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ON USE OF MIXED RULE IN AN ADAPTIVE INTEGRATION SCHEME

Dwiti Krushna Behera*, Debasish Das* & Rajani Ballav Dash*

Author Affiliation:
*Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India.
E-mail: dwiti78@gmail.com, debasisdas100@gmail.com and rajani_bdash@rediffmail.com

Corresponding Author:
Dwiti Krushna Behera, Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India.
E-mail: dwiti78@gmail.com

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Abstract

In this paper, we introduce a mixed quadrature rule using Fejer's second rule and Gaussian rule. This rule is taken as the base rule to develop an adaptive integration scheme. Using this scheme, some test integrals have been evaluated. The results are found to be more encouraging as compared to those obtained by using some other quadratures in this integration scheme.

Keywords: Fejer's second quadrature rule; Gaussian rule; mixed quadrature rule; Clenshaw-Curtis 5-point; Adaptive quadrature.
2000 Mathematics Subject Classification: 65D30, 65D32

1. INTRODUCTION

Many quadrature rules are found literature [1-6, 9] for numerical evaluation of real definite integrals. The first paper on mixed quadrature rule for numerical evaluation of real definite integrals appeared in 1996 by R. N. Das and G. Pradhan [2]. This method of precession enhancement is found to be more simple and easy as compared to other methods such as Konrod [8] extension and Richardson's extrapolation. They formed a mixed quadrature rule of higher precision by linear/convex combination of the rules of lower(equal) precision. Not much work have been done on implementation of mixed rules in adaptive integration scheme. Some authors such as: D. K. Behera, D. Das and R. B. Dash have successfully implemented different mixed rules in adaptive integration scheme. Very recently, R. B. Dash et.al.[4] formed a mixed quadrature rule by using average Gauss-Legendre rule to evaluate some real definite Integrals in adaptive environment.

In this paper, we get motivation for successfully forming an open type mixed rule of precision seven taking linear combination of Fejer's second rule and Gaussian quadrature rules, each of precision five. The mixed quadrature rule is implemented to form an efficient adaptive integration scheme. Using this scheme, some test integrals have been numerically evaluated. These results are compared with those obtained by implementing other rules in this adaptive integration scheme (Table-1 and Table-2).

2. CONSTRUCTION OF THE MIXED QUADRATURE RULE OF PRECISION SEVEN

Expressing the integrand in terms of Chebyshev polynomials one can derive Fejer's second $n-$ point quadrature rule as

$$\int_{-1}^{1} f(x)dx \approx R_{2Fn}(f) = \sum_{k=1}^{n} w_k f\left(Cos\frac{k}{n+1}\pi\right)$$

Where $w_k = \frac{4\, Sin\theta_k}{n+1} \sum_{m=1}^{\left[\frac{(n+1)}{2}\right]} \frac{Sin(2m-1)\theta_k}{2m-1}$

Taking $n = 5$ one can write 5-point Fejer's second rule as

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{2F5}(f) = \frac{2}{45}\left[7f\left(\frac{\sqrt{3}}{2}\right) + 9f\left(\frac{1}{2}\right) + 13f(0) + 9f\left(-\frac{1}{2}\right) + 7f\left(-\frac{\sqrt{3}}{2}\right)\right]. \tag{2.1}$$

We have well known Gauss- Legendre rule as:

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{GL3}(f) = \frac{1}{9}\left[5\left\{f\left(\sqrt{\frac{3}{5}}\right) + f\left(-\sqrt{\frac{3}{5}}\right)\right\} + 8f(0)\right]. \tag{2.2}$$

Each of the rules (2.1) and (2.2) are degree of precision is five.

Let $E_{2F5}(f)$ and $E_{GL3}(f)$ denote the errors in approximating the integrals $I(f)$ by the rules (2.1) and (2.2) respectively.

Then

$$I(f) = R_{2F5}(f) + E_{2F5}(f) \tag{2.3}$$

$$I(f) = R_{GL3}(f) + E_{GL3}(f) \tag{2.4}$$

Using Maclurin's expansion of functions in equations (2.1) and (2.2) we have

$$E_{2F5}(f) = I(f) - R_{2F5}(f)$$

$$= \frac{1}{67200} f^{vi}(0) + \frac{1}{1814400} f^{viii}(0) + \dots \tag{2.5}$$

and $E_{GL3}(f) = I(f) - R_{GL3}$

$$= \frac{8}{126000} f^{vi}(0) + \frac{88}{45360000} f^{viii}(0) + \dots \tag{2.6}$$

Now multiplying the equations (2.3) and (2.4) $-\frac{1}{8}$ and $\frac{8}{15}$ respectively, and then adding the resulting equations, we have

$$I(f) = \frac{1}{49}[64R_{2F5}(f) - 15R_{GL3}(f)] + \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$$

or $I(f) = R_{2F5GL3}(f) + E_{2F5GL3}(f)$ (2.7)

Where $R_{2F5GL3}(f) = \frac{1}{49}[64R_{2F5}(f) - 15R_{GL3}(f)]$

and $E_{2F5GL3}(f) = \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$

Hence

$$R_{2F5GL3}(f) = \frac{1}{2205}\left[\begin{array}{c} 896f\left(-\frac{\sqrt{3}}{2}\right) - 375f\left(-\sqrt{\frac{3}{5}}\right) + 1152f\left(-\frac{1}{2}\right) + 1064f(0) + \\ 1152f\left(\frac{1}{2}\right) - 375f\left(\sqrt{\frac{3}{5}}\right) + 896f\left(\frac{\sqrt{3}}{2}\right) \end{array}\right]. \tag{2.8}$$

This is the desired mixed quadrature rule of precision seven for the approximate evaluation of $I(f)$.

The truncation error associated to this rule is given by

$$E_{2F5GL3}(f) = \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$$

or $$E_{2F5GL3}(f) = \frac{1}{5! \times 68600} f^{viii}(0) + \dots \quad (2.9)$$

3. ERROR ANALYSIS

An asymptotic error estimate and an error bound of the rule (2.8) are given in theorems $(3.1(a))$ and $(3.1(b))$ respectively.

Theorem $3.1(a)$: Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $E_{2F5GL3}(f)$ associated with the rule $R_{2F5GL3}(f)$ is given by

$$|E_{2F5GL3}(f)| \approx \frac{1}{5! \times 68600} |f^{viii}(0)|.$$

Proof: From eq. (2.7)

$$I(f) = R_{2F5GL3}(f) + E_{2F5GL3}(f)$$

Where $$R_{2F5GL3}(f) = \frac{1}{49}[64R_{2F5}(f) - 15R_{GL3}(f)]$$

and $$E_{2F5GL3}(f) = \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$$

Hence $$E_{2F5GL3}(f) = \frac{1}{5! \times 68600} f^{viii}(0) + \dots$$

So, $$|E_{2F5GL3}(f)| \approx \frac{1}{5! \times 68600} |f^{viii}(0)|.$$

Theorem $3.1(b)$: The bound for the truncation error $E_{2F5GL3}(f) = I(f) - R_{2F5GL3}(f)$ is given by

$$|E_{2F5GL3}(f)| \leq \frac{M}{51450} |(\eta_2 - \eta_1)|; \; \eta_1, \eta_2 \in [-1,1],$$

Where $M = \underset{-1 \leq x \leq 1}{Max} \left| f^{vii}(x) \right|$.

Proof: We have $$E_{2F5}(f) \approx \frac{1}{67200} f^{vi}(\eta_1), \eta_1 \in [-1,1]$$

$$E_{GL3}(f) \approx \frac{8}{175 \times 6!} f^{vi}(\eta_2), \eta_2 \in [-1,1]$$

Hence $$E_{2F5GL3}(f) = \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$$

$$= \frac{1}{51450} \int_{\eta_1}^{\eta_2} f^{vii}(x)dx, \qquad \eta_1 < \eta_2$$

From this we have

$$|E_{2F5GL3}(f)| = \left| \frac{1}{51450} \int_{\eta_1}^{\eta_2} f^{vii}(x)dx \right| \leq \frac{1}{51450} \int_{\eta_1}^{\eta_2} |f^{vii}(x)dx|$$

$$\therefore |E_{2F5GL3}(f)| \leq \frac{M}{51450} |(\eta_2 - \eta_1)|$$

Where $M = \underset{-1 \leq x \leq 1}{Max} \left| f^{vii}(x) \right|$.

This shows that the error bound as η_1, η_2 are unknown points in $[-1, 1]$. Also, it gives the error in approximation will be minimize if the points η_1, η_2 are very close to each other.

Corollary: The error bounds for the truncation error $E_{2F5GL3}(f)$ is given by $|E_{2F5GL3}(f)| \leq \frac{2M}{51450}$
.

Proof: From $3.1(b)$, we have

$$|E_{2F5GL3}(f)| \leq \frac{M}{51450} |(\eta_2 - \eta_1)|; \ \eta_1, \eta_2 \in [-1, 1]$$

Where $M = \underset{-1 \leq x \leq 1}{Max} \left| f^{vii}(x) \right|$.

Choosing $|(\eta_2 - \eta_1)| \leq 2$, we have

$$|E_{2F5GL3}(f)| \leq \frac{2M}{51450}.$$

4. ALGORITHM FOR ADAPTIVE QUADRATURE ROUTINE

Applying the constituent rules $R_{CC5}(f)$, $R_{2F5}(f), R_{GL3}(f)$ and the mixed quadrature rules $R_{CC5GL3}(f), R_{2F5GL3}(f)$; one can evaluate real definite integrals of the type $I(f) = \int_a^b f(x)$ in adaptive integration scheme. In the adaptive integration scheme, the desired accuracy is sought by progressively subdividing the interval of integration according to the computed behavior of the integrand and applying the same formula over each sub interval. The algorithm for adaptive integration scheme is outlined here using the mixed quadrature rules $R_{CC5GL3}(f) \ and \ R_{2F5GL3}(f)$ in the following four steps.

Input: Function $F: [a, \quad b] \rightarrow R$ and the prescribed tolerance $\in$.

Output: An approximation $Q(f)$ to the integral $I(f) = \int_a^b f(x)\,dx$ such that $|Q(f) - I(f)| \leq \in$.

Step-1: The interval $[a, \quad b]$ is divided into two subintervals, $[a, \quad c]$ and $[c, \quad b]$, where $c = \frac{a+b}{2}$. The length of the interval $[a, \quad b]$ is $b - a$ which has a fixed value. The length of any subinterval is denoted by $\bar{h}$ which changes when the subinterval width changes.

Step-2: The quadrature rule/mixed quadrature rule is applied to compute $Q[a, \quad c] \approx \int_a^c f(x)\,dx$ and

$Q[c, \quad b] \approx \int_c^b f(x)dx.$ Since the anti-derivatives of the integrands in the chosen integrals exist, the exact values $I[a, \quad c] = \int_a^c f(x)\,dx$ and $I[c, \quad b] = \int_c^b f(x)dx$ are computed.

Step-3: $|Q[a, \quad c] - I[a, \quad c]|$ and $|Q[c, \quad b] - I[c, \quad b]|$ are estimated. If $|Q[a, \quad c] - I[a, \quad c]| \leq \frac{\bar{h}}{b-a} \in$ (termination criterion) then the value of $Q[a, \quad c]$ is added
to the SUM register, otherwise the step-1 is repeated. Similarly, If $|Q[c, \quad b] - I[c, \quad b]| \leq \frac{\bar{h}}{b-a} \in$, then the value of $Q[c, \quad b]$ is added to the SUM register, otherwise the step-1 is repeated.

Step-4: This process is continued till the termination criterion in all sub-intervals is satisfied. When the process stops, then addition of all $Q(f)$ values yields the approximate value $Q(f)$ of the integral

$I(f)=\int_a^b f(x)\,dx$ such that $|Q(f)-I(f)|\leq\epsilon$.

N:B: In this algorithm we can use any quadrature rule to evaluate real definite integrals in adaptive integration scheme.

5. NUMERICAL VERIFICATION

The adaptive integration scheme of this paper has been C-Programmed by the authors. The results of the Tables 1 and 2 have been obtained using this software.

Table 1:

Comparison of results obtained implementing the quadrature rules $(R_{CC5}, R_{2F5}$ and $R_{GL3}(f))$ in adaptive integration scheme.

Integrals	Exact Value(I(f))	Approximate Values($R(f)$)					
		$R_{CC5}(f)$	No. of Steps	$R_{2F5}(f)$	No. of Steps	$R_{GL3}(f)$	No. of Steps
$I_1=\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}dx$	1	0.99999999705	05	0.99999999842	05	0.99999999262	05
$I_2=\int_0^{\pi}\frac{1}{5+4\cos x}dx$	1.047197551196503	1.047197550178	11	1.0471975508049	11	1.047197548486	13
$I_3=\int_0^{1}\frac{1}{1+25x^2}dx$	0.274680153389003	0.274680163709	09	0.2746801594025	09	0.2746801533341	13
$I_4=\int_0^{\frac{\pi}{2}}\cos^3 x\,dx$	0.6666666666…	0.666666665946	07	0.6666666662623	07	0.6666666649406	07
$I_5=\int_0^{\frac{\pi}{4}}\frac{1}{1+\sin x}dx$	0.5857864372626905	0.58578643696	03	0.5857864372559	03	0.58578643604	03
$I_6=\int_0^{1}\frac{1}{1+x}dx$	0.693147180559945	0.6931471751150 7	03	0.693147177465	03	0.693147179351	05
$I_7=\int_0^{1}\frac{1}{1-0.5x^4}dx$	1.143667254069416	1.14366724238	09	1.1436672397903	07	1.143667244384	11
$I_8=\int_0^{1}\frac{1}{1+100x^2}dx$	0.147112767430373	0.1471127721417	13	0.1471127479136	09	0.147112778997	13
$I_9=\int_1^{2}\frac{\ln x}{x}dx$	0.240226506959101	0.240226507959	05	0.240226513986	03	0.240226509367	05
$I_{10}=\int_1^{2}\frac{1}{e^x-1}dx$	0.313261687518223	0.3132616820735	03	0.3132616844242	03	0.3132616863104	05

Note: Here the prescribed tolerance ϵ =0.000001

Table 2:

Comparison of results obtained implementing the quadrature rules $(R_{CC5GL3}(f)$ and $R_{2F5GL3}(f))$ for approximation of some real definite integrals using adaptive integration scheme.

Integrals	Exact Value (I(f))	Approximate Values($R(f)$)			
		$R_{CC5GL3}(f)$ Absolute Error	No. of Steps	$R_{2F5GL3}(f)$ Absolute Error	No. of Steps
$I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$	1	1.0000000007701 77×10^{-11}	03	1.0000000000426 42×10^{-12}	03
$I_2 = \int_0^{\pi} \frac{1}{5+4\cos x} dx$	1.047197551196503	1.0471975512116 151×10^{-13}	07	1.0471975517286 532×10^{-12}	07
$I_3 = \int_0^{1} \frac{1}{1+25x^2} dx$	0.274680153389003	0.27468015344808 59×10^{-12}	07	0.27468015337389 15×10^{-12}	07
$I_4 = \int_0^{\frac{\pi}{2}} \cos^3 x\, dx$	0.6666666666…	0.666666666387 279×10^{-12}	03	0.666666666746 796×10^{-13}	03
$I_5 = \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} dx$	0.5857864372626905	0.5857864381774 914×10^{-12}	01	0.5857864374704 207×10^{-12}	01
$I_6 = \int_0^{1} \frac{1}{1+x} dx$	0.693147180559945	0.69314718061706 571×10^{-13}	03	0.693147177509 305×10^{-11}	01
$I_7 = \int_0^{1} \frac{1}{1-0.5x^4} dx$	1.143667254069416	1.1436672545501 48×10^{-11}	07	1.1436672539307 13×10^{-11}	07
$I_8 = \int_0^{1} \frac{1}{1+100x^2} dx$	0.147112767430373	0.147112768379 949×10^{-12}	09	0.14711277295 552×10^{-11}	07
$I_9 = \int_1^{2} \frac{\ln x}{x} dx$	0.240226506959101	0.2402265068126 14×10^{-11}	03	0.24022650700118 42×10^{-12}	03
$I_{10} = \int_1^{2} \frac{1}{e^x - 1} dx$	0.313261687518223	0.313261687575338 571×10^{-13}	03	0.3132616844679 305×10^{-11}	01

$$Absolute\ Error = |Exact\ Value - Approximation\ Value| = |I(f) - R(f)|$$

Note: Here the prescribed tolerance ϵ =0.000001

6. CONCLUSION

Before drawing conclusion, we categories the results obtained in the above two tables(Table-1 and Table-2) as RI, RII and RIII.

RI: The results of the test integrals using adaptive integration scheme constructed in this paper are found in columns 4 and 5 of Table-2.

RII: The results of the test integrals using the adaptive integration scheme based on rules Clenshaw-Curtis 5-point $(R_{CC5}(f))$, Fejer's second rule $(R_{2F5}(f))$ and Gauss Legendre 3-point rule $(R_{GL3}(f))$ are given in Table-1.

RIII: The results of the test integrals using adaptive integration scheme based on a mixed rule (blending Clenshaw-Curtis 5-point and Gauss Legendre 3-pont rule) i.e. R_{CC5GL3} are given in Table-2 of columns 3 and 4. There results have been obtained in a paper authored by R. B. Dash and D. Das[3].

We observe that our results RI are much more encouraging than those in RII. Though our results RI are at par with RIII for some integrals, but in some other integrals (I_6, I_9, I_{10}) results are highly encouraging.

Thus the adaptive integration scheme developed in this paper using the mixed quadrature of Fejer's second rule and Gauss Legendre 3-point rule is no doubt an efficient integration scheme.

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