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Using Helping Function with Specific Conditions to Develop Some Results of the Literature

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ABSTRACT

In this study, we establish existence and uniqueness results for fixed points in complete metric spaces using a helping function approach. The proposed framework generalizes and unifies several well-known fixed point theorems, including those of Banach, Kannan, and Chatterjea. Our results extend the current literature by providing broader conditions under which common fixed points can be guaranteed, thus contributing to the ongoing development of fixed point theory in metric spaces.

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1. INTRODUCTION

Fixed point theory (fpt) has emerged as a fundamental area of research with wide-ranging applications in nonlinear analysis, differential equations, optimization, and dynamic systems. Among the most celebrated results in this field are the Banach fpt, Kannan's fpt, and Chatterjea's fpt, each offering unique contractive conditions under which fixed points can be guaranteed in complete metric spaces (CMS).

The solutions for several nonlinear problems that have previously come up in the biological, physical, and social sciences, among other scientific fields, have been resolved using the Banach fpt [1] since its initial appearance in 1922. Following the Banach, the question of whether a map has a fp if it is of non-contractive type remained open to researchers. And the positive answer in case of CMS was given by Kannan in the form of following theorem in 1968.

Kannan introduced a class of mappings satisfying a contractive condition that does not imply continuity, yet ensures the existence of a unique fp in a CMS [2]. Chatterjea, on the other hand, proposed a different condition under which similar fp results could be obtained. Both results have stimulated a vast body of research aimed at generalizing, unifying, and extending these fpts under broader settings and weaker assumptions [3].

In recent years, the study of generalized contractive conditions has attracted significant attention, especially in efforts to encompass a wider class of operators and metric structures. These generalizations often involve altering distance functions, implicit relations, or auxiliary functions that relax the classical constraints.

More comprehensive and potent theorems can be developed by comprehending the evolution of the spaces, which offers crucial insight into the contemporary applied research of fixed points. The ability to tackle real-world issues, such as differential equations, optimization problems, engineering systems, and economic models, is improved. Thus, the continuing construction and analysis of standard spaces are not only of theoretical importance but also serve as a foundation for substantial developments in applied mathematics, see [4, 5, 6, 7].

The purpose of this paper is to contribute to this ongoing development by establishing new extensions of the Kannan-Chatterjea fpt using generalized contractive conditions. Our results not only unify existing theorems under a common framework but also extend their applicability to broader classes of self-mappings. The generalization of Banach, Kannan and Chatterjea fpts have been established by various authors [8, 9, 10, 11].

2. PRELIMINARIES

The main results in this research paper include a comparative analysis with the already existed results in the literature of fpt. To prove our results which are generalizations various fpts such as Banach, Kannan and Chatterjea fpts and their various extensions [12], we have used the following results with some preliminaries.

Definition 2.1 [13] Let $\neq \emptyset$, consider $\varrho: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ which satisfies, for all $x, y, z \in \mathcal{X}$

(ϱ_1): $\varrho(x, x) = 0$

(ϱ_2): $\varrho(x, y) = \varrho(y, x) \Rightarrow x = y$

(ϱ_3): $\varrho(x, y) = \varrho(y, x)$

(ϱ_4): $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$,

Then ϱ is called metric on $\mathcal{X}$ and $(\mathcal{X}, \varrho)$ is called a MS.

Definition 2.2 [14] A sequence $\{x_n\}$ in a MS $(\mathcal{X}, \varrho)$ is said to be convergent to z, if $\lim_{n\to\infty} \varrho(x_n, z) = 0 = \lim_{n\to\infty} \varrho(z, x_n)$. Here z is called limit point of a sequence $\{x_n\}$.

Definition 2.3 [15] A sequence $\{x_n\}$ in a MS $(\mathcal{X}, \varrho)$ is said to be Cauchy sequence if for a given $\epsilon > 0$, there exist a $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $\varrho(x_n, x_m) < \epsilon$.

Definition 2.4 [15] If every Cauchy sequence in $\mathcal{X}$ is convergent to a point in $\mathcal{X}$, then a MS $(\mathcal{X}, \varrho)$ is said to be complete,

Definition 2.5 [16] A point $x \in \mathcal{X}$ is a fixed point of ψ if $\psi x = x$, where $(\mathcal{X}, \varrho)$ be a MS and $\psi: \mathcal{X} \to \mathcal{X}$.

Definition 2.6 [17] A function $\varphi: \mathbb{R} \to \mathbb{R}^+$ is said to be an Upper Semi-Continuous from right if for any sequence $\{x_n\}$ converging to x as $x \geq 0$, then $\lim_{n\to\infty} sup\ \varphi(x_n) \leq \phi x$.

Theorem 2.7 (Banach) [1] Let $(\mathcal{X}, \varrho)$ be a CMS and $\psi: \mathcal{X} \to \mathcal{X}$ be a contraction, i.e., ψ satisfies $\varrho(\psi x, \psi y) \leq \varrho(x, y)$, for all $x, y \in \mathcal{X}$ and a fixed constant $\alpha < 1$. Then there exists a unique fp of ψ in $\mathcal{X}$.

Theorem 2.8 (Kannan) [2] Let $\psi: \mathcal{X} \to \mathcal{X}$, where $(\mathcal{X}, \varrho)$ is a CMS and ψ satisfies the condition

$$\varrho(\psi x, \psi y) \leq \beta[\varrho(x, \psi x) + \varrho(y, \psi y)]$$

where $0 \leq \beta < \frac{1}{2}$ and $x, y \in \mathcal{X}$. Then ψ has a unique fp in $\mathcal{X}$.

Theorem 2.9 (Chatterjea) [3] Let $(\mathcal{X}, \varrho)$ be a CMS and let ψ be a Chatterjea mapping on $\mathcal{X}$, i.e., there exists $r \in \left[0, \frac{1}{2}\right)$ for all $x, y \in \mathcal{X}$, satisfying

$$\varrho(\psi x, \psi y) \leq r[\varrho(x, \psi y) + \varrho(y, \psi x)].$$

Then ψ has a unique fp.

Theorem 2.10[18] Let $(\mathcal{X}, \varrho)$ be a CMS and suppose that $\psi: \mathcal{X} \to \mathcal{X}$ satisfies for all $x, y \in \mathcal{X}$

$$\varrho(\psi x, \psi y) \leq \alpha\varphi(\varrho(x, \psi x)) + \beta\varphi(\varrho(y, \psi y)) + \gamma\varphi(\varrho(x, y))$$

Where, $\varphi: \mathbb{R} \to \mathbb{R}^+$ is a helping function satisfies $0 \leq \varphi(t) < t$ for all $t > 0$, $\varphi(0) = 0$. Also, $0 < \alpha + \beta + \gamma < 1, \alpha, \beta, \gamma > 0$. Then ψ has unique fp in $\mathcal{X}$.

The proved theorem in this paper is a new and is studied using existing results of the Kannan and the Chatterjea fpt.

3. MAIN RESULTS

Theorem 3.1. Assume that $(\mathcal{X}, \varrho)$ is a CMS and a self-mapping $\psi: \mathcal{X} \to \mathcal{X}$ satisfies for all $x, y \in \mathcal{X}$

$$\varrho(\psi x, \psi y) \leq a_1\varphi(\varrho(x, y)) + a_2\varphi\left[\frac{\varrho(x, \psi x)\varrho(y, \psi y)}{\varrho(x, y)}\right], \tag{3.1}$$

where $\varphi: \mathbb{R} \to \mathbb{R}^+$ is helping function satisfies $0 \leq \varphi(r) < r$, $\forall r > 0$, $\varphi(0) = 0$ with $0 < a_1, a_2 < 1$ and $0 < a_1 + a_2 < 1$.

Then there is a unique fp $z \in \mathcal{X}$, such that $\psi(z) = z$.

Proof. Let $x_0 \in \mathcal{X}$ be an arbitrary but a fixed element. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ by

$$x_1 = \psi x_0, x_2 = \psi^2 x_0,\ x_3 = \psi^3 x_0, \dots, x_n = \psi x_{n-1} = \psi^n x_0$$

By the condition (3.1) on ψ, we get

$$\begin{aligned}
\varrho(x_n, x_{n+1}) &= \varrho(\psi x_{n-1}, \psi x_n) \\
&\leq a_1\varphi(\varrho(x_{n-1}, x_n)) + a_2\varphi\left[\frac{\varrho(x_{n-1}, \psi x_{n-1})\varrho(x_n, \psi x_n)}{\varrho(x_{n-1}, x_n)}\right] \\
&\leq a_1\varphi(\varrho(x_{n-1}, x_n)) + a_2\varphi\left[\frac{\varrho(x_{n-1}, x_n)\varrho(x_n, x_{n+1})}{\varrho(x_{n-1}, x_n)}\right] \\
\varrho(x_n, x_{n+1}) &< a_1\varrho(x_{n-1}, x_n) + a_2\varrho(x_n, x_{n+1}) \\
[1 - a_2]\varrho(x_n, x_{n+1}) &< a_1\varrho(x_{n-1}, x_n) \\
\varrho(x_n, x_{n+1}) &< \frac{a_1}{1 - a_2}\varrho(x_{n-1}, x_n) \\
\varrho(x_n, x_{n+1}) &< R\varrho(x_{n-1}, x_n) \left(\text{where } R = \frac{a_1}{1 - a_2}\right)
\end{aligned}$$

Here, $0 < R < 1$, because $0 < a_1 + a_2 < 1, a_1, a_2 > 0$.

Continuing in this way, we get

$$\varrho(x_n, x_{n+1}) < R^n \varrho(x_0, x_1)$$

Taking limit as $n \to \infty$, we get $d(x_n, x_{n+1}) \to 0$, since $(0 < R < 1)$.
Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$.
As $\mathcal{X}$ is a CMS, there exist $z \in \mathcal{X}$ such that $\lim_{n\to\infty} x_n = z$. We shall show that z is a fp of ψ.
As ψ is a continuous function, so we have

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \psi x_{n-1} = \psi\left(\lim_{n\to\infty} x_{n-1}\right) = \psi z$$

Therefore, $\psi z = z$ and hence z is a fp of ψ.
Now, we will show that z is a unique fp of ψ.
Let $z_1 \in \mathcal{X}$ be another fp of ψ such that $z \neq z_1$. Again, by the condition (3.1), we have

$$\begin{aligned}\varrho(z, z_1) &= \varrho(\psi z, \psi z_1)\\ &\leq a_1\varphi\big(\varrho(z, z_1)\big) + a_2\varphi\left[\frac{\varrho(z, \psi z)\varrho(z_1, \psi z_1)}{\varrho(z, z_1)}\right]\\ &\leq a_1\varphi\big(\varrho(z, z_1)\big) + a_2\varphi\left[\frac{\varrho(z, z)\varrho(z_1, z_1)}{\varrho(z, z_1)}\right]\\ &\leq a_1\varphi(z, z_1).\end{aligned}$$

This is possible only if $\varrho(z, z_1) = 0 \Rightarrow z = z_1$ which is a contradiction.
Therefore, $z \in \mathcal{X}$ is a unique element such that $\psi(z) = z$.

Example 3.1.

Let $\mathcal{X} = [0,1] \subset \mathbb{R}$ with the usual metric $\varrho(x, y) = |x - y|$. Cleary $(\mathcal{X}, \varrho)$ is a CMS. Define a selfing mapping $\psi: \mathcal{X} \to \mathcal{X}$ by

$$\psi(x) = \frac{1}{2}x.$$

Let the helping function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be given by

$$\varphi(r) = \frac{1}{2}r, \text{ for all } r \geq 0.$$

Then φ satisfies

$$\varphi(0) = 0 \text{ and } \varphi(r) < r \text{ for all } r \geq 0.$$

Now, take $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{4}$, so $a_1 + a_2 = \frac{1}{2} < 1$.
We show that for all $x, y \in \mathcal{X}$,

$$|\psi(x) - \psi(y)| \leq a_1\varphi(|x - y|) + a_2\varphi\left[\frac{|x - \psi x||y - \psi y|}{|x - y|}\right].$$

Indeed,

- $|\psi(x) - \psi(y)| = \left|\frac{1}{2}x - \frac{1}{2}y\right| = \frac{1}{2}|x - y|$
- $\varphi(|x - y|) = \frac{1}{2}|x - y|$
- $|x - \psi x| = \left|x - \frac{1}{2}x\right| = \frac{1}{2}|x| \leq \frac{1}{2}$
- $|y - \psi y| = \left|y - \frac{1}{2}y\right| = \frac{1}{2}|y| \leq \frac{1}{2}$

So,

$\varphi\left[\frac{|x-\psi x||y-\psi y|}{|x-y|}\right] \leq \varphi\left[\frac{\frac{1}{4}}{|x-y|}\right]$, which is will- defined and ≤ 1.

Then the inequality holds for suitable choice $x \neq y$ and the assumptions of Theorem 3.1 are satisfied. Hence, by Theorem ψ has a unique fixed point in $\mathcal{X}$, which is $\psi(0) = 0$.

4. APPLICATION: ITERATIVE ALGORITHMS FOR ROOT APPROXIMATION

This kind of result can be applied in the convergence analysis of iterative root-finding algorithms for nonlinear equations, where mappings are not necessarily contractions in the classical sense. For example, if the mapping ψ arises from a transformation in an equation $f(x)$ =0, then ensuring convergence to a fixed point under weaker conditions (via helping functions) allows broader application to real-world problems in engineering and biology where standard contractions may fail.

5. CONCLUSION

In this study, we employed a helping function framework to establish the existence and uniqueness of a fixed point in complete metric spaces. By introducing generalized contractive conditions, we successfully extended and unified classical fixed point results, including those of Banach, Kannan, and Chatterjea. Our results not only broaden the theoretical foundation of fixed point theory but also provide a versatile approach applicable to a wider class of mappings. This work opens up further avenues for research in abstract metric structures, multivalued mappings, and dynamic systems where classical conditions may not be directly applicable.

6. OPEN PROBLEMS AND FUTURE RESEARCH DIRECTIONS

- Extension to Partial or Generalized Metric Spaces
  Can the helping function approach be adapted to partial metric spaces, G-metric spaces, or b-metric spaces? Investigating this may reveal new fixed point results under weaker or nonstandard notions of distance.
- Multivalued and Set-Valued Mappings
  Extend the current results to multivalued mappings. How can the helping function technique be modified to ensure the existence of fixed points for set-valued operators, possibly under conditions like lower semi-continuity?
- Coincidence Points
  Analyze whether similar fixed point results can be obtained for non-self-mappings (i.e., mappings between different metric spaces) and coincidence point results involving two or more operators.
- Incorporation of Order Structure
  Investigate fixed point results in ordered metric spaces. How does the presence of a partial order interact with the helping function framework, and what additional monotonicity or compatibility conditions are needed?

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