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Possible regression models for the municipal finances of the municipal corporations of various Indian states II *

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Abstract In continuation of the ongoing long term research work of the first author on the detailed mathematical and statistical analysis of the budget of the Bangalore Municipal Corporation (see, A. Sathyavathi https://ssrn.com/abstract=4403863, https://ssrn.com/abstract=4461494, https://ssrn.com/abstract=4518454 and https://ssrn.com/abstract=4568499), we carry ahead our previous study (A. Sathyavathi and L. M. Upadhyaya, Possible regression models for the municipal finances of the municipal corporations of various Indian states, Bull. Pure Appl. Sci. Sect. E Math. Stat. 42E(1), 72–93 (2023))to propose some possible regression models to account for the observed trends of the revenue receipts and the revenue expenditures of the municipal corporations of the various Indian States including the Union Territory of Chandigarh for the 2018-19 Budget Estimates, which pertains to the data of finances of municipal corporations of the various Indian States during the financial year 2018-2019. We also present the comparisons of our different models for each of the categories of our present explorations. The entire data used by us in this study is gratefully taken by us from the Report on Municipal Finances which was published by the Reserve Bank of India recently on 10 November, 2022 (see, https://m.rbi.org.in/scripts/AnnualPublications.aspx?head=Report%20on%20Municipal%20Finances,https://rbidocs.rbi.org.in/rdocs/Publications/PDFs/RMF101120223A34C4F7023A4A9E99CB7F7FEF6881D0.PDF). We find that a Polynomial Degree 4 model best describes the financial data and in addition we also give a Rational Model and an Exponential Association 2 model for describing the trend of finances for the municipal corporations of India during the year 2018-19 in the Budget Estimates category and we also present our computational results of the Polynomial Degree 4 Model and the Rational Model using MATLAB.

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1 Introduction

When talking about the municipal finances in India, in particular, and at the global level, in general, the budgeting is one of the four major pillars of the public financial management, according to Venkateswaran [20], the other three being accounting, reporting and auditing. Writing on the application of public financial management to the local bodies, Venkateswaran (see, [20, p. 93]) writes: "Budgets provide operational and financial plans for the attainment of the local government's goals." He also presents a copy of the snapshot of the Line-item budget of the Bruhat Bengaluru Municipal Corporation (BBMP) titled 'Abstract of Budget Estimates for 2009-10' of the Bangalore Municipal Corporation ([20, Table 3.1, p. 96]). We would like to mention here that the first author is already working on her long term project work of the detailed analysis of the Budget of the Bangalore Municipal Corporation (see, Sathyavathi [24–28]). Talking about the present great pace of urbanization in India, it would be more better in this context to focus one's attention to the growing phenomenon of urbanization the world over. In this context we are reminded of the following lines of Bahl ([21, p.3 of 40]): "Asia and the Pacific is witnessing the world's fastest urbanization in history. In the period 2000-2025, 1.1 billion people are projected to migrate into Asian cities and the region is now home to more than half of the megacities worldwide. Providing quality jobs, housing, urban infrastructure and public services for these new urban migrants and supporting sustainable development of the region's large metropolitan areas would be a significant fiscal challenge for many governments. The current approach of revenue mobilization for cities and municipal fiscal reform efforts are unlikely to meet the substantial financing needs. Instead, there is a need for a metropolitan public financing strategy that is integrated into national urban development plans and matches national development objectives." The need to provide a strong financing structure for the local governments is a world wide problem. In most of the countries, the Urban Local Governments find it difficult to meet the need of adequately financing their basic functions - like providing the basic amenities of life like pure drinking water, sanitation, housing, medical facilities, educational and employment avenues for the people inhabiting their territories. Delving into the issue of the autonomy in financial matters of the ULBs in India, Bagchi (see, [22, p.3 of 28]) studies "the impact of decentralization on resource mobilization capacity of municipal bodies both w.r.t. tax and non-tax revenue". ULBs in India also receive some grants from the Central and their respective State Governments in the form of 'tied grants' which the ULBs are required to spend only for that specific purpose for which that particular grant is meant for, the ULBs are not free to spend the tied grants for any other purpose than specified for the grant. This reduces the financial autonomy of the ULBs in India. In this context Bagchi ([22, p. 4 of 28]) aptly remarks that "The type of tied grant, on the other hand, limits local governments' discretion in using it and thus hampers the very essence of the process of 'independent decision making' ". If we talk about the financial position of the ULBs of India, many studies have shown that they are in a poor condition. Even if we look into the fact that the upper two tiers of the governments - the Central Government and the State Governments have authorized the municipalities to levy and collect various taxes, like the property tax, advertisement tax, entertainment tax, penalties, license fees, user charges for the services provided by the ULBs, etc., still the own revenue generated by the Indian ULBs is very less than expected. Chaubey (see, [23, p. 24 of 45]) highlights this situation as: "Except for four states of Gujarat, Maharashtra, Orissa and Punjab where revenue position of municipalities is somewhat better, the compensation paid in lieu of this tax by the states such as Haryana and Madhya Pradesh has been quite low and payment is delayed." The pathetic condition of the state of municipal finances in India vis-a-vis the other developing or developed countries is also highlighted by Ahluwalia et al. in their

recent study ([7, p. 7]) by emphasizing that: “While cities are expected to act as engines of growth in the coming decades, municipal finances in India remain underdeveloped. Municipal revenue continues to account for a small share of GDP in India, and has remained stagnant at around 1 per cent of GDP during the period from 2007-08 to 2017-18 (Chart 2.1). The same ratio was 4.5 per cent for Poland, 6.0 per cent for South Africa, 7.4 per cent for Brazil, 13.9 per cent for the United Kingdom and 14.2 per cent for Norway in 2010 (OECD 2012).” The importance of stressing the issue of municipal finances in India was also most strongly felt by the Reserve Bank of India, and this authoritative organization of the country has published its first ever report [6] on the municipal finances of our country. This report (see, [6]) is the main source of the financial data utilized by us in this paper for our study, which augments our previous study (see, [29]) and *follows as a natural extension of the currently ongoing long term research work of the analysis of the finances and the budget of the Bangalore Municipal Corporation (also known as the Bruhat Bengaluru Mahanagara Palike (BBMP)) (see, A. Sathyavathi [24–28])*. We find it appropriate to mention a few other pertinent literary references dealing with the issue of municipal finances [1–19], *which are also already quoted by us as references in the first part of this study of ours (see, Sathyavathi and Upadhyaya [29])* .
We employ the statistical tools of curve fitting and regression as detailed in a very wide source of authentic literary sources, out of which we refer only to a quite few just for the sake of the interested readers ([30–39]). Our scheme of the paper includes the description of the data under analysis, which is given in Section 2 of the paper and is identical to the Section 2 of our previous paper (see, [29]). Because the analysis is becoming detailed for each category of the financial year, therefore, in this paper we only present our possible best fitting regression models for the data concerning the 2018-19 Budget Estimates category. In Section 3 of this paper we present a degree 4 polynomial regression model, a Rational Model an Exponential Association-2 Model for describing the trends of the budget data of this category. The comparisons of these three models are detailed in Section 4 and in Section 5 we present the results of our computations of the Polynomial Degree 4 Model and the Rational Model to the data under study using MATLAB. The conclusions of our study in this paper are summarized in Section 6.

2 Description of the data

We reproduce below the data for the sake of ready reference for the readers, with the mention that the same was also presented by us in of our precursor paper [29, Table 1 and Table 2, pp. 74–76]. The data of the revenue receipts in Table 1 and the revenue expenditures in Table 2 of the municipal corporations of the various States and Union Territories of India presented underneath is taken respectively from the Appendix I and the Appendix II of the Reserve Bank of India’s Report on Municipal Finances [6] and we gratefully acknowledge that source.

Table 1: Revenue Receipts of Municipal Corporations (in Rs. Lakhs).
Source: Report on Municipal Finances, The Reserve Bank of India, dated 10 November, 2022 [6, Appendix-I].

S. No.	State/ U.T.	2018-19 B.E.	2018-19 R.E.	2019-20 B.E.
1.	Andhra Pradesh	310,267.4	330,663.3	367,270.3
2.	Assam	22,717.2	22,882.4	25,628.3
3.	Bihar	76,853.6	68,109.6	128,265.8
4.	Chhattisgarh	140,923.1	117,838.6	174,391.7
5.	Goa	4,769.9	4,101.8	6,146.7
6.	Gujarat	1,057,062.3	987,043.9	1,167,004.8
7.	Haryana	290,213.6	290,147.8	357,421.3
8.	Himachal Pradesh	19,718.8	9,470.5	13,950.7

Continued on the next page

Table 1 – Continued from the previous page

S. No.	State / U.T.	2018-19 B.E.	2018-19 R.E.	2019-20 B.E.
9.	Jammu and Kashmir	38,844.5	38,844.5	42,008.8
10.	Jharkhand	51,493.4	50,049.3	47,752.4
11.	Karnataka	165,249.9	140,777.7	167,416.6
12.	Kerala	220,371.3	161,664.9	210,570.2
13.	Madhya Pradesh	309,051.2	309,051.2	340,266.3
14.	Maharashtra	5,245,861.0	4,686,343.2	5,786,325.1
15.	Mizoram	2,527.1	2,527.1	3,069.0
16.	Odisha	49,598.9	45,050.0	57,195.9
17.	Punjab	235,775.4	214,886.4	251,363.3
18.	Rajasthan	266,363.4	216,671.7	275,667.8
19.	Sikkim	1,750.8	1,953.5	2,289.9
20.	Tamil Nadu	596,614.8	578,100.9	679,189.5
21.	Telangana	424,327.3	382,657.4	420,324.2
22.	Tripura	21,775.7	21,775.7	20,706.5
23.	Uttar Pradesh	952,227.2	892,135.4	818,754.9
24.	Uttarakhand	29,419.2	30,794.5	37,295.1
25.	West Bengal	538,361.8	501,110.8	543,568.4
26.	Chandigarh	17,093.0	21,270.0	27,686.0
27.	Delhi	1,981,183.3	1,959,108.6	2,180,172.5

Abbreviations used in the table above: U.T. = Union Territory, Acc. = Accounts, B.E. = Budget Estimates, R.E. = Revised Estimates.

Table 2: Revenue Expenditures of Municipal Corporations (in Rs. Lakhs). *Source: Report on Municipal Finances, The Reserve Bank of India, dated 10 November, 2022 [6, Appendix-II].*

S. No.	State/ U.T.	2018-19 B.E.	2018-19 R.E.	2019-20 B.E.
1.	Andhra Pradesh	264,788.2	269,252.3	295,408.5
2.	Assam	34,588.7	36,877.5	41,302.8
3.	Bihar	144,940.1	114,048.3	178,718.0
4.	Chhattisgarh	174,513.5	131,923.2	199,207.2
5.	Goa	4,958.5	4,259.0	5,626.8
6.	Gujarat	916,796.7	889,215.1	993,408.0
7.	Haryana	388,350.7	386,773.7	470,735.2
8.	Himachal Pradesh	22,629.7	11,223.5	12,438.7
9.	Jammu and Kashmir	14,738.2	14,738.2	12,929.2
10.	Jharkhand	9,198.0	8,801.5	10,392.4
11.	Karnataka	143,549.8	121,141.4	141,929.1
12.	Kerala	179,834.6	136,188.7	198,039.8

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Table 2 – Continued from the previous page

S. No.	State / U.T.	2018-19 B.E.	2018-19 R.E.	2019-20 B.E.
13.	Madhya Pradesh	571,316.9	571,316.9	582,536.1
14.	Maharashtra	3,649,951.8	3,182,861.2	3,931,060.7
15.	Mizoram	2,991.0	2,597.5	3,399.2
16.	Odisha	66,004.4	68,081.5	77,283.1
17.	Punjab	160,259.8	150,027.0	177,364.1
18.	Rajasthan	186,264.1	185,050.5	220,279.6
19.	Sikkim	532.2	693.8	689.5
20.	Tamil Nadu	660,257.1	695,438.7	767,200.1
21.	Telangana	295,967.7	279,620.9	315,543.8
22.	Tripura	9,394.8	9,394.8	7,851.7
23.	Uttar Pradesh	681,307.5	658,777.1	539,027.8
24.	Uttarakhand	41,927.4	37,107.0	35,491.6
25.	West Bengal	515,960.9	492,488.8	506,576.3
26.	Chandigarh	65,351.0	64,734.0	79,441.0
27.	Delhi	1,988,092.3	1,992,980.9	2,146,466.1

Abbreviations used in the table above: U.T. = Union Territory, Acc. = Accounts, B.E. = Budget Estimates, R.E. = Revised Estimates.

3 Regression analysis of the municipal finances for the category 2018-19 Budget Estimates

In this section we fit regression models for the revenue receipts and the revenue expenditures of the municipal corporations of the twenty seven States and Union Territories of India for the category 2018-19 Budget estimates, the financial data pertaining to which is displayed in the third columns of Table 1 and Table 2 respectively. We present our results of investigations as below:

3.1 Polynomial regression model for the 2018-19 Budget Estimates category

As in our previous study [29], for an accurate analysis we also scale here the data of both the variables, viz., the revenue receipts (our predictor variable) and the revenue expenditures (our response variable) by a factor of 10^{-6}. The outcomes of our numerical experiments indicate that the data given in the third columns of Table 1 and Table 2 displaying the revenue receipts and the revenue expenditures of the municipal corporations of the Indian States can be modeled with a great measure of accuracy by a *fourth degree polynomial* given below in (3.1) with the predictor variable x representing the revenue receipts and the response variable y representing the revenue expenditures for this category:

$$y = a + bx + cx^2 + dx^3 + ex^4, \tag{3.1}$$

in which model the coefficients $a, b, \ldots, e$ (regression parameters) are given by

$a = -5.40488060256723\text{E} - 03$; $b = 1.30852791260652\text{E} + 00$; $c = -9.64604014383582\text{E} - 01$;
$d = 5.60145416765319\text{E} - 01$; $e = -7.59637319267380\text{E} - 02$

where the number $a = -5.40488060256723\text{E} - 03$; means the number $a = -5.40488060256723 \times 10^{-3}$, etc. The standard errors of the values of the regression parameters and their 95% confidence intervals are as shown below:

Parameter	Value	Std. Error	Range (95% Confidence)
a	-0.005405	0.027128	-0.061665 to 0.050855
b	1.308528	0.245529	0.799333 to 1.817723
c	-0.964604	0.455012	-1.908242 to -0.020966
d	0.560145	0.234152	0.074543 to 1.045748
e	-0.075964	0.029846	-0.137861 to -0.014066

The following display gives the standard deviations of the values of the parameters of regression in the Polynomial Degree 4 model of (3.1) and their uncertainties at 95% confidence level:

Parameter Standard Deviations	Parameter Uncertainties at 95% Confidence Level
a_stddev = 2.71281334061448E − 02	a_unc = 5.62603052535109E − 02
b_stddev = 2.45528632826750E − 01	b_unc = 5.09195219018691E − 01
c_stddev = 4.55012492659462E − 01	c_unc = 9.43638154086330E − 01
d_stddev = 2.34152443544330E − 01	d_unc = 4.85602446450501E − 01
e_stddev = 2.98462214583401E − 02	e_unc = 6.18972748611505E − 02

The other relevant information about the polynomial regression model of (3.1) is as shown below:

Name	Polynomial Regression (degree = 4)
Kind	Regression
Family	Linear Regressions
Equation	$y = a + b * x + c * x^2 + \ldots$
# of Indep. Vars	1
Standard Error	0.082273
Correlation Coeff. (r)	0.995170
Coeff. of Determination (r^2)	0.990363
DOF	22
AICc	− 130.587702

In Fig. 1 below we show the graphical representation of the fourth degree polynomial fit (3.1) to the data set and its Residual Plot is exhibited in Fig. 2 for the P-Value of 57.47% at 95% confidence level (it being understood that we have scaled the values of both the variables by a factor of 10^{-6} for convenience as already mentioned earlier). The Residuals Plot of Fig. 2 exhibits a pattern of randomly distributed pattern of the residuals, which indicates that our Polynomial Degree 4 Fit is a good fit to the data with fourteen number of runs and the P-Value of 0.5747 for the Wald-Wolfowitz runs test performed on the residuals (being greater than the critical value of 0.05) shows that the pattern is not unlikely.
A sample prediction table for predicting the response variable (revenue expenditure) in (3.1)) for given values of the predictor variable (revenue receipt) is shown in Table 3. Due to the inherent oscillating character of the polynomials at many points within and outside the range of the data values, the occurrence of some negative response values in Table 3 is neither unusual nor unexpected.
During the course of our investigations for the best possible polynomial fits to the financial data of the 2018-19 Budget Estimates category, the polynomial model of (3.1) gives the best fit to this data. To further corroborate this stand we present in Table 4 the predicted values of the revenue expenditures (the response variable) for the minimum and maximum values of the revenue receipts (the predictor variable) of data set in question. As observed in our previous study (see, Sathyavathi and Upadhyaya [29]), here again a negative predicted value at the lower end (minimum) of the data set is obtained, while at the upper end (maximum) of the data set, the predicted value is in excellent agreement with the observed (actual) value. In order to further appreciate the properties of the fitted polynomial model of degree 4 of (3.1), we graphically present the the evaluation for the maximum value of the predictor by this model in Fig. 3. In Fig. 4 we show that the slope of the curve of (3.1) at $x = 2.6238059$ is $\frac{dy}{dx} = 2.32678657452$. Fig. 5 demonstrates that the area under the entire

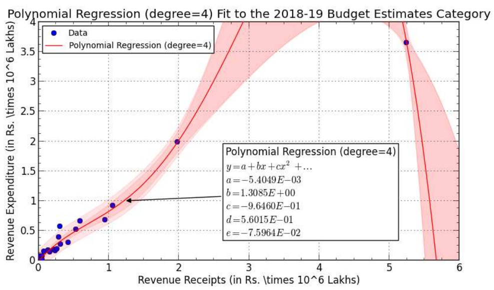

Fig. 1: Polynomial Degree 4 Fit to the 2018-19 Budget Estimates Category with Values of the Variables in Rs. $\times 10^6$ Lakhs.

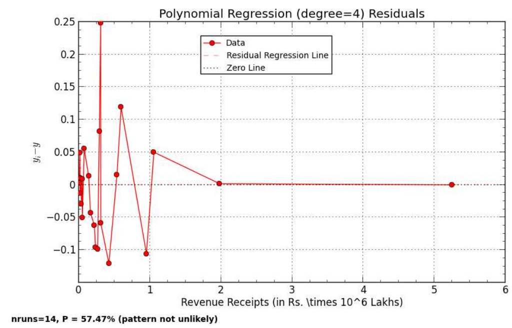

Fig. 2: Residual Plot of Polynomial Degree 4 Fit to the 2018-19 Budget Estimates Category with P-Value = 0.5747 at 95% Confidence Level.

Table 3: A representative prediction table for the Polynomial Degree 4 Fit of (3.1) to the 2018-19 Budget Estimates Category.

S. No.	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Revenue Expenditure y (in Rs. $\times 10^6$Lakhs)
1.	0.0000000000000000E+00	-5.4048806025672261E-03
2.	5.9999999999999998E-01	5.4360093214685623E-01
3.	1.2000000000000000E+00	9.8621171946008013E-01
4.	1.7999999999999998E+00	1.6939595537875682E+00
5.	2.3999999999999999E+00	2.8020989157948586E+00
6.	3.0000000000000000E+00	4.2096066943625647E+00
7.	3.5999999999999996E+00	5.5791821865863689E+00
8.	4.2000000000000002E+00	6.3372470977770341E+00
9.	4.7999999999999998E+00	5.6739455414603936E+00
10.	5.3999999999999995E+00	2.5431440393773599E+00

Table 4: Prediction table for the Polynomial Degree 4 Fit of (3.1) to the minimum and the maximum values of the 2018-19 Budget Estimates Category.

	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Observed Revenue Expenditure y_{obs} (in Rs. $\times 10^6$Lakhs)	Predicted Revenue Expenditure y_{pred} (in Rs. $\times 10^6$Lakhs) by (3.1)	Difference $= y_{\text{obs}} - y_{\text{pred}}$ (in Rs. $\times 10^6$Lakhs)
Minimum	0.001751	0.000532	-0.003116602698	0.003648602698
Maximum	5.245861	3.649952	3.64997165257	-0.00001965257

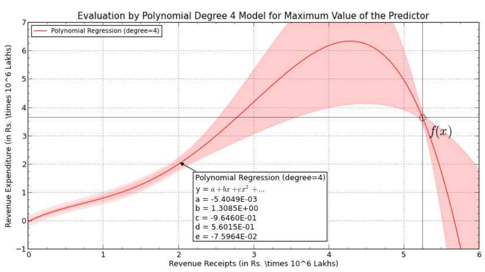

Fig. 3: Evaluation by Polynomial Degree 4 Model for the Maximum Value of the Predictor.

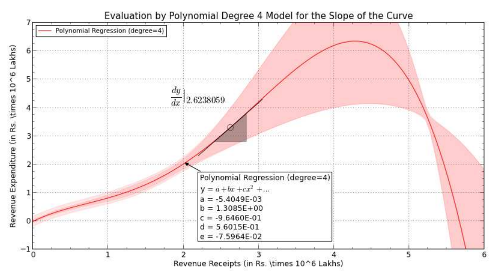

Fig. 4: Evaluation by Polynomial Degree 4 Model for the Slope of the Curve.

curve of (3.1) ranging from the minimum value of the revenue receipts (predictor) to its maximum value, i.e., from $x = 0.0017508$ to $x = 5.245861$ is 17.2526291472 (Rs. $\times$ 10^6Lakhs)^2, meaning thereby that $\int_{0.0017508}^{5.245861} f(x)\,dx = 17.2526291472$(Rs. $\times$ 10^6Lakhs)^2. The total arc length of the curve from $x = 0.0017508$ (minimum value) to $x = 5.245861$ (maximum value) is 10.8199111604 (in Rs. $\times 10^6$ Lakhs) as shown in Fig. 6. The polynomial model of (3.1) has a 5×5 covariance matrix of parameters which we give in Table 5 for the prospective researchers who may utilize the same for further researches on this model fit.

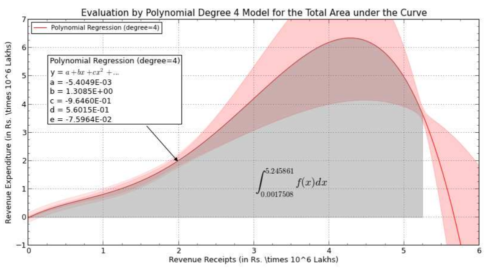

Fig. 5: Evaluation by Polynomial Degree 4 Model for the Total Area under the Curve.

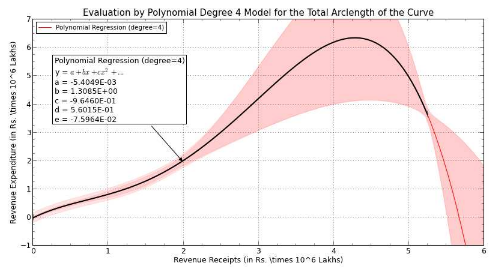

Fig. 6: Evaluation by Polynomial Degree 4 Model for the Total Arclength of the Curve.

Table 5: (5×5) Covariance Matrix of the Polynomial Degree 4 Fit given by (3.1) to the Financial Data of 2018-19 Budget Estimates Category.

Parameters	Entries of Covariance Matrix	Description
a	1.0872381442086132E−01 −6.9477771838185221E−01 1.0214348962816810E+00 −4.6455364132122895E−01 5.6108153488989607E−02	First row of the (5×5) covariance matrix
b	−6.9477771838190117E−01 8.9061323923382840E+00 −1.5650904504419712E+01 7.6161882184699445E+00 −9.4389624941813066E−01	Second row of the (5×5) covariance matrix
c	1.0214348962821003E+00 −1.5650904504423249E+01 3.0586620661590395E+01 −1.5565442175584225E+01 1.9627922950787626E+00	Third row of the (5×5) covariance matrix
d	−4.6455364132143773E−01 7.6161882184727698E+00 −1.5565442175586263E+01 8.0999482509870688E+00 −1.0305972344462471E+00	Fourth row of the (5×5) covariance matrix
e	5.6108153489037353E−02 −9.4389624941854544E−01 1.9627922950791270E+00 −1.0305972344463017E+00 1.3160232739928363E−01	Fifth row of the (5×5) covariance matrix

3.2 Nonlinear regression models for the category 2018-19 Budget Estimates

Now we discuss two nonlinear regression models that provide two best fits to the financial data of the 2018-19 Budget estimates category. The first is the Rational Model and the other is the Exponential Association 2 Model (which was also fit by us to the financial data of the 2017-18 Accounts category in our previous paper (see Sathyavathi and Upadhyaya [29]). The *Rational Model* belongs to the category of *Miscellaneous Models* and the *Exponential Association 2 Model* belongs to the general class of *Growth Models*.

3.2.1 Rational Model fit to the category 2018-19 Budget Estimates

This sub subsection discusses the details of the Rational Model fit to the category 2018-19 Budget Estimates. The model is described by (3.2) below:

$$y = \frac{a + bx}{1 + cx + dx^2} \tag{3.2}$$

in which a, b, c, d are the four regression parameters. In (3.2) y denotes the response (revenue expenditures) variable and x is the predictor (revenue receipts) variable. The values of the four regression parameters for the data of the 2018-19 Budget Estimates category are given by:

$$a = 3.05088001669627\text{E} - 02,\ \ b = 7.42495903948858\text{E} - 01,$$
$$c = -1.93663808920809\text{E} - 01,\ \ d = 3.96359305987297\text{E} - 02.$$

The standard errors and the 95% confidence intervals for the parameters of the model of (3.2) are given by

Parameter	Standard Error	Range (95% confidence)
a	0.025373	− 0.021980 to 0.082997
b	0.108054	0.518968 to 0.966024
c	0.079240	− 0.357583 to − 0.029744
d	0.010310	0.018309 to 0.060963

and the covariance matrix of this model is as given below in Table 6.

Table 6: (4×4) Covariance Matrix of the Rational Model Fit given by (3.2) to the Financial Data of 2018-19 Budget Estimates Category.

Parameters	Entries of Covariance Matrix	Description
a	7.7613138178333083E−02 −2.2655271574698488E−01 −1.2877422087979337E−01 1.3514010225522299E−02	First row of the (4×4) covariance matrix
b	−2.2655271574698968E−01 1.4075612453682171E+00 9.5860904653330992E−01 −1.1179419878674424E−01	Second row of the (4×4) covariance matrix
c	−1.2877422087979778E−01 9.5860904653331569E−01 7.5694973106220165E−01 −9.6069415765991636E−02	Third row of the (4×4) covariance matrix
d	1.3514010225522892E−02 −1.1179419878674535E−01 −9.6069415765991928E−02 1.2813770287026922E−02	Fourth row of the (4×4) covariance matrix

In Table 7 we mention the remaining relevant details of the Rational Model of (3.2). Fig. 7 demonstrates

Table 7: Description of the Rational Fit of (3.2) to the 2018-19 Budget Estimates Category.

Name: Rational Model
Kind: Regression
Family: Miscellaneous
Equation: $y = (a + b * x)/(1 + c * x + d * x^2)$
No. of Independent variables: 1
Standard Error: 9.10770455486030E-02
Correlation Coefficient (r): 9.93807495937196E-01
Coefficient of Determination (r^2): 0.987653
DOF 23
AICc -126.672445
Parameter Standard Deviations:
a_stddev=2.53732767274356E$-$02
b_stddev=1.08054431931130E$-$01
c_stddev = 7.92396326639633E$-$02
d_stddev = 1.03097325964646E$-$02
Parameter Uncertainties, 95%:
a_unc = 5.24886220034779E$-$02, b_unc = 2.23527622953839E-01
c_unc = 1.63919669157117E$-$01, d_unc = 2.13273067970617E-02

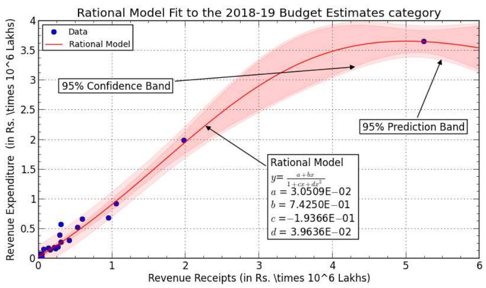

Fig. 7: Rational Model Fit to the 2018-19 Budget Estimates Category with the values of the Variables in Rs. $\times 10^6$ Lakhs.

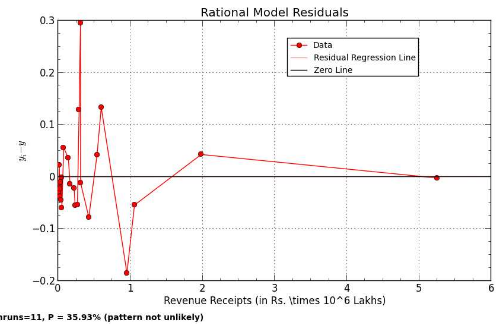

Fig. 8: Residual Plot of the Rational Model Fit to the 2018-19 Budget Estimates Category with P-Value =0.3593 at the 95% Confidence Level.

the plot of the Rational Model fit to the data along with the Confidence Band and the Prediction Band at 95% confidence level, (the narrower Confidence Band in darker pink color and the wider Prediction Band in lighter pink color), while Fig. 8 presents the residual plot for this model with a P-Value of 0.3593 as the outcome of a Wald-Wolfowitz runs test performed on the residuals showing eleven number of runs. Fig. 8 shows that the residuals are randomly scattered around the Residual Regression Line (which in this case again coincides with the Zero Line) hence showing that the Rational Model is a good fit to the data. The theoretical interpretation of the Confidence Band and the Prediction Band is that on repeated sampling the Confidence Band represents the region that has a 95% chance that the true Rational Model curve fitting the data in question will lie in this region, while the Prediction Band denotes the region in which there is a 95% probability that the values predicted by the fitted model will lie.

The graphs of the Residual History and the Parameter Histories for the Rational Model fit are shown by us in Fig. 9 and Fig. 10 respectively. From the plot of the Residual History (Fig. 9) we observe that the norm of the residual (the red colored line) attains a constant value just after the second iteration and the change in residual has fallen below the preset tolerance limit of 10^{-8} after the fifth iteration which indicates that the process of computation has converged, where we remark here that in our machine computations we preset the tolerance limit of 10^{-8} for both the change in the residual as well as the change in the parameters and the maximum number of iterations is set at 100. The Parameter Histories Plot of Fig. 10 shows that the parameters a (red colored line), c (blue colored line) and d (pink colored line) attain constancy in their values just after the first iteration, while the parameter b (green colored line) achieves this state after the second iteration. Hence we conclude that the proposed Rational Model Fit of (3.2) provides a good and robust fit to the 2018-19 Budget Estimates Category data.
The Table 8 below shows a representative prediction table for predicting the values of the revenue expenditures for given values of the response variable (the revenue receipts) based on the Rational Model of (3.2). Similarly in Table 9 the predicted values of the revenue expenditures for the given values of the minimum and the maximum revenue receipts based on the Rational Model Fit of (3.2). The agreement of the observed and the predicted values by the Rational Model fit of (3.2) is more closer at the upper (maximum) value of the data set as compared to that at the lower (minimum) value as was also seen above for the Polynomial Degree 4 Fit of (3.1). We can ascribe this to the presence of outside points and far outside points in the samples of the Revenue Receipts and the the Revenue Expenditures. The box-and-whisker plot of the Revenue Receipts sample under consideration presented in Fig. 11

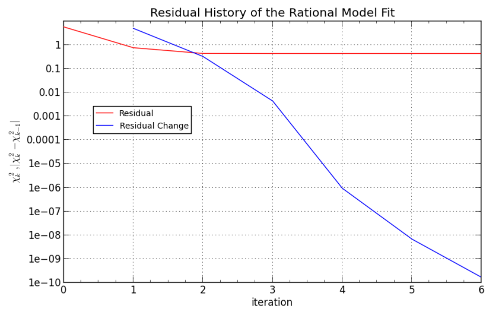

Fig. 9: The Residual History of the Rational Model Fit to the 2018-19 Budget Estimates Category.

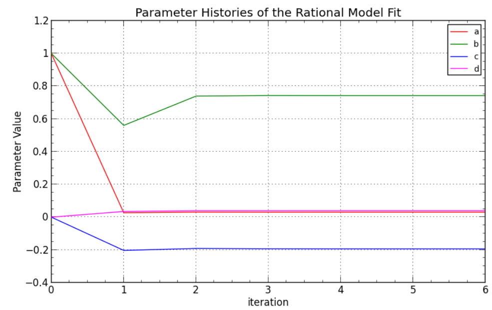

Fig. 10: The Parameter Histories of the Rational Model Fit to the 2018-19 Budget Estimates Category.

shows the presence of one *outside point* of Rs. 1.0570623×10^6 Lakhs (for Gujarat) and two *far outside points* of Rs. 1.9811833×10^6 Lakhs (for Delhi) and Rs. 5245861 Lakhs (for Maharashtra). Similarly the box-and-whisker plot of the sample of Revenue Expenditure in Fig. 12 indicates an *outside point* of Rs. 1.9880923×10^6 Lakhs (for Delhi) and a *far outside point* of Rs. 3.6499518×10^6 Lakhs (for Maharashtra). A detailed analysis of these sample values for possible *outliers* is reserved for our later study and investigations.

The evaluation for the maximum value of the predictor revenue receipts is shown in Fig. 13 for the Rational Model Fit of (3.2). The slope of the curve for the Rational Model Fit at $x = 2.6238059$ (i.e., for a value of Revenue Receipts of Rs. 2.6238059 $\times 10^6$ Lakhs is $\frac{dy}{dx} = 0.922438436746$ as seen from Fig. 14. The total area under the Rational Model Fit curve from the lower limit of Revenue Receipts of Rs. $x = 0.001751 \times 10^6$ Lakhs to its upper limit of Rs. $x = 5.245861 \times 10^6$ Lakhs is 12.0877301796 (Rs. $\times\, 10^6$ Lakhs)^2, i.e., $\int_{0.0017508}^{5.245861} f(x)\, dx = 12.0877301796 \big(\text{Rs.} \times 10^6 \text{Lakhs}\big)^2$ as shown in Fig. 15. It is evident from Fig. 16 that the entire arclength of the Rational Model curve from the lower limit of $x = 0.001751 \times 10^6$ Lakhs to its upper limit of $x = 5.245861 \times 10^6$ Lakhs is Rs. 6.58842211222 $\times 10^6$ Lakhs.

3.2.2 Exponential Association 2 Model fit to the category 2018-19 Budget Estimates

In this subsection we present another competing model, the *Exponential Association 2 Model* which belongs to the general class of *Growth Models* which has also emerged as one of the best nonlinear models that can be used to describe the behavior of the predictor (revenue receipts) and the response (revenue expenditures) variables of the 2018-19 Budget Estimates category. The Exponential Association 2 Model which we fit to the 2018-19 Budget Estimates category is defined by

$$y = a(1 - e^{-bx}), \tag{3.3}$$

in which a and b are the *two* regression parameters. Our numerical investigations led us to the following values of the parameters a and b:

$$a = 6.42132707294925E + 00, b = 1.62065935302595E - 01.$$

Table 10 details the necessary information about the Exponential Association 2 Model of (3.3) along with its covariance matrix, parameter standard deviations and the parameter uncertainties at the 95% confidence level. The 95% confidence intervals for the parameters of this model are given in Table 11. Fig. 17 shows the graphical representation of the Exponential Association 2 Model Fit of (3.3) to the 2018-19 Budget Estimates category and Fig. 18 demonstrates the Residual Plot of this model.

Table 8: A representative prediction table for the Rational Model Fit of (3.2) to the 2018-19 Budget Estimates Category.

S. No.	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Revenue Expenditure y (in Rs. $\times 10^6$Lakhs)
1.	0.0000000000000000E+00	3.0508800166962696E-02
2.	5.9999999999999998E-01	5.3003217810855729E-01
3.	1.2000000000000000E+00	1.1174089502271076E+00
4.	1.7999999999999998E+00	1.7529579677836811E+00
5.	2.3999999999999999E+00	2.3739039432406206E+00
6.	3.0000000000000000E+00	2.9107947867978852E+00
7.	3.5999999999999996E+00	3.3111092638854136E+00
8.	4.2000000000000002E+00	3.5550099264138284E+00
9.	4.7999999999999998E+00	3.6543266990654173E+00
10.	5.3999999999999995E+00	3.6396303687820750E+00
11.	6.0000000000000000E+00	3.5460878053702602E+00

Table 9: Prediction table for the Rational Model Fit of (3.2) for the minimum and the maximum values of the 2018-19 Budget Estimates Category.

	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Observed Revenue Expenditure y_{obs} (in Rs. $\times 10^6$Lakhs)	Predicted Revenue Expenditure y_{pred} (in Rs. $\times 10^6$Lakhs) by (3.2)	Difference= $y_{\text{obs}} - y_{\text{pred}}$ (in Rs. $\times 10^6$Lakhs)
Minimum	0.001751	0.000532	0.0318196968567	-0.0312876968567
Maximum	5.245861	3.649952	3.65230968594	-0.00235768594

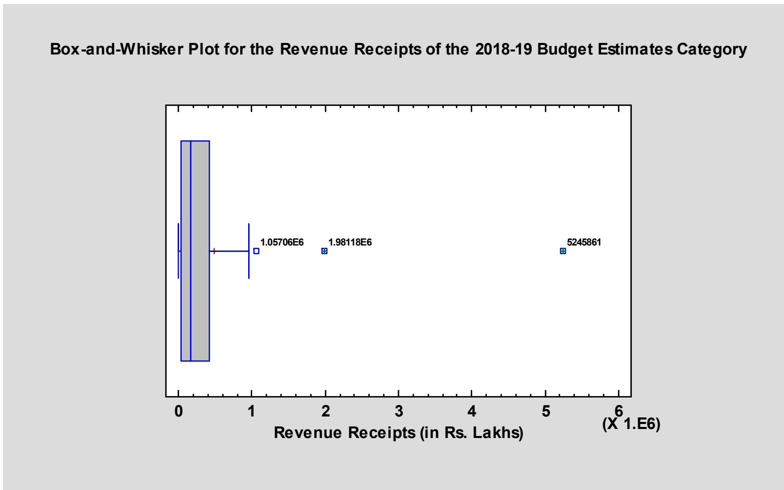

Fig. 11: The box-and-whisker plot for the Revenue Receipts sample corresponding to the 2018-19 Budget Estimates Category.

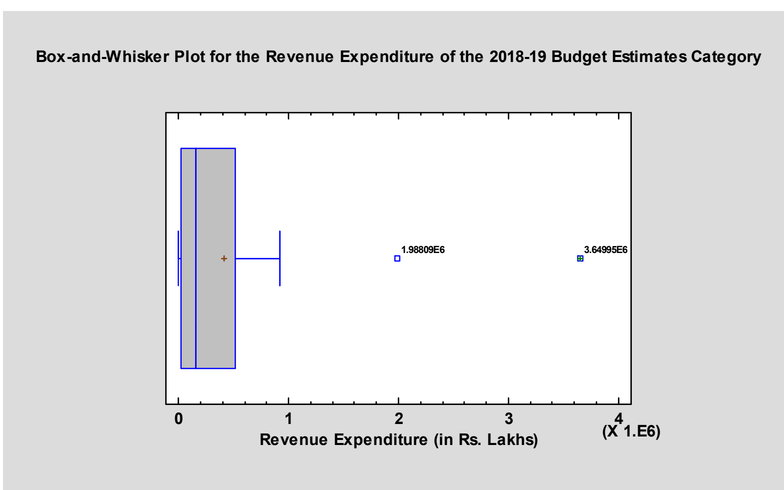

Fig. 12: The box-and-whisker plot for the Revenue Expenditures sample corresponding to the 2018-19 Budget Estimates Category.

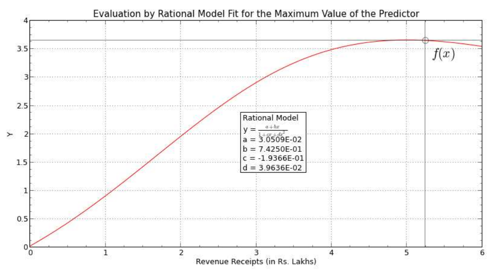

Fig. 13: Evaluation by Rational Model for the Maximum Value of the Predictor.

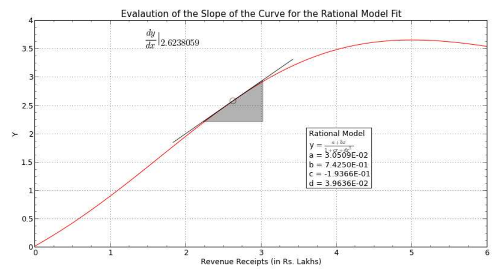

Fig. 14: Evaluation by Rational Model for the Slope of the Curve.

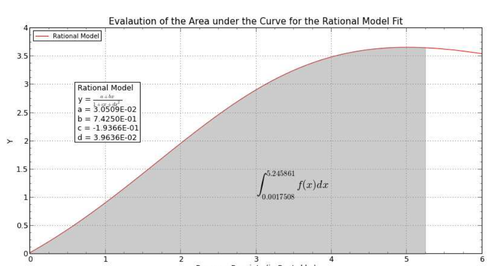

Fig. 15: Evaluation by Rational Model for the Area under the Predictor.

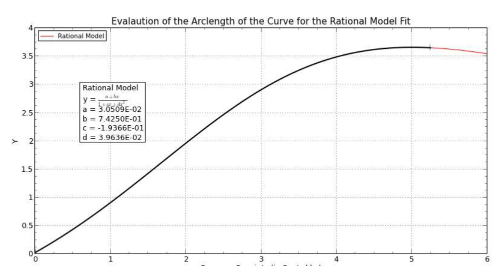

Fig. 16: Evaluation by Rational Model for the Arclength of the Curve.

The randomly scattered pattern of the residuals about the zero line in Fig. 18 indicates that the model of (3.2) is also a good fit to the data. The Residual History and the Parameter Histories of this model are depicted in Fig. 19 and Fig. 20. Fig. 19 shows that the norm of the residual (red colored line) has attained constancy after the second iteration and the change in residual (blue colored line) has declined below the preset tolerance limit of 10^{-8} after the fifth iteration. The Parameter Histories plot of Fig. 20 shows that the parameter a (red colored line) attains a constant value after the fourth iteration and the parameter b (green colored line) attains constancy most strictly after the third iteration. These conclusions lead us to the inference that the Exponential Association 2 Model of (3.3) also provides a robust fit to the data under of the 2018-19 Budget Estimates Category. Table 12 gives a representative table of prediction values made by the Exponential Asociation Model of (3.3) for the sample of 2018-19 Budget Estimates category. Table 13 gives the prediction estimates for the minimum and the maximum values of the predictor (revenue receipts) according to the Exponential Association 2 Model of (3.3). A comparison of Table 9 and Table 13 at once shows that the predicted values by the Exponential Association 2 Model for the minimum and the maximum values of the predictor (Revenue Receipts) are poorer than those given by the Rational Model, thereby, *indicating that the Exponential Association 2 Model is not as good as a fit to the data in question as the Rational Model is.* Yet we chose to describe the Exponential Association 2 Model fit here in detail, because *we found it a competing model* besides the Rational Model, among a host of other nonlinear regression models examined by us for this purpose during our investigations. This fact is also substantiated by the conclusion given in Table 16 below, where the result of the F test indicates that *there is a 6.15453% probability that the the simpler regression (Exponential Association 2) is the better fit to the data. A host of other available nonlinear regression models with us were also simultaneously examined by us for this purpose and as a result of comparison of the different nonlinear regression models, the Exponential Association 2 Model excelled the other nonlinear regression models available with us to describe the trend of the sample under consideration. This model of* (3.3) *is second only to the Rational Model of* (3.2)*, though the inference of the F-test during the comparison of these two models (see Table 16) weighed in favor of the Exponential Association 2 Model, which fact led us to describe the outcome of this model in detail here, so that the interested researchers may decide for themselves which model of these two would be better to study the behavior of the given sample of the 2018-19 Budget Estimates category.*

As in the case of the fits of (3.1) and (3.2), Fig. 21 demonstrates the evaluation for the maximum value of the revenue receipts (predictor). Fig. 22 presents the slope of the Exponential Association 2 curve at $x = 2.6238059$, which yields a value of $\frac{dy}{dx} = 0.680207901116$. Fig. 23 shows the total area under the Exponential Association 2 curve from the lower limit of revenue receipts $x =$ Rs 0.0017508×10^6 Lakhs to its upper limit of $x =$ Rs. 5.245861×10^6 Lakhs is $10.9956270821\ \left(\text{Rs. } 10^6 \text{ Lakhs}\right)^2$, i.e., $\int_{0.0017508}^{5.245861} f(x)\,dx = 10.9956270821\left(\text{Rs.} \times 10^6 \text{Lakhs}\right)^2$. Lastly, Fig. 24 gives the entire arclength of the Exponential Association 2 curve from the lower limit of revenue receipts $x =$ Rs 0.0017508×10^6 Lakhs to its upper limit of $x =$ Rs. 5.245861×10^6 Lakhs is Rs. $6.44519223765 \times 10^6$ Lakhs.

4 Comparison of the Polynomial Degree 4, the Rational and the Exponential Association 2 models

In this Section we compare the three models of Section 3 fitted by us to the data set of the 2018-19 Budget Estimates Category.

4.1 Comparison of the Polynomial Degree 4 and the Rational models

Table 14 presents the details of the results of the comparison between the Polynomial Degree 4 of (3.1) and the Rational Model of (3.2). Fig. 25 depicts the results of this comparison graphically. The Akaike Information Criterion (AIC) test (more precisely, the AICc, which is the AIC corrected for small samples) (for example, see [39]) and the usual F-test are our tools of comparison of these two models. The results of both the AIC test and the F test clearly indicate that *the Polynomial Degree Model of* (3.1) *is the better fit to the data of 2018-19 Budget Estimates category as compared to the Rational Model given by* (3.2).

Table 10: Description of the Exponential Association 2 Model Fit of (3.3) to the 2018-19 Budget Estimates Category.

Name: Exponential Association 2
Kind: Regression
Family: Growth Models
Equation: $a * (1 - \exp(-b * x))$
No. of Independent variables: 1
Parameters:
a = 6.42132707294925E+00
b = 1.62065935302595E-01
Standard Error: 9.86158131566630E-02
Coefficient of Determination (r^2): 9.84266079574036E-01
Correlation Coefficient (r): 9.92101849395532E-01
DOF 25
AICc -125.010225
Covariance matrix:
7.1253134755295008E+01 −2.4340923946529722E+00
−2.4340923946529722E+00 8.5648156378119220E-02
Parameter Standard Deviations:
a_stddev = 8.32431580561327E-01
b_stddev = 2.88606142192433E-02
Parameter Uncertainties, 95%:
a_unc = 1.71442493269541E+00
b_unc = 5.94395476406715E-02

Table 11: 95% Confidence Intervals for the Parameters of the Exponential Association 2 Model Fit of (3.3) to the 2018-19 Budget Estimates Category.

Parameter	Value	Standard Error	Range (95% Confidence)
a	6.421327	0.832432	4.706902 to 8.135752
b	0.162066	0.028861	0.102626 to 0.221505

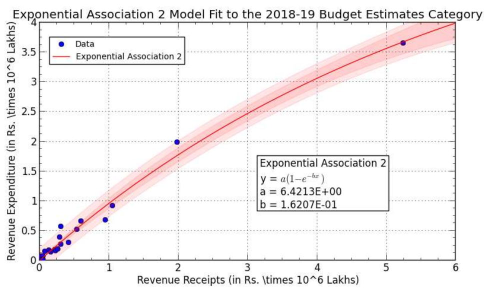

Fig. 17: Exponential Association 2 Model Fit to the 2018-19 Budget Estimates Category with the values of the Variables in Rs. $\times 10^6$ Lakhs.

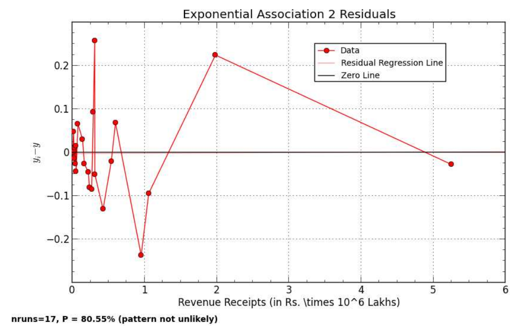

Fig. 18: Residual Plot of the Exponential Association 2 Model Fit to the 2018-19 Budget Estimates Category with P-Value =0.7472 at 95% Confidence Level.

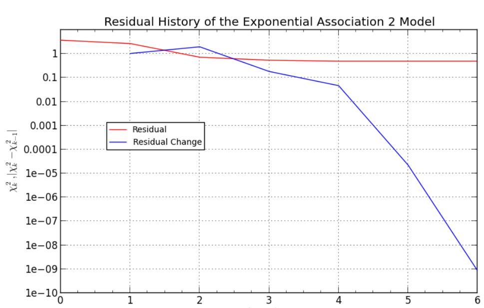

Fig. 19: The Residual History of the Exponential Association 2 Model Fit to the 2018-19 Budget Estimates Category.

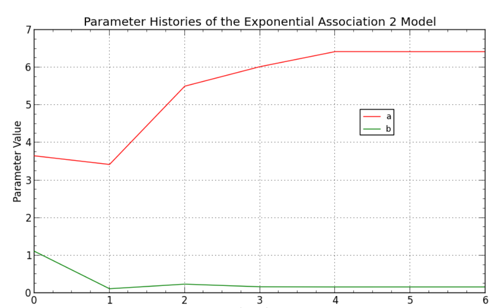

Fig. 20: The Parameter Histories of the Exponential Association 2 Model Fit to the 2018-19 Budget Estimates Category.

Table 12: A representative prediction table for the Exponential Association 2 Model Fit of (3.3) to the 2018-19 Budget Estimates Category.

S. No.	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Revenue Expenditure y (in Rs. $\times 10^6$Lakhs)
1.	0.0000000000000000E+00	0.0000000000000000E+00
2.	5.9999999999999998E-01	5.9500904737656934E-01
3.	1.2000000000000000E+00	1.1348837335162851E+00
4.	1.7999999999999998E+00	1.6247328845830502E+00
5.	2.3999999999999999E+00	2.0691919358428299E+00
6.	3.0000000000000000E+00	2.4724667967189680E+00
7.	3.5999999999999996E+00	2.8383736512508002E+00
8.	4.2000000000000002E+00	3.1703750705867355E+00
9.	4.7999999999999998E+00	3.4716127792437788E+00
10.	5.3999999999999995E+00	3.7449373852001355E+00
11.	6.0000000000000000E+00	3.9929353551563174E+00

Table 13: Prediction table for the Exponential Association 2 Model Fit of (3.3) for the minimum and the maximum values of the 2017-18 Accounts Category.

	Revenue Receipt x (in Rs. $\times 10^6$Lakhs)	Observed Revenue Expenditure y_{obs} (in Rs. $\times 10^6$Lakhs)	Predicted Revenue Expenditure y_{pred} (in Rs. $\times 10^6$Lakhs) by (3.2)	Difference= $y_{\text{obs}} - y_{\text{pred}}$ (in Rs. $\times 10^6$Lakhs)
Minimum	0.001751	0.000532	0.00182196931068	-0.00128996931068
Maximum	5.245861	3.649952	3.67723727071	-0.02728527071

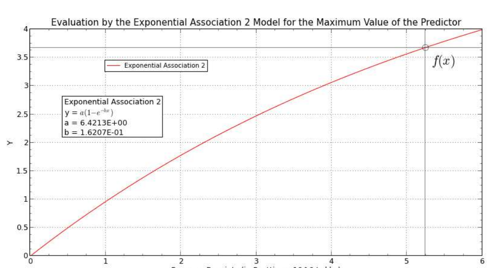

Fig. 21: Evaluation by the Exponential Association 2 Model for the Maximum Value of the Predictor.

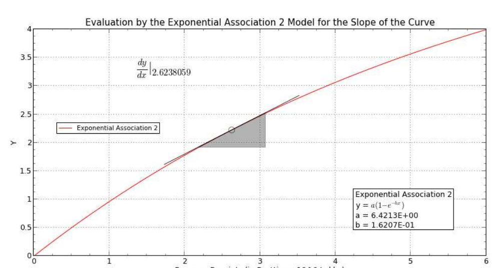

Fig. 22: Evaluation by Exponential Association 2 Model for the Slope of the Curve.

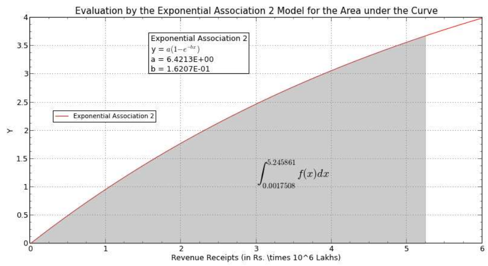

Fig. 23: Evaluation by Exponential Association 2 Model for the Area under the Curve.

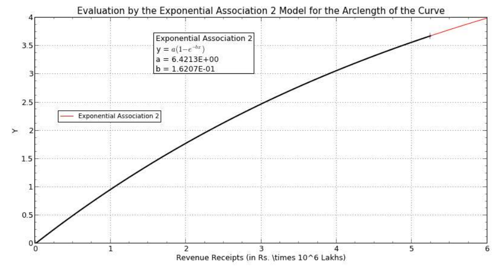

Fig. 24: Evaluation by Exponential Association 2 Model for the Arclength of the Curve.

Table 14: Comparison of the Polynomial Degree 4 and the Rational Models for the data set of the 2018-19 Budget Estimates Category.

Overview

Model	**Sum of Squares**	**DOF**	**AICc**
Polynomial Regression (degree=4)	0.148915	22	−130.588
Rational Model	0.190786	23	−126.672

Comparison

The best regression is (chosen via the AIC test): **Polynomial Regression (degree=4)**

The best regression is (chosen via the F-Test): **Polynomial Regression (degree=4)**

Justification

Akaike's Information Criterion

Model	**AICc**
Polynomial Regression (degree=4)	**−130.588**
Rational Model	**−126.672**

Delta = 3.91526

Probability = 0.123724

The likelihood that Polynomial Regression (degree=4) is the better model is 87.6276%.

Continued on the next page

Table 14 – Continued from the previous page

F-Test

The more complicated model (Polynomial Regression (degree=4)) has a better sum of squares than the simpler model (Rational Model). Performing an F test. Since the result of an F test is strictly not valid unless the simpler model is a subset of the more complex model; i.e., the simpler model can be obtained by setting some parameters in the more complex model to certain constants. As the software cannot currently tell if this is the case, so it presents F-Test results regardless of this fact.

F = 6.18581

P = 0.0209472

If the simpler model were correct, the sum of squares would increase by approximately the gain in degrees of freedom when moving from the complex model to the simple one (i.e., F = 1). There is a 2.09472% probability that the the simpler regression (Rational Model) is the better fit to the data.

4.2 Comparison of the Polynomial Degree 4 and the Exponential Association 2 models

Table 15 presents the details of the results of the comparison between the Polynomial Degree 4 of (3.1) and the Exponential Association 2 Model of (3.3). Fig. 26 depicts the results of this comparison graphically. Once again, as in the Subsection 4.1, *the outcomes of the AIC test and the F test indicate that the Polynomial Degree 4 Model of* (3.1) *is the better fit to the data as compared to the Exponential Association 2 Model of* (3.3).

Table 15: Comparison of the Polynomial Degree 4 and the Exponential Association 2 Models for the data set of the 2018-19 Budget Estimates Category.

Overview

Model	**Sum of Squares**	**DOF**	**AICc**
Polynomial Regression (degree=4)	0.148915	22	−130.588
Exponential Association 2	0.243127	25	−125.01

Comparison

The best regression is (chosen via the AIC test): **Polynomial Regression (degree=4)**

The best regression is (chosen via the F-Test): **Polynomial Regression (degree=4)**

Justification

Akaike's Information Criterion

Model	**AICc**
Polynomial Regression (degree=4)	**−130.588**
Exponential Association 2	**−125.01**

Delta = 5.57748

Probability = 0.0579358

The likelihood that Polynomial Regression (degree=4) is the better model is 94.2064%.

Continued on the next page

Table 15 – Continued from the previous page

F-Test
The more complicated model (Polynomial Regression (degree=4)) has a better sum of squares than the simpler model (Exponential Association 2). Performing an F test. Since the result of an F test is strictly not valid unless the simpler model is a subset of the more complex model; i.e., the simpler model can be obtained by setting some parameters in the more complex model to certain constants. As the software cannot currently tell if this is the case, so it presents F-Test results regardless of this fact.
F = 4.63949
P = 0.0116439
If the simpler model were correct, the sum of squares would increase by approximately the gain in degrees of freedom when moving from the complex model to the simple one (i.e., F = 1). There is a 1.16439% probability that the the simpler regression (Rational Model) is the better fit to the data.

4.3 Comparison of the Rational and the Exponential Association 2 models

Table 16 presents the details of the results of the comparison between the Rational Model of (3.2) and the Exponential Association 2 Model of (3.3). Fig. 27 depicts the results of this comparison graphically. The result of the AIC test indicates that the Rational Model is the better model, whereas, the result of the F test declares that the Exponential Association 2 Model is the better fit to the data. We must mention here that while the results of the AIC test are valid in all cases, the results of the F test are valid only when the simpler model can be obtained from the more complex model by making a suitable choice of some of its parameter values. This thing may not be possible in all the cases, and may require further considerations and probing of another related factors depending on the cases in hand, thus, without going into the details of the same, we present our results of the F test in Table 16. As already remarked by us in Sathyavathi and Upadhyaya [29, Subsection 3.3, p. 85], in case of a contradiction between the results of the AIC test and the F test, *we personally prefer to rely on the results of the AIC test to infer that the Rational Model is a better fit to the sample under study as compared to the Exponential Association 2 Model. However, we also mention that since, the P-Value of the F test is, 6.15453% at the 95% confidence level, so we leave it for the interested researcher to consider also the Exponential Association 2 Model for the sample of the 2018-19 Budget Estimates presented in Table 1 and Table 2. Our decision to include the details of the Exponential Association 2 Model Fit to the sample under study was also much motivated by the fact that the P-Value of the F test is, 6.15453% at the 95% confidence level, in contrast to the P-Value of the F-test conducted by us in the previous study of ours (see, Sathyavathi and Upadhyaya [29, Table 10, p. 89]) which was reported as 2.917% at the 95% confidence level .*

Table 16: Comparison of the Rational and the Exponential Association 2 Models for the data set of the 2018-19 Budget Estimates Category.

Overview

Model	**Sum of Squares**	**DOF**	**AICc**
Rational Model	0.190786	23	−126.672
Exponential Association 2	0.243127	25	−125.01

Comparison

The best regression is (chosen via the AIC test): **Rational Model**

The best regression is (chosen via the F-Test): **Exponential Association 2**

Continued on the next page

Table 16 – Continued from the previous page

Justification

Akaike's Information Criterion

Model	**AICc**
Rational Model	**−126.672**
Exponential Association 2	**−125.01**

Delta = 1.66222

Probability = 0.30341

The likelihood that Polynomial Regression (degree=4) is the better model is 69.659%.

F-Test

The more complicated model (Rational Model) has a better sum of squares than the simpler model (Exponential Association 2). Performing an F test. Since the result of an F test is strictly not valid unless the simpler model is a subset of the more complex model; i.e., the simpler model can be obtained by setting some parameters in the more complex model to certain constants. As the software cannot currently tell if this is the case, so it presents F-Test results regardless of this fact.

F = 3.15498

P = 0.0615453

If the simpler model were correct, the sum of squares would increase by approximately the gain in degrees of freedom when moving from the complex model to the simple one (i.e., F = 1). There is a 6.15453% probability that the the simpler regression (Exponential Association 2) is the better fit to the data.

5 Fitting of the above models using MATLAB

In this section we fit the above three models, viz., the Polynomial Degree 4 Model, the Rational Model and the Exponential Association 2 Model to the financial data of 2018-19 Budget Estimates Category using MATLAB.

5.1 Fitting the Polynomial Degree 4 Model using MATLAB

This subsection is devoted to the discussion of our results obtained by fitting the Polynomial Degree 4 Model using MATLAB.

The model fitted by us here is described by the following equation:

$$f(x) = p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5 \tag{5.1}$$

The description of this fit is given in Fig. 28, from where we can easily read the values of the five parameters $p_1, \ldots, p_5$ along with their respective 95% confidence intervals. The goodness of fit parameters listed include the SSE (the sum of squares due to error, which 'measures the deviation of the responses (Revenue Expenditures) from the fitted values of the responses (Revenue Rxpenditures)'), R-square (the coefficient of multiple determination, which 'measures how successful the fit is in explaining the variation of the data'). We observe that the value of R-square is 0.9904, which is close to 1, this indicates that our proposed Polynomial Degree 4 Fit of (5.1) is a very good fit to the data of 2018-19 Budget Estimates. The Adjusted R-square (which is 'the degree of freedom adjusted R-square') has a value of 0.9886, which is also close to 1, signifying that our proposed fit of (5.1) is a very good fit to the data under consideration. The Adjusted R-square is considered to be 'generally the best indicator of the fit quality when we add additional coefficients to our model'. The RMSE (i.e., the root mean squared error) has a value 8.227e+004 (i.e., 8.227×10^4), *which being far from the value of zero , does indicate the fact that the proposed Polynomial Degree 4 Fit of* (5.1) *is not a good fit to the data. But we mention here that we also investigated a number of other Polynomial Fits to this data of 2018-19 Budget Estimates Category, and we found that a Polynomial of Degree 4 provides the best fit to this data*

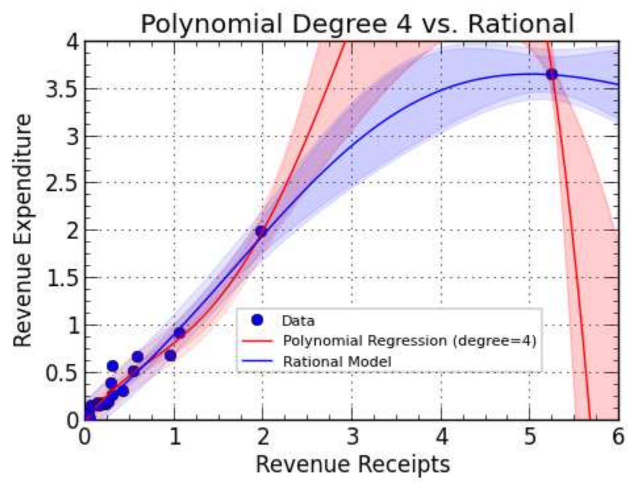

Fig. 25: The comparison of the Polynomial Degree 4 Model and the Rational Model (where both the variables are in Rs. $\times 10^6$ Lakhs) for the data of 2018-19 Budget Estimates Category.

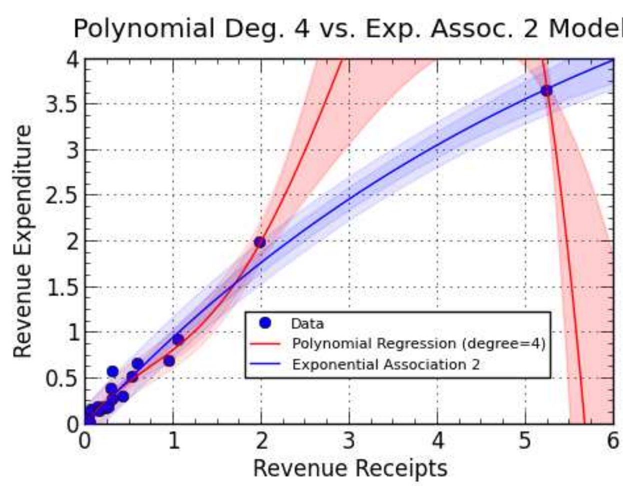

Fig. 26: The comparison of the Polynomial Degree 4 Model and the Exponential Association 2 Model (where both the variables are in Rs. $\times 10^6$ Lakhs) for the data of 2018-19 Budget Estimates Category.

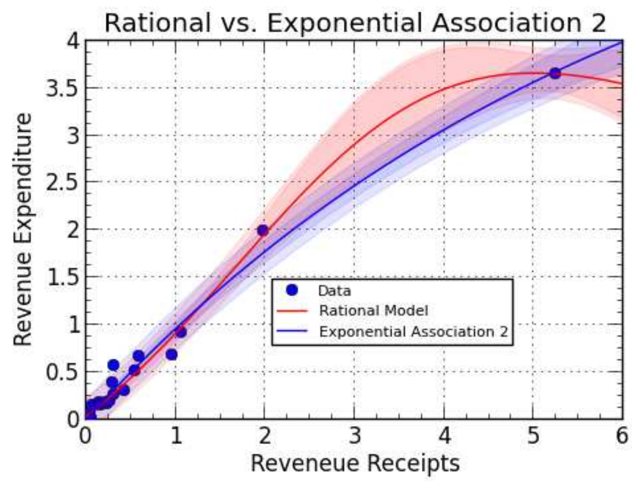

Fig. 27: The comparison of the Rational Model and the Exponential Association 2 Model (where both the variables are in Rs. $\times 10^6$ Lakhs) for the data of 2018-19 Budget Estimates Category.

as compared to the other possible Polynomials as is also supported by our arguments in the Subsection 3.1. The remarkable similarity between Fig. 1 and Fig. 2 and the two plots of Fig. 30 produced by MATLAB, specially the fact that the portions of the Residual Plots of Fig. 2 and the lower Plot of Fig. 30 for the two largest values of the predictor (Revenue Receipts) lend sufficient credence to the validity of our conclusions drawn in Subsection 3.1. Fig. 29 gives the Table of Fits, which summarizes the various characteristics of the fit of (5.1) to the data of this study. The various Plots of the Analysis for the Polynomial Degree 4 Fit of (5.1) are shown in Fig. 31, which include the Plots of the Fit with 95% Prediction Bounds, the first and second derivatives of the Fit as well as the plot of the Integral of the Fit from the minimum value of the predictor starting at Rs. 1750.8 Lakhs corresponding to the Revenue Receipts of Sikkim in the second column of Table 1 under the 2018-19 Budget Estimates Category. The table in Fig. 32 gives the values of the response (Revenue Expenditures) for some representative values of the predictor (Revenue Receipts) starting from the minimum value of the predictor from Rs. 1750.8 Lakhs at steps of Rs. 524411 Lakhs along with the other necessary parameters whose graphical plot is depicted in Fig. 31.

5.2 Rational Model Fit using MATLAB

Now we proceed to fit a Rational Model using MATLAB to the 2018-19 Budget Estimates Category, similar to the Rational Model Fit of (3.2). We choose in MATLAB a rational expression which has linear polynomial as its numerator and a quadratic polynomial as its denominator. One such expression is given by

$$f(x) = \frac{p_1 x + p_2}{x^2 + q_1 x + q_2} \tag{5.2}$$

which we shall fit to the data in hand. The description of the Rational Model of (5.2) is shown in Fig. 33 below and the Table of Fits for this model is displayed in Fig. 34. Fig. 35 depicts the Plot of the Model and its Residuals and the different types of Analysis Plots of this model are shown in Fig. 36. Finally Fig. 37 presents the Prediction Table for the Rational Model of (5.2). *We must mention here that the Rational Model Fit of (5.2) using MATLAB is not in agreement with the Rational Model Fit of (3.2) as can be seen from the comparison of Fig. 7, Fig. 8 and Fig. 35.*

Description of the Polynomial Degree 4 Fit for the 2018-19 Budget Estimates Category using MATLAB

Fit Name: Polynomial Degree 4 Fit
Data Set: REVEXP vs. REVREC
Exclusion Rule: None
Type of Fit: Polynomial

Results:
Linear model Poly4:
f(x) = p1*x^4 + p2*x^3 + p3*x^2 + p4*x + p5
where x is normalized by mean 4.841e+005 and std 1.047e+006
Coefficients (with 95% confidence bounds):
p1 = -9.115e+004 (-1.654e+005, -1.688e+004)
p2 = 4.735e+005 (5.39e+004, 8.932e+005)
p3 = -2.825e+005 (-6.578e+005, 9.269e+004)
p4 = 7.682e+005 (5.868e+005, 9.495e+005)
p5 = 4.614e+005 (3.993e+005, 5.235e+005)

Goodness of fit:
SSE: 1.489e+011
R-square: 0.9904
Adjusted R-square: 0.9886
RMSE: 8.227e+004

Fig. 28: The description of the Polynomial Degree 4 Model Fit of (5.1) for the data of 2018-19 Budget Estimates Category using MATLAB.

Table of Fits for the Polynomial Degree 4 Fit for the 2018-19 Budget Estimates using MATLAB

Name	Data set	Type	SSE	R-squared	Adj R-squared	RMSE	#Coeff	DFE
Polynomial Degree 4 Fit	REVEXP vs. REVREC	Polynomial	1.4891478718326154E11	0.9903630047341944	0.9886108237767751	82273.04517808688	5.0	22.0

Fig. 29: The Table of Fits for the Polynomial Degree 4 Model Fit of (5.1) for the data of 2018-19 Budget Estimates Category using MATLAB.

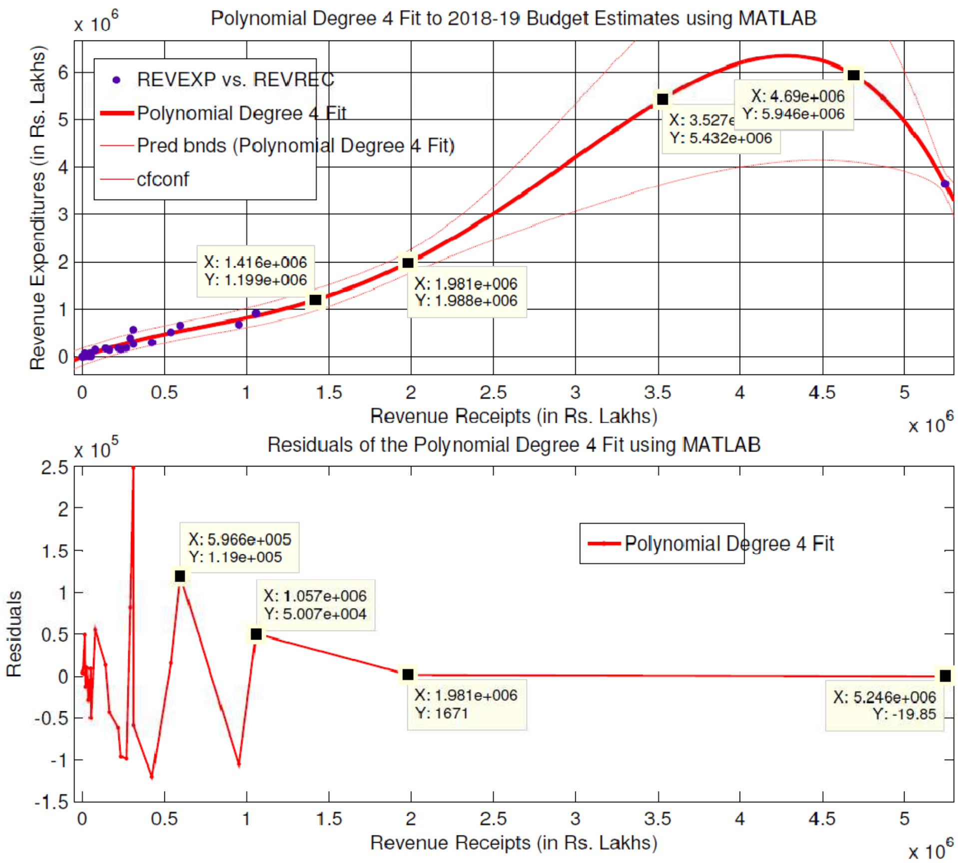

Fig. 30: The Plot and the Residual Plot of the Polynomial Degree 4 Model Fit of (5.1) for the data of 2018-19 Budget Estimates Category using MATLAB.

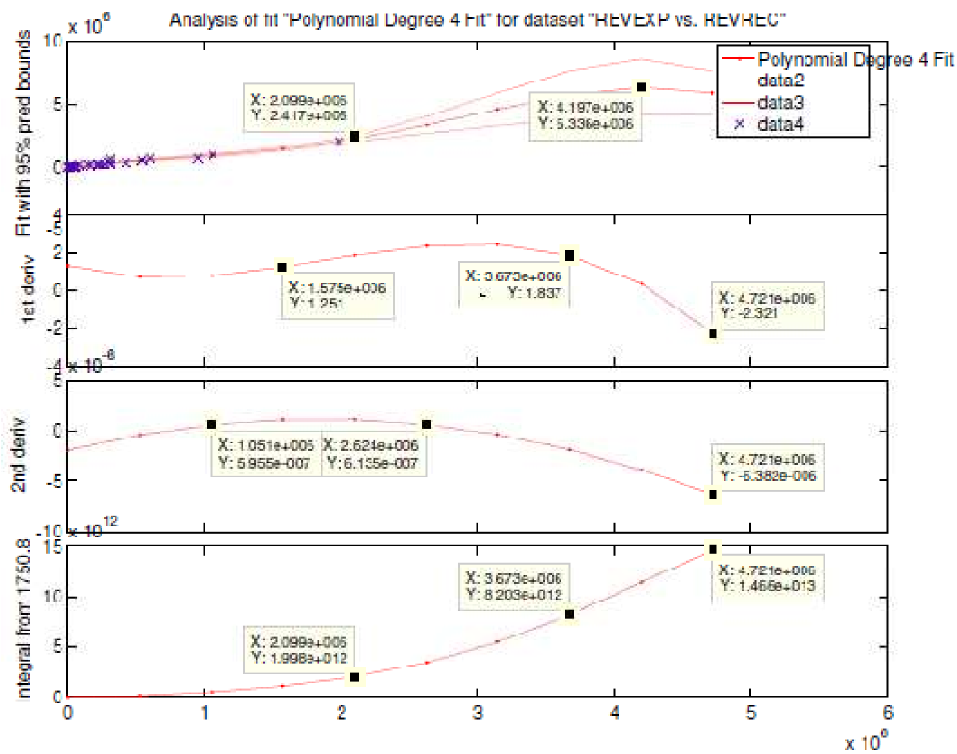

Fig. 31: The various Analysis Plots for the Polynomial Degree 4 Model Fit of (5.1) for the data of 2018-19 Budget Estimates Category using MATLAB.

Prediction Table for the Polynomial Degree 4 Model for 2018-19 Budget Estimates using MATLAB

Xi	lower f(Xi)	f(Xi)	upper f(Xi)	df(Xi)/dX	d2f(Xi)/dX2	Integral f(Xi)
1750.8	-58752.9	-3116.86	52519.2	1.30516	-1.92333e-006	0
526162	428996	491817	554639	0.714413	-4.13209e-007	1.41578e+011
1.05057e+006	739967	861627	983286	0.784131	5.95536e-007	4.9476e+011
1.57498e+006	1.2334e+006	1.38372e+006	1.53405e+006	1.25138	1.10291e-006	1.07269e+012
2.09939e+006	1.97787e+006	2.19763e+006	2.4174e+006	1.85324	1.10891e-006	1.99785e+012
2.62381e+006	2.63863e+006	3.30501e+006	3.97139e+006	2.32679	6.13531e-007	3.42972e+012
3.14822e+006	3.25358e+006	4.56963e+006	5.88568e+006	2.40909	-3.83217e-007	5.49251e+012
3.67263e+006	3.77284e+006	5.71737e+006	7.66189e+006	1.83722	-1.88134e-006	8.20282e+012
4.19704e+006	4.09939e+006	6.33623e+006	8.57307e+006	0.348257	-3.88083e-006	1.13974e+013
4.72145e+006	4.10665e+006	5.87635e+006	7.64606e+006	-2.32073	-6.3817e-006	1.46606e+013

Fig. 32: The Prediction Table for the Polynomial Degree 4 Model Fit of (5.1) for the data of 2018-19 Budget Estimates Category using MATLAB.

5.3 Fitting the Exponential Association 2 Model using MATLAB

Our attempts to fit the Exponential Association 2 Model to the data of 2081-19 Budget Estimates Category using MATLAB showed that the 'fit computation did not converge', therefore, we do not discuss those results here.

6 Conclusion

In this study we put forward three possible regression models for the financial data of the Municipal Corporations of the Indian States, belonging to the 2018-19 Budget Estimates Category - which includes a Polynomial Degree 4 Model, a Rational Model and an Exponential Association 2 Model. Going by the comparative results of the AIC test (which is valid in all the cases) as discussed in Section 4, we conclude that the Polynomial Degree 4 Model of (3.1) is the best fit to the sample of the 2018-19 Budget Estimates Category, followed by the Rational Model of (3.2) and then followed by the Exponential Association 2 Model of (3.3). Because the P-Value of the F test performed by us as reported in Table 16 is 0.0615453, which being greater than the critical value of 0.05 at the 95% confidence level, we also decided to include the full details of the Exponential Association 2 Model in this paper so that the interested readers may further explore the problem at hand. The presence of possible outliers as mentioned in the sub subsection 3.2.1 and demonstrated in Fig. 11 and Fig. 12 would be a matter of future investigation of ours. In Section 5 we also presented our computations of the Polynomial Degree 4 Model and the Rational Model using MATLAB and found that the Polynomial Degree 4 Model fitted by using MATLAB is in perfect agreement with the results of computation of this Model discussed by us in Subsection 3.1.

We also intend to propose further regression models for the financial data of the municipal corporations of the Indian Sates for the 2018-19 Revised Estimates and the 2019-20 Budget Estimates Categories as mentioned in the fourth, and fifth columns respectively of Table 1 and Table 2 of this paper.

Description of the Rational Model Fit for the 2018-19 Budget Estimates Category using MATLAB

Fit Name: Rational Model Fit
Data Set: REVEXP vs. REVREC
Exclusion Rule: None
Type of Fit: Rational

Results:
General model Rat12:
f(x) = (p1*x + p2) / (x^2 + q1*x + q2)
where x is normalized by mean 4.841e+005 and std 1.047e+006
Coefficients (with 95% confidence bounds):
p1 = 1.83 (-2.956e+007, 2.956e+007)
p2 = -2.151 (-3.595e+007, 3.595e+007)
q1 = 0.8969 (-8.108e+007, 8.108e+007)
q2 = -10.1 (-1.449e+008, 1.449e+008)

Goodness of fit:
SSE: 2.009e+013
R-square: -0.3004
Adjusted R-square: -0.47
RMSE: 9.347e+005

Fig. 33: The Description of the Rational Model Fit of (5.2) for the data of 2018-19 Budget Estimates Category using MATLAB.

Table of Fits for the Rational Model Fit for the 2018-19 Budget Estimates using MATLAB

Name	Data set	Type	SSE	R-squared	Adj R-squared	RMSE	#Coeff	DFE
Rational Fit to 2018-19 Budget Estimates Category	REVEXP vs. REVREC	Rational	2.00937389436255E13	-0.3003629171745481	-0.4699754715886195	934687.5540178608	4.0	23.0

Fig. 34: The Table of Fits for the Rational Model Fit of (5.2) for the data of 2018-19 Budget Estimates Category using MATLAB.

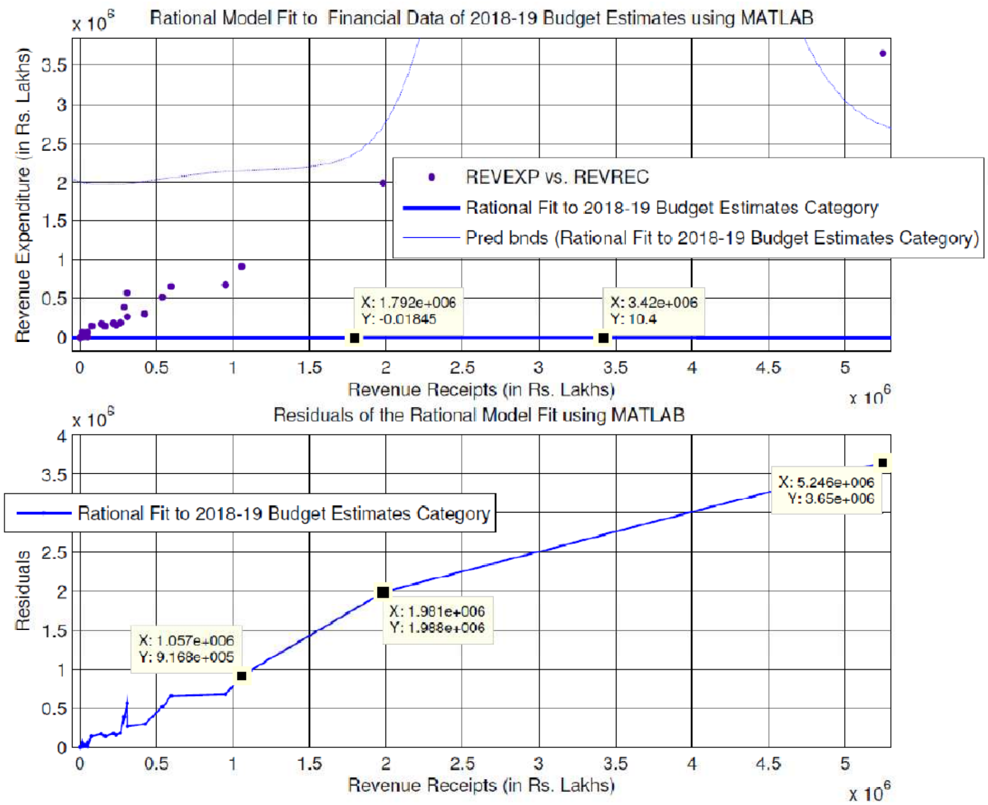

Fig. 35: The Plot and the Residual Plot for the Rational Model Fit of (5.2) for the data of 2018-19 Budget Estimates Category using MATLAB.

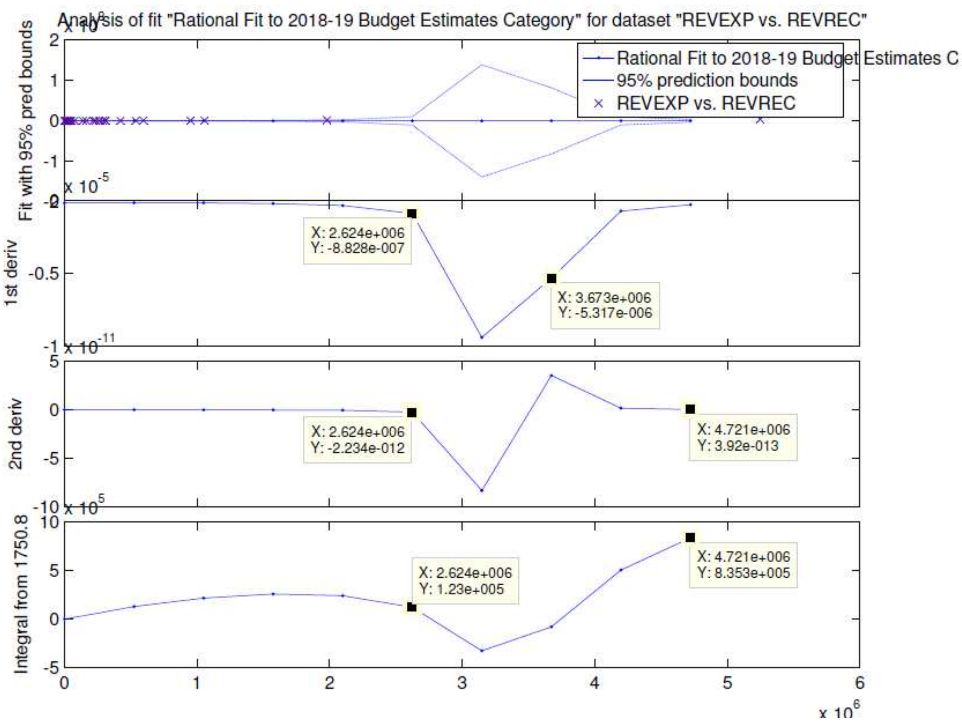

Fig. 36: The various Analysis Plots for the Rational Model Fit of (5.2) for the data of 2018-19 Budget Estimates Category using MATLAB.

Prediction Table for the Rational Model for 2018-19 Budget Estimates using MATLAB

From Xi= 1750.8:524411:5.24586e+006

Warning: Imaginary parts of complex X and/or Y arguments ignored

Xi	lower f(Xi)	f(Xi)	upper f(Xi)	df(Xi)/dX	d2f(Xi)/dX2	Integral f(Xi)
1750.8	-549123	0.290738	549124	-1.70494e-007	5.23291e-014	0+0i
526162	-650857	0.206464	650857	-1.5469e-007	8.75398e-015	130008+0i
1.05057e+006	-943562	0.124457	943562	-1.62416e-007	-4.15408e-014	216950+0i
1.57498e+006	-1.0855e+006	0.0300451	1.0855e+006	-2.05959e-007	-1.38508e-013	258442+0i
2.09939e+006	-2.49199e+006	-0.106539	2.49199e+006	-3.40296e-007	4.39895e-013	241386+0i
2.62381e+006	-1.03444e+007	-0.389732	1.03444e+007	-8.82836e-007	-2.23378e-012	123010+0i
3.14822e+006	-1.38719e+008	-1.87955	1.38719e+008	-9.36639e-006	-8.28632e-011	324702+0i
3.67263e+006	-8.17869e+007	1.78761	8.17869e+007	-5.3169e-006	3.53536e-011	-78202.8-1.48704e+006i
4.19704e+006	-1.00419e+007	0.765993	1.00419e+007	-7.23167e-007	1.70123e-012	505064-1.48704e+006i
4.72145e+006	-3.3915e+006	0.529979	3.3915e+006	2.82658e-007	3.91968e-013	835252-1.48704e+006i

Fig. 37: The Prediction Table for the Rational Model Fit of (5.2) for the data of 2018-19 Budget Estimates Category using MATLAB.

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