-S)^{1/2} (I-T)^{1/2} (M-N)^{1/2} (\Lambda-N)^{-1/2} \left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1},$$
$$-4\left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1} (\Lambda-N)^{-1/2} (M-N)^{1/2} (I-S)^{1/2} T^{1/2} S^{1/2} ZS^{1/2} T^{1/2} (I-S)^{1/2} \times$$
$$(M-N)^{1/2} (\Lambda-N)^{-1/2} \left\{I+(M-\Lambda)(\Lambda-N)^{-1} S\right\}^{-1}\Bigg] dTdS \tag{3.12}$$

for $\mathrm{Re}(a_1, a_2, a_3) > (p-1)/2$. Then, for $p=2$, the M-transform of the double integral defined by G above in (3.12) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively is given by

$$M(G) = \left|I+(M-\Lambda)(\Lambda-N)^{-1}\right|^{2\rho_1+\rho_2+\rho_3} M(H_B) \tag{3.13}$$

where $M-\Lambda > 0, M-N > 0$, $\Lambda - N > 0$ and $F_C^{(3)}$ represents the Lauricella function F_C of three variables (see Exton [11], or Srivastava and Karlsson [14], for instance) .

Proof: We first recall here that the M-transform of the Lauricella function F_C of n variables with n real symmetric positive definite matrices as arguments is defined by Mathai [4, (6.3), p.76] as follows:

For

$$F_C = F_C(a, b; c_1, \ldots, c_n; -X_1, \ldots, -X_n)$$

$$M\left(F_C\right)=\frac{\left\{\prod_{j=1}^{n}\left\{\Gamma_p\left(c_j\right)\Gamma_p\left(\rho_j\right)\right\}\right\}\Gamma_p\left(a-\rho_1-\ldots-\rho_n\right)\Gamma_p\left(b-\rho_1-\ldots-\rho_n\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)\left\{\prod_{j=1}^{n}\Gamma_p\left(c_j-\rho_j\right)\right\}} \tag{3.14}$$

where $\mathrm{Re}\left(\rho_j,c_j-\rho_j,a-\rho_1-\ldots-\rho_n,b-\rho_1-\ldots-\rho_n\right)>\left(p-1\right)/2,\ j=1,\ldots,n$. From where it can be readily inferred that for the function involved in the integrand of (3.12) we can easily write form (3.14) that

$$M\left(F_C^{(3)}\right)=\frac{\left\{\prod_{j=1}^{3}\left\{\Gamma_p\left(c_j\right)\Gamma_p\left(\rho_j\right)\right\}\right\}\Gamma_p\left(\frac{a_1+a_2+a_3+1}{2}-\rho_1-\rho_2-\rho_3\right)}{\Gamma_p\left(\frac{a_1+a_2+a_3+1}{2}\right)\Gamma_p\left(\frac{a_1+a_2+a_3}{2}+\frac{1}{4}\right)\left\{\prod_{j=1}^{3}\Gamma_p\left(c_j-\rho_j\right)\right\}}\times$$
$$\Gamma_p\left(\frac{a_1+a_2+a_3}{2}+\frac{1}{4}-\rho_1-\rho_2-\rho_3\right) \tag{3.15}$$

Now taking the M-transform of the right side of (3.12) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we have

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\left|X\right|^{\rho_1-(p+1)/2}\left|Y\right|^{\rho_2-(p+1)/2}\left|Z\right|^{\rho_3-(p+1)/2}F_C^{(3)}\left[\frac{a_1+a_2+a_3+1}{2},\frac{a_1+a_2+a_3}{2}+\frac{1}{4};c_1,c_2,c_3;\right.$$
$$-4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(M-N\right)\left(\Lambda-N\right)^{-1}\left(I-T\right)^{1/2}T^{1/2}SXST^{1/2}\left(I-T\right)^{1/2}\left(\Lambda-N\right)^{-1}\times$$
$$\left(M-N\right)\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1},-4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(\Lambda-N\right)^{-1/2}\left(M-N\right)^{1/2}\times$$
$$\left(I-T\right)^{1/2}\left(I-S\right)^{1/2}S^{1/2}YS^{1/2}\left(I-S\right)^{1/2}\left(I-T\right)^{1/2}\left(M-N\right)^{1/2}\left(\Lambda-N\right)^{-1/2}\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1},$$
$$-4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(\Lambda-N\right)^{-1/2}\left(M-N\right)^{1/2}\left(I-S\right)^{1/2}T^{1/2}S^{1/2}ZS^{1/2}T^{1/2}\left(I-S\right)^{1/2}\times$$
$$\left.\left(M-N\right)^{1/2}\left(\Lambda-N\right)^{-1/2}\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\right]dXdYdZ \tag{3.16}$$

Applying the transformations

$$X_1=4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(M-N\right)\left(\Lambda-N\right)^{-1}\left(I-T\right)^{1/2}T^{1/2}SXST^{1/2}\left(I-T\right)^{1/2}\left(\Lambda-N\right)^{-1}\times$$
$$\left(M-N\right)\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}$$

$$Y_1=4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(\Lambda-N\right)^{-1/2}\left(M-N\right)^{1/2}\left(I-T\right)^{1/2}\left(I-S\right)^{1/2}S^{1/2}YS^{1/2}\left(I-S\right)^{1/2}\times$$
$$\left(I-T\right)^{1/2}\left(M-N\right)^{1/2}\left(\Lambda-N\right)^{-1/2}\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}$$

$$Z_1=4\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}\left(\Lambda-N\right)^{-1/2}\left(M-N\right)^{1/2}\left(I-S\right)^{1/2}T^{1/2}S^{1/2}ZS^{1/2}T^{1/2}\left(I-S\right)^{1/2}\times$$
$$\left(M-N\right)^{1/2}\left(\Lambda-N\right)^{-1/2}\left\{I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right\}^{-1}$$

with

$$dX_1=4^{p(p+1)/2}\left|I+\left(M-\Lambda\right)\left(\Lambda-N\right)^{-1}S\right|^{-(p+1)}\left|M-N\right|^{p+1}\left|\Lambda-N\right|^{-(p+1)}\left|I-T\right|^{(p+1)/2}\left|T\right|^{(p+1)/2}\left|S\right|^{p+1}dX$$

$$dY_1 = 4^{p(p+1)/2} \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-(p+1)} |M-N|^{(p+1)/2} |\Lambda - N|^{-(p+1)/2} |I-T|^{(p+1)/2} |I-S|^{(p+1)/2} |S|^{(p+1)/2} dY$$

$$dZ_1 = 4^{p(p+1)/2} \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-(p+1)} |M-N|^{(p+1)/2} |\Lambda - N|^{-(p+1)/2} |I-S|^{(p+1)/2} |T|^{(p+1)/2} |S|^{(p+1)/2} dZ$$

and

$$|X_1| = 4^p \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-2} |M-N|^2 |\Lambda - N|^{-2} |I-T||T||S|^2 |X|$$

$$|Y_1| = 4^p \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-2} |M-N||\Lambda - N|^{-1} |I-T||I-S||S||Y|$$

$$|Z_1| = 4^p \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-2} |M-N||\Lambda - N|^{-1} |I-S||T||S||Z|$$

in (3.16), and then writing the M-transform of the $F_C^{(3)}$ function with the help of (3.15), it gives

$$4^{-p(\rho_1+\rho_2+\rho_3)} \left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{2(\rho_1+\rho_2+\rho_3)} |M-N|^{-2\rho_1-\rho_2-\rho_3} |\Lambda - N|^{2\rho_1+\rho_2+\rho_3} |I-T|^{-\rho_1-\rho_2} \times$$

$$|T|^{-\rho_1-\rho_3} |S|^{-2\rho_1-\rho_2-\rho_3} |I-S|^{-\rho_2-\rho_3} \frac{\left\{ \prod_{j=1}^{3} \left\{ \Gamma_p(c_j) \Gamma_p(\rho_j) \right\} \right\} \Gamma_p\left(\frac{a_1+a_2+a_3+1}{2} - \rho_1 - \rho_2 - \rho_3 \right)}{\Gamma_p\left(\frac{a_1+a_2+a_3+1}{2} \right) \Gamma_p\left(\frac{a_1+a_2+a_3}{2} + \frac{1}{4} \right) \left\{ \prod_{j=1}^{3} \Gamma_p(c_j - \rho_j) \right\}} \times$$

$$\Gamma_p\left(\frac{a_1+a_2+a_3}{2} + \frac{1}{4} - \rho_1 - \rho_2 - \rho_3 \right). \tag{3.17}$$

The above expression on simplification with the help of (1.8) yields

$$\left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{2(\rho_1+\rho_2+\rho_3)} |M-N|^{-2\rho_1-\rho_2-\rho_3} |\Lambda - N|^{2\rho_1+\rho_2+\rho_3} |I-T|^{-\rho_1-\rho_2} |T|^{-\rho_1-\rho_3} |S|^{-2\rho_1-\rho_2-\rho_3} \times$$

$$|I-S|^{-\rho_2-\rho_3} \frac{\left\{ \prod_{j=1}^{3} \left\{ \Gamma_p(c_j) \Gamma_p(\rho_j) \right\} \right\} \Gamma_p(a_1+a_2+a_3-2\rho_1-2\rho_2-2\rho_3)}{\Gamma_p(a_1+a_2+a_3) \left\{ \prod_{j=1}^{3} \Gamma_p(c_j - \rho_j) \right\}}. \tag{3.18}$$

Substituting this expression on the right hand side of (3.12) and integrating out T in the resulting expression by using (1.6) leads us to

$$\frac{\left\{ \prod_{j=1}^{3} \left\{ \Gamma_p(c_j) \Gamma_p(\rho_j) \right\} \right\} \Gamma_p(a_1+a_2+a_3-2\rho_1-2\rho_2-2\rho_3) \Gamma_p(a_1-\rho_1-\rho_3) \Gamma_p(a_2-\rho_1-\rho_2)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(a_3) \left\{ \prod_{j=1}^{3} \Gamma_p(c_j - \rho_j) \right\} \Gamma_p(a_1+a_2-2\rho_1-\rho_2-\rho_3)} \times$$

$$|M-N|^{a_1+a_2} |\Lambda - N|^{-a_1-a_2} \int_0^I |S|^{a_1+a_2-2\rho_1-\rho_2-\rho_3-(p+1)/2} |I-S|^{a_3-\rho_2-\rho_3-(p+1)/2} \times$$

$$\left| I + (M-\Lambda)(\Lambda - N)^{-1} S \right|^{-a_1-a_2-a_3+2(\rho_1+\rho_2+\rho_3)} dS \tag{3.19}$$

We now return to (1.9) and observe that if in this equation we set $\alpha = a_1 + a_2 - 2\rho_1 - \rho_2 - \rho_3$, $\gamma - \alpha = a_3 - \rho_2 - \rho_3$, $\beta = a_1 + a_2 + a_3 - 2\rho_1 - 2\rho_2 - 2\rho_3$ which gives that $\gamma = \beta$ and further observing that ${}_2F_1(\alpha, \gamma; \gamma; -X) = {}_1F_0(\alpha;\ ;-X) = |I + X|^{-\alpha}$, where, $X = (M-\Lambda)(\Lambda - N)^{-1}$ we can write the S - integral of (3.19) as

$$\Gamma_p\left(a_3-\rho_2-\rho_3\right)\left[\frac{\Gamma_p\left(a_1+a_2+a_3-2\rho_1-2\rho_2-2\rho_3\right)}{\Gamma_p\left(a_1+a_2-2\rho_1-\rho_2-\rho_3\right)\Gamma_p\left(a_3-\rho_2-\rho_3\right)}\int_0^I |S|^{a_1+a_2-2\rho_1-\rho_2-\rho_3-(p+1)/2}\times\right.$$

$$\left.|I-S|^{a_3-\rho_2-\rho_3-(p+1)/2}\left|I+(M-\Lambda)(\Lambda-N)^{-1}S\right|^{-a_1-a_2-a_3+2(\rho_1+\rho_2+\rho_3)}dS\right]$$

$$=\Gamma_p\left(a_3-\rho_2-\rho_3\right){}_1F_0\left[a_1+a_2-2\rho_1-\rho_2-\rho_3;\ ;-(M-\Lambda)(\Lambda-N)^{-1}\right]$$

$$=\Gamma_p\left(a_3-\rho_2-\rho_3\right)\left|I+(M-\Lambda)(\Lambda-N)^{-1}\right|^{-a_1-a_2+2\rho_1+\rho_2+\rho_3} \tag{3.20}$$

We also observe that

$$|M-N|^{a_1+a_2}=\left|(M-\Lambda)+(\Lambda-N)\right|^{a_1+a_2}=\left|(M-\Lambda)(\Lambda-N)^{-1}+I\right|^{a_1+a_2}|\Lambda-N|^{a_1+a_2}$$

whence follows that

$$|M-N|^{a_1+a_2}|\Lambda-N|^{-a_1-a_2}=\left|I+(M-\Lambda)(\Lambda-N)^{-1}\right|^{a_1+a_2} \tag{3.21}$$

Finally it can be seen that (3.19) simplifies to the right hand side of (3.13) with the aid of (3.20), (3.21) and (2.1). ■

4. CORRESPONDING RESULTS FOR THE FUNCTIONS OF COMPLEX MATRIX ARGUMENT

We now list the corresponding results for the above cases when the argument matrices are Hermitian positive definite (see also section 4, pp. 213-215 of [8]) in this section. We mention that, all the matrices are Hermitian positive definite matrices of order $(p\times p)$ in this concluding section of the paper. We prefer to use the same notation for matrices (complex matrices) in this section as we have used in the previous sections of this paper, but, for the sake of the interested reader, we do point out here that Mathai [6] has denoted the matrices having complex entries by placing a tilde ($\sim$) sign over the notation of the matrix concerned. The results concerning the Jacobians of matrix transformations in the case of matrices when their elements are complex quantities are available in the Chapter 3 of Mathai [6]. The complex analogues of all the results mentioned in (1.1) to (1.9) (except (1.8)) above can be found in Chapters 3 and 6 of Mathai [6]. We also remark here that for the result developed in the Theorem 3.8, which is valid only for (2×2) real symmetric positive definite matrices and because the complex analogue of the result in (1.8) (in fact, the complex case of the analogue of Lemma 5.4 of Chapter 5 p. 340 of Mathai [6]) has a different structure (see also Mathai [6], Chapter 6, p.399) therefore the corresponding complex analogue of the result given in the Theorem 3.8 has a different structure therefore it is not given here. The key point to be noted here is that the complex analogues of the results in the Definition 2.1 and the Theorems 3.1 through 3.7 can be most easily written down with the help of the results in (2.1), (3.1), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) respectively, by replacing the expression $(p+1)/2$ appearing in the power of the determinant of the matrix by p and in the condition of convergence of the integral, the expression $\text{Re}(.)>(p-1)/2$ has to be replaced by $\text{Re}(.)>(p-1)$ (see Mathai [6], pp. 364-365 and see also Mathai and Provost [7]). With the help of these observations, which are already very well established in the literature beforehand (e.g., Mathai [6] and Mathai and Provost [7]), we state below (without proofs) the complex analogues of the definition and results in (2.1), (3.1), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) in the form of the statements of the Definition 4.1, Theorems 4.2 through 4.8 respectively. Appealing to the complex analogues of the corresponding results in (1.1) to (1.9) (except (1.8)) which are available from the Chapters 3 and 6 of Mathai [6] and following the same parallel steps as we have done for deducing these results in Section 3 of this paper (see Chapters 5 and 6 of Mathai [6]) we can prove these corresponding complex analogues. It is also to be borne in mind here that in this section of the paper $|A|$ now represents the absolute value of the determinant of the matrix A of complex elements.

Definition 4.1: The Srivastava function H_B of complex matrix arguments

$$H_B=H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$M\left(H_B\right)=\int_{X>0}\int_{Y>0}\int_{Z>0}\left|X\right|^{\rho_1-p}\left|Y\right|^{\rho_2-p}\left|Z\right|^{\rho_3-p}H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)dXdYdZ$$
$$=\frac{\Gamma_p\left(a_1-\rho_1-\rho_3\right)\Gamma_p\left(a_2-\rho_1-\rho_2\right)\Gamma_p\left(a_3-\rho_2-\rho_3\right)}{\Gamma_p\left(a_1\right)\Gamma_p\left(a_2\right)\Gamma_p\left(a_3\right)}\times$$
$$\frac{\Gamma_p\left(c_1\right)\Gamma_p\left(c_2\right)\Gamma_p\left(c_3\right)\Gamma_p\left(\rho_1\right)\Gamma_p\left(\rho_2\right)\Gamma_p\left(\rho_3\right)}{\Gamma_p\left(c_1-\rho_1\right)\Gamma_p\left(c_2-\rho_2\right)\Gamma_p\left(c_3-\rho_3\right)} \tag{4.1}$$

for $\mathrm{Re}\left(a_1-\rho_1-\rho_3,a_2-\rho_1-\rho_2,a_3-\rho_2-\rho_3,c_i-\rho_i,\rho_i\right)>\left(p-1\right)$, where, $i=1,2,3.$

Theorem 4.2:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(s\right)}{\Gamma_p\left(a_1\right)\Gamma_p\left(s-a_1\right)}\int_0^I\left|T\right|^{a_1-p}\left|I-T\right|^{s-a_1-p}\times$$
$$H_B\left(s,a_2,a_3;c_1,c_2,c_3;-T^{1/2}XT^{1/2},-Y,-T^{1/2}ZT^{1/2}\right)dT \tag{4.2}$$

for $\mathrm{Re}\left(s-a_1,a_1\right)>\left(p-1\right)$.

Theorem 4.3:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(s\right)}{\Gamma_p\left(a_2\right)\Gamma_p\left(s-a_2\right)}\int_0^I\left|T\right|^{a_2-p}\left|I-T\right|^{s-a_2-p}\times$$
$$H_B\left(a_1,s,a_3;c_1,c_2,c_3;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2},-Z\right)dT \tag{4.3}$$

for $\mathrm{Re}\left(s-a_2,a_2\right)>\left(p-1\right)$.

Theorem 4.4:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(s\right)}{\Gamma_p\left(a_3\right)\Gamma_p\left(s-a_3\right)}\int_0^I\left|T\right|^{a_3-p}\left|I-T\right|^{s-a_3-p}\times$$
$$H_B\left(a_1,a_2,s;c_1,c_2,c_3;-X,-T^{1/2}YT^{1/2},-T^{1/2}ZT^{1/2}\right)dT \tag{4.4}$$

for $\mathrm{Re}\left(s-a_3,a_3\right)>\left(p-1\right)$.

Theorem 4.5:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(c_1\right)}{\Gamma_p\left(s\right)\Gamma_p\left(c_1-s\right)}\int_0^I\left|T\right|^{s-p}\left|I-T\right|^{c_1-s-p}\times$$
$$H_B\left(a_1,a_2,a_3;s,c_2,c_3;-T^{1/2}XT^{1/2},-Y,-Z\right)dT \tag{4.5}$$

for $\mathrm{Re}\left(c_1-s,s\right)>\left(p-1\right)$.

Theorem 4.6:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(c_2\right)}{\Gamma_p\left(s\right)\Gamma_p\left(c_2-s\right)}\int_0^I\left|T\right|^{s-p}\left|I-T\right|^{c_2-s-p}\times$$
$$H_B\left(a_1,a_2,a_3;c_1,s,c_3;-X,-T^{1/2}YT^{1/2},-Z\right)dT \tag{4.6}$$

for $\mathrm{Re}\left(c_2-s,s\right)>\left(p-1\right)$.

Theorem 4.7:

$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z\right)=\frac{\Gamma_p\left(c_3\right)}{\Gamma_p\left(s\right)\Gamma_p\left(c_3-s\right)}\int_0^I\left|T\right|^{s-p}\left|I-T\right|^{c_3-s-p}\times$$
$$H_B\left(a_1,a_2,a_3;c_1,c_2,s;-X,-Y,-T^{1/2}ZT^{1/2}\right)dT \tag{4.7}$$

for $\text{Re}(c_3 - s, s) > (p-1)$.

Theorem 4.8:

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;-X,-Y,-Z) = \frac{\Gamma_p(a_1+a_2)}{\Gamma_p(a_1)\Gamma_p(a_2)} \int_0^I |T|^{a_1-p} |I-T|^{a_2-p} \times$$

$$X_4 \left[\begin{array}{c} a_1+a_2, a_1+a_2; a_1+a_2, a_3; a_1+a_2, a_3 \\ c_1; \qquad c_2; \qquad c_3 \end{array} \right.$$

$$\left. -(I-T)^{1/2} T^{1/2} X T^{1/2} (I-T)^{1/2}, -(I-T)^{1/2} Y (I-T)^{1/2}, -T^{1/2} Z T^{1/2} \right] dT \qquad (4.8)$$

for $\text{Re}(a_1, a_2) > (p-1)$.

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