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ON EXTON'S TRIPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS-I

Lalit Mohan Upadhyaya

Author Affiliation:
Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, (Uttarakhand), India, 248179.

Corresponding Author:
Lalit Mohan Upadhyaya, Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, (Uttarakhand), India, 248179.
E-mail: lmupadhyaya@rediffmail.com

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Abstract

In the present paper I have defined the Exton's X_1 and X_3 triple hypergeometric functions of matrix arguments and have established two integral representations for them (one for each) using the Mathai's matrix transform technique.

Keywords: Exton's functions, triple hypergeometric functions, matrix arguments, matrix - transform.

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1. INTRODUCTION

The very broad character of the theory of the Gauss's hypergeometric function ${}_2F_1$ paved the way for extension of this fruitful theory to the case of two and more variables. The first widely acclaimed generalization of this theory to the corresponding functions is attributed to Appell [1] who gave four generalizations of this function to the two variables case which have been hailed in the literature as the Appell's functions. Later on Horn devoted his life extensively to the paramount task of the very systematic examination of all the double hypergeometric functions of the second order and in a series of papers, (i.e., Horn [4,5,6]) he gave a final list of thirty four functions which included fourteen complete series (for instance, see Exton [2], Srivastava and Karlsson [10]). It is in this terminal list of the complete functions given by Horn, there appear the two Horn's functions H_3 and H_4. Later on, Exton [3] in 1982 gave twenty triple hypergeometric functions $X_1, \cdots, X_{20}$ which according to him [3, p.113] are the generalizations of the Horn's double hypergeometric functions H_3 and H_4 (see also Srivastava and Karlsson [10], p.287). The present paper aims at defining the Exton's X_1 and X_3 triple hypergeometric functions of matrix arguments and some properties of these functions will be studied by using the Mathai's matrix transform technique. The matrices appearing in this paper are all real symmetric and positive definite with order $(p \times p)$. $A > 0$ Will mean that the matrix A is positive definite. $A^{1/2}$ will represent the symmetric square root of the matrix A. While integrating over matrices $\int_X f(X)dX$ represents integral over X of the scalar function $f(X)$. Re(.) denotes the real part of (.).

I begin with quoting some preliminary results and definitions which occur in the literature. Mathai [7] in 1978 defined the matrix transform (M- transform) of a function $f(X)$ of a $(p\times p)$ real symmetric positive definite matrix X as follows:

$$M_f(\rho)=\int_{X>0}\left|X\right|^{\rho-(p+1)/2} f(X)dX \tag{1.1}$$

for $X>0$ and $\mathrm{Re}(\rho)>(p-1)/2$ whenever $M_f(s)$ exists.

The following results and definitions will be used by me at various places in this paper.

Theorem 1.1: (Mathai [8], eq.(2.24), p.23)- Let X and Y be $(p\times p)$ symmetric matrices of functionally independent real variables and A be a $(p\times p)$ non singular matrix of constants. Then,

$$Y=AXA'\Rightarrow dY=\left|A\right|^{p+1}dX \tag{1.2}$$

and

$$Y=aX\Rightarrow dY=a^{p(p+1)/2}dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2: Type-2 Beta integral (Mathai [9], (eq. (2.2.4), p.36 and eq. (2.1.2), p.32)-

$$B_p(\alpha,\beta)=\int_{Y>0}\left|Y\right|^{\alpha-(p+1)/2}\left|I+Y\right|^{-(\alpha+\beta)}dY=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.4}$$

for $\mathrm{Re}(\alpha)>(p-1)/2, \mathrm{Re}(\beta)>(p-1)/2$, where,

$$\Gamma_p(\alpha)=\pi^{p(p-1)/4}\Gamma\left(\alpha\right)\Gamma\left(\alpha-\tfrac{1}{2}\right)\cdots\Gamma\left(\alpha-\tfrac{p-1}{2}\right) \tag{1.5}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$.

Theorem 1.3: Type-1 Beta integral (Mathai [9], (eq. 2.2.2, p.34 and eq. (2.1.2), p.32)-

$$B_p(\alpha,\beta)=\int_{0<X<I}\left|X\right|^{\alpha-(p+1)/2}\left|I-X\right|^{\beta-(p+1)/2}dX=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\mathrm{Re}(\alpha)>(p-1)/2, \mathrm{Re}(\beta)>(p-1)/2$.

Theorem 1.4: Gamma integral (Mathai [9], (eq. 2.1.3, p.33)- For real symmetric positive definite matrices X and B of order $(p\times p)$

$$\int_{X>0}\left|X\right|^{\alpha-(p+1)/2}e^{-tr(BX)}dX=\left|B\right|^{-\alpha}\Gamma_p(\alpha) \tag{1.7}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$ where $tr\left(X\right)$ denotes the trace of the matrix X.

Theorem 1.5: (Mathai [9], eq.(3.2.7), p.55) The M-transform of

$${}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X_1-\cdots-X_n)$$

is the following, where $X_1,\cdots,X_n$ are $(p\times p)$ real symmetric positive definite matrices.

$$M({}_rF_s)=\int_{X_1>0}\cdots\int_{X_n>0}\left|X_1\right|^{\rho_1-(p+1)/2}\cdots\left|X_n\right|^{\rho_n-(p+1)/2}{}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X_1-\cdots-X_n)\times dX_1\cdots dX_n$$

$$=\frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}\left\{\prod_{k=1}^{r}\Gamma_p(a_k-\rho_1-\cdots-\rho_n)\right\}\left\{\prod_{m=1}^{n}\Gamma_p(\rho_m)\right\}}{\left\{\prod_{k=1}^{r}\Gamma_p(a_k)\right\}\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho_1-\cdots-\rho_n)\right\}} \tag{1.8}$$

for $\mathrm{Re}(a_k-\rho_1-\cdots-\rho_n, b_j-\rho_1-\cdots-\rho_n,\rho_m)>(p-1)/2$ and $k=1,\cdots,r;\ j=1,\cdots,s;$ $m=1,\cdots,n$.

Theorem 1.6: (Mathai [9], (eq.(6.13), p. 84) - For $p=2$,

$$4^{-p\rho}\frac{\Gamma_p[(a+1)/2-\rho]\Gamma_p[(2a+1)/4-\rho]}{\Gamma_p[(a+1)/2]\Gamma_p[(2a+1)/4]}=\frac{\Gamma_p(a-2\rho)}{\Gamma_p(a)} \tag{1.9}$$

Definition 1.1: The Appell's F_4 function of matrix arguments $F_4=F_4(a,b;c,c';-X,-Y)$ is defined as that class of functions for which the matrix transform (M-transform) is the following:

$$\begin{aligned}M(F_4)&=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_4(a,b;c,c';-X,-Y)dXdY\\&=\frac{\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(a)\Gamma_p(b)}\frac{\Gamma_p\left(b-\rho_1-\rho_2\right)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)}\end{aligned} \tag{1.10}$$

for $\mathrm{Re}(a-\rho_1-\rho_2,b-\rho_1-\rho_2,c-\rho_1,c'-\rho_2,\rho_1,\rho_2)>(p-1)/2$.

Definition 1.2: The ${}_rF_s$ function of matrix arguments, ${}_rF_s={}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)$ is defined as that class of functions which has the following matrix transform:

$$\begin{aligned}M\left({}_rF_s\right)&=\int_{X>0}|X|^{\rho-(p+1)/2}{}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)dX\\&=\frac{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m-\rho\right)\right\}}{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k-\rho\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m\right)\right\}}\Gamma_p\left(\rho\right)\end{aligned} \tag{1.11}$$

for $\mathrm{Re}(\rho,a_m-\rho,b_k-\rho)>(p-1)/2$ where, $m=1,\cdots,r;k=1,\cdots,s$.

2. DEFINITIONS OF THE EXTON'S FUNCTIONS

Definition 2.1: The Exton's function $X_1=X_1\left[\begin{matrix}a_1,a_1;a_1,a_1;a_1,a_2\,|\\ c_2;\quad c_1;\quad c_1\quad |\end{matrix}-X,-Y,-Z\right]$ of matrix arguments is defined as that class of functions which has the following matrix transform (M-transform):

$$\begin{aligned}M\left(X_1\right)&=\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times\\&\qquad X_1\left[\begin{matrix}a_1,a_1;a_1,a_1;a_1,a_2\,|\\ c_2;\quad c_1;\quad c_1\quad |\end{matrix}-X,-Y,-Z\right]dXdYdZ\\&=\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_2-\rho_3)\Gamma_p(c_2-\rho_1)}\end{aligned} \tag{2.1}$$

for $\mathrm{Re}(a_1-2\rho_1-2\rho_2-\rho_3,a_2-\rho_3,c_1-\rho_2-\rho_3,c_2-\rho_1,\rho_1,\rho_2,\rho_3)>(p-1)/2$.

Definition 2.2: For Exton's function $X_3=X_3\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2\,|\\ c_1;\quad c_1;\quad c_2\quad |\end{matrix}-X,-Y,-Z\right]$

$$\begin{aligned}M\left(X_3\right)&=\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times\\&\qquad X_3\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2\,|\\ c_1;\quad c_1;\quad c_2\quad |\end{matrix}-X,-Y,-Z\right]dXdYdZ\end{aligned}$$

$$=\frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1-\rho_2)\Gamma_p(c_2-\rho_3)} \quad (2.2)$$

for $\mathrm{Re}(a_1-2\rho_1-\rho_2-\rho_3, a_2-\rho_2-\rho_3, c_1-\rho_1-\rho_2, c_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2$.

3. INTEGRAL REPRESENTATIONS FOR THE EXTON'S FUNCTIONS X_1 AND X_3 OF MATRIX ARGUMENTS

Now we proceed to derive two integral representations for the Exton's X_1 and X_3 functions of matrix arguments in the form of the two succeeding theorems:

Theorem 3.1:

$$X_1\left[\begin{matrix} a_1,a_1;a_1,a_1;a_1,a_2 \\ c_2; \quad c_1; \quad c_1 \end{matrix}\middle| -X,-Y,-Z\right] = \frac{\Gamma_p(c_1)}{\Gamma_p(a_2)\Gamma_p(c_1-a_2)}\int_0^I |U|^{a_2-(p+1)/2}|I-U|^{c_1-a_2-(p+1)/2}\left|I+U^{1/2}ZU^{1/2}\right|^{-a_1} \times$$

$$F_4\left[\frac{a_1+1}{2},\frac{2a_1+1}{4};c_2,c_1-a_2;-4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}X\left(I+U^{1/2}ZU^{1/2}\right)^{-1},\right.$$

$$\left.-4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}(I-U)^{1/2}Y(I-U)^{1/2}\left(I+U^{1/2}ZU^{1/2}\right)^{-1}\right]dU \quad (3.1)$$

for $p=2$, $0<U<I$ and for $\mathrm{Re}(a_2, c_1-a_2) > (p-1)/2$.

Proof: Taking the M-transform of the left side of eq.(3.1) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively we obtain,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\left|I+U^{1/2}ZU^{1/2}\right|^{-a_1}\times$$

$$F_4\left[\frac{a_1+1}{2},\frac{2a_1+1}{4};c_2,c_1-a_2;-4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}X\left(I+U^{1/2}ZU^{1/2}\right)^{-1},\right.$$

$$\left.-4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}(I-U)^{1/2}Y(I-U)^{1/2}\left(I+U^{1/2}ZU^{1/2}\right)^{-1}\right]dXdYdZ \quad (3.2)$$

Applying the transformations

$X_1=4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}X\left(I+U^{1/2}ZU^{1/2}\right)^{-1}$,

$Y_1=4\left(I+U^{1/2}ZU^{1/2}\right)^{-1}(I-U)^{1/2}Y(I-U)^{1/2}\left(I+U^{1/2}ZU^{1/2}\right)^{-1}$, and $Z_1=U^{1/2}ZU^{1/2}$ so that,

$X_1=4(I+Z_1)^{-1}X(I+Z_1)^{-1}$, $Y_1=4(I+Z_1)^{-1}(I-U)^{1/2}Y(I-U)^{1/2}(I+Z_1)^{-1}$, and

$dX_1=4^{p(p+1)/2}|I+Z_1|^{-(p+1)}dX$, $dY_1=4^{p(p+1)/2}|I+Z_1|^{-(p+1)}|I-U|^{(p+1)/2}dY$,

$dZ_1=|U|^{(p+1)/2}dZ$ (from theorem (1.1)) with,

$|X_1|=4^p|I+Z_1|^{-2}|X|, |Y_1|=4^p|I+Z_1|^{-2}|I-U||Y|, |Z_1|=|U||Z|$ renders the eq.(3.2) as below,

$$4^{-p(\rho_1+\rho_2)}|I-U|^{-\rho_2}|U|^{-\rho_3}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}\times$$

$$|I+Z_1|^{-(a_1-2\rho_1-2\rho_2)}F_4\left[\frac{a_1+1}{2},\frac{2a_1+1}{4};c_2,c_1-a_2;-X_1,-Y_1\right]dX_1dY_1dZ_1 \quad (3.3)$$

Writing down the M-transform of an F_4 function (definition (1.1)) and integrating out Z_1 by using a type-2 Beta integral (theorem 1.2), we get,

$$4^{-p(\rho_1+\rho_2)}|I-U|^{-\rho_2}|U|^{-\rho_3}\frac{\Gamma_p(a_1-2\rho_1-2\rho_2-\rho_3)\Gamma_p(c_2)\Gamma_p(c_1-a_2)}{\Gamma_p(a_1-2\rho_1-2\rho_2)\Gamma_p\left(\frac{a_1+1}{2}\right)\Gamma_p\left(\frac{2a_1+1}{4}\right)}\times$$
$$\frac{\Gamma_p\left(\frac{a_1+1}{2}-\rho_1-\rho_2\right)\Gamma_p\left(\frac{2a_1+1}{4}-\rho_1-\rho_2\right)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-a_2-\rho_2)\Gamma_p(c_2-\rho_1)} \quad (3.4)$$

Substituting this expression on the right side of eq.(3.1) and integrating out U by means of a type-1 Beta integral (theorem 1.3) and utilizing the theorem (1.6) we finally arrive at $M\left(X_1\right)$ as given by eq.(2.1), thereby finishing the proof.

Theorem 3.2:

$$X_3\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ c_1; \quad c_1; \quad c_2 \quad \mid \end{matrix} -X,-Y,-Z\right]=\frac{1}{\Gamma_p(a_1)\Gamma_p(a_2)}\int_{S_1>0}\int_{S_2>0} e^{-tr(S_1+S_2)}|S_1|^{a_1-(p+1)/2}|S_2|^{a_2-(p+1)/2}\times$$

$$ {}_0F_1\left(\ ;c_1;-S_1XS_1^{'}-S_2^{1/2}S_1^{1/2}YS_1^{1/2}S_2^{1/2}\right){}_0F_1\left(\ ;c_2;-S_2^{1/2}S_1^{1/2}ZS_1^{1/2}S_2^{1/2}\right)dS_1dS_2 \quad (3.5)$$

for $\mathrm{Re}(a_1,a_2)>(p-1)/2$.

Proof: Taking the M-transform of the left side of eq.(3.5) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively we arrive at,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$
$${}_0F_1\left(\ ;c_1;-S_1XS_1^{'}-S_2^{1/2}S_1^{1/2}YS_1^{1/2}S_2^{1/2}\right){}_0F_1\left(\ ;c_2;-S_2^{1/2}S_1^{1/2}ZS_1^{1/2}S_2^{1/2}\right)dXdYdZ \quad (3.6)$$

On the application of the transformations,

$$X_1=S_1XS_1^{'},Y_1=S_2^{1/2}S_1^{1/2}YS_1^{1/2}S_2^{1/2},Z_1=S_2^{1/2}S_1^{1/2}ZS_1^{1/2}S_2^{1/2}$$

(with $dX_1=|S_1|^{(p+1)}dX, dY_1=|S_2|^{(p+1)/2}|S_1|^{(p+1)/2}dY,\ dZ_1=|S_2|^{(p+1)/2}|S_1|^{(p+1)/2}dZ$ (from theorem (1.1)) and $|X_1|=|S_1|^2|X|,|Y_1|=|S_2||S_1||Y|,|Z_1|=|S_2||S_1||Z|$) in the expression (3.6) then utilizing the definition (1.2) and the theorem (1.5) leads us to,

$$|S_1|^{-2\rho_1-\rho_2-\rho_3}|S_2|^{-\rho_2-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1-\rho_2)\Gamma_p(c_2-\rho_3)} \quad (3.7)$$

Substituting this expression on the right side of eq.(3.5) and integrating out the variables S_1 and S_2 by using a Gamma integral (theorem (1.4)) ultimately generates $M\left(X_3\right)$ in conformity with eq.(2.2), thereby concluding the proof.

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