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A Note on Banach Summability of a Fourier series

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ABSTRACT

The present research paper investigates the Banach summability of Fourier series, focusing on conditions under which the Fourier series of a function converges in the Banach space setting. We explore the key concepts, including the Banach summability method, its relation to traditional summability criteria, and its impact on the convergence of Fourier series. Through examples and theoretical results, the paper clarifies the role of Banach summability in ensuring improved convergence for specific classes of functions.

Keywords: Absolute Summability, Indexed Summability, Banach Summability,Big O,Small O, Fourier Series.

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1. Introduction, Preliminaries & Motivation

In the realm of mathematical analysis, Banach summability emerges as the centre of attraction for concept that enriches our understanding of series. It surpass the limitations of classical convergence, inviting us to explore the intricate landscapes of series that may diverge or converge only conditionally. This elegant framework, rooted in analysis, empowers mathematicians to assign meaningful limits to these series, unveiling the hidden harmonies within seemingly chaotic expressions.

Equally enchanting is the world of Fourier series, where periodic functions are elegantly expressed as infinite sums of sines and cosines. This powerful tool serves as a bridge between abstract theory and practical application, finding resonance in diverse fields such as signal processing, physics, and engineering. Yet, the journey of Fourier series is not without its challenges; functions with discontinuities often thwart traditional convergence, prompting a quest for deeper understanding.

Herein lies the synergy between Banach summability and Fourier series. By employing the principles of Banach summability, we can navigate the complexities of Fourier series, granting us insight into their behavior even when conventional convergence falters. This interplay not only enhances our theoretical knowledge but also equips us with versatile tools to tackle real-world problems.

The theory of summability was first discovered by great mathematician Godfrey Harold Hardy [1]. He was worked on the development the theory of divergent theory (1970). His famous & evergreen book "Divergent Series "influenced many great mathematicians like Stefan Banach [2], Salomor Bochaner [3], Ram Chandran [4], Shyam Lal Singh [5]. The groundbreaking work was done on Banach limits & Banach summability by Stefan Banach [6] revolutionized the field, providing a rigorous framework for studying convergence & divergence of a series. Later S.K Paikray et al. [7] has initiated absolute indexed summability factor of an infinite series using quasi monotone sequence., R. K Jati etal [8] have adopted absolute indexed matrix summability of an infinite series. Mishra & Mishra [8] G. D. Dikshit [9], L. McFadden [10], T. Pati [11] have been extending different methods of absolute summability methods & Fourier Series.

Definition 1.1.

Let $\sum u_n$ be an infinite series and let s_n be the sequence of partial sums.Let the sequence { t_r(n)} defined by ^{n}

t_r(n)=$\frac{1}{r}\sum_{v=0}^{r-1} s_{n+v}$ (1.1)

then we say t_r(n) is to be the r-th element of the Banach transformed sequence.

If $\lim\limits_{t\to\infty} t_r(n) = s$, a definite number uniformly for n∈N , (1.2)

then $we\ say$ $\sum u_n$ is said to be Banach summable to s[1]. Further, if

$\sum\{t_r(n) - t_r{+}_1(n)\}$ <∞ uniformly for $n \in N$ (1.3)

∞

then, the series $\sum u_n$ said to be absolutely Banach summable or simply

Again, for $k \geq 1$, if

$$\sum_{r=1}^{\infty} r^{k-1}\ |t_r(n) - t_{r+1}(n)| < \infty\ ,\quad \text{uniformly for n} \in \text{N}\quad .\quad u - 1 = 0 \qquad (1.4)$$

Then, we shall say that $\sum u_n$ absolutely Banach summable of index k .

Definition 1.2.

Big O notation provides an upper bound on the growth rate of a function. If we write f(n) =O(g(n)), it means that there exists a positive constant C & a value n such that for all n>n_0

$$|f(n)|\leq C|g(n)|$$

Example: Let f(n)=3n^2+2n+1, We can say that:

f(n)=O(n^2)

because for sufficiently large n, 3n^2+2n+1 is less than or equal to Cn^2 for some constant C.

Small o notation describes a function that grows slower than another function.

If f(n)=o(g(n)),

It means that $\lim\limits_{n\to\infty}\frac{f(n)}{g(n)}$=1

His indicates that f(n) becomes insignificant relative to g(n) as n →∞

Example: If f(n)=n & g(n)=n^2 hen f(n)=o(g(n)) because n becomes insignificant compared to n^2 as n increases.

Let *f(t)* be a 2π -periodic function which is L-integrable in $(-\pi, \pi)$. then the series

$$\sum_{n=0}^{\infty} A_n\,(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n sinnx) \qquad (1.5)$$

is called the Fourier series of $f\ (t)$ where

$$\left.\begin{aligned} a_o &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{dt} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, sinntdt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, sinnt \end{aligned}\right] \qquad (1.6)$$

We denote

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

2. Know Result On approaching with absolute Banach summability of a Fourier

Series, Dikshit, Mishra e tal prove the following theorem
Theorem 1. Let $\alpha \geq 1$ & let the function q_n the following conditions
(i)$\int_0^1 q_\sigma$(t)dt=1
(ii) for 0<σ<1, q_σ(t) is increasing for0<t<1.
(iii) $\frac{Q_{\sigma(t)}}{t^{\sigma-p+1}}$ ∈L(0,1) with $\sigma = \alpha$. If φ_α(t) ∈ $BV(0, \pi)$, then at t=x, the Fourier series of f is summable by the method $|N(_{q_\alpha})|$.

Theorem 2. If ø(t) is a bonded variation function in the interval (0,π) ,then the Fourier series $\sum A_n(x)$ *of f(t*) is absolute Banach summable.

Theorem-3. If $\{\boldsymbol{\lambda_n}\}$ non negative sequence of numbers such that $\sum \frac{\boldsymbol{\lambda_n}}{\boldsymbol{n}} \leq \infty$ & if $\int_0^\pi \frac{\boldsymbol{\phi(t)}}{\boldsymbol{t}} \leq \infty$,then the factor Fourier Series $\sum \lambda_n A_n\,(x)$ is absolute Banach summable.

In the view of above, we shall generalize absolute indexed summability of Fourier series.

3. MAIN RESULT

Theorem 3. If$\emptyset$(t) is a bonded variation function in the interval (0,π) ,then the Fourier series $\sum^{A_n}$ (x) of f(t) is absolute indexed Banach summability.

We need the following lemma to establish the above theorem.
Lemma1. Let t_r(n) defined by (1.1). then

$$t_r(\mathrm{n}) - t_{r+1}(n) = \frac{-1}{r(r+1)}\sum_{v=1}^{v=r} v u_{n+v} \qquad (3.1)$$

Lemma2. Let $\{p_n\}$ be a positive non-decreasing sequence of real numbers &let $\tau=\left\{\frac{1}{t}\right\}$, then for all a,b$\epsilon$N

$$\sum_{n=a}^{b} aP_n cosnt \quad = O(p_\tau)$$

$$\sum_{n=a}^{b} aP_n sinnt= \ O(p_\tau) \qquad (3.2)$$

Lemma3. Let $\{\boldsymbol{\lambda_n}\}$ non negative sequence of numbers such that $\sum\frac{\boldsymbol{\lambda_n}}{\boldsymbol{n}}$<∞, then $\{\boldsymbol{\lambda_n}\}$ is monotonically decreasing.

Proof of the theorem:

Since from definition of Fourier series $A_n(x)=\frac{2}{\pi}\int_0^{\pi}\varphi(t)cosntdt$

$$=\frac{2}{n\pi}\int_0^{\pi}\varphi(t)sinntdt \quad \text{for all} \quad n=1,2,\ 3,\ldots.$$

For the series □ $A_n(x)$ $=\frac{2}{\pi}$*, we have by from given Lemma1*

$$\square \ r^{k-1}|t_r(n)-t_{r+1}(n)|^k=\sum_{r=1}^{\infty} r^{k-1}\left|\frac{1}{r(r+1)}\sum_{v=1}^{r} vA_{n+v(x)}\right|^k$$

$$=\left(\frac{2}{n}\right)^k\sum_{r=1}^{\infty}\frac{1}{r(r+1)^k}\left|\sum_{r=1}^{r=\infty}\int_0^{\pi}\frac{v}{n+v}sin(n+v)d\varphi(t)\right|k$$

Since ø(t) is a bonded variation function in the interval (0,π)

$$\Rightarrow \int_0^{\pi} d\text{ø}(t)<\infty$$

$$\Rightarrow \sum_{r=1}^{\infty} r^{k-1}\,|t_r(n)-t_{r+1}(n)|<\infty \ \textbf{is} \text{ uniformly in n.}$$

Again, if for all 0<t<π, $\sum_{r=1}^{\infty}\frac{1}{r(r+1)^k}\left|\sum_{r=1}^{r=\infty}\int_0^{\pi}\frac{v}{n+v}sin(n+v)\,t\right|^K < \infty$ *is uniformly for all t*

Now

$$\sum_{r=1}^{r}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{r}\frac{v}{n+v}sin(n+v)t\right|\text{k}+\sum_{r=r+1}^{\infty}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{v=\infty}\int_0^{\pi}\frac{v}{n+v}sin(n+v)\,t\right|^{\text{K}}$$

$$=\Sigma_1+\Sigma_2$$

Where $\Sigma_1=\sum_{r=1}^{\tau}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{r}\frac{v}{n+v}sin(n+v)t\right|^{\text{K}}$

$$\leq\sum_{r=1}^{r}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{r}\frac{v}{n+v}(n+v)t\right|^{\text{K}} \qquad (\therefore \text{sin}\leq\text{x})$$

$$=\ t^k\ \sum_{r=1}^{r}\frac{1}{r(r+1^k}\left|\sum_{v=1}^{r} v\right|^{\text{K}}$$

$$=t^k\sum_{r=1}^{\tau}\frac{1}{r(r+1)^k}\left(\frac{r(r+1)^k}{2}\right)$$

$$=\frac{t^k}{2^k}\sum_{r=1}^{\tau} r^{k-1}$$

$$=\frac{t^k}{2^k}\text{O}(\tau^k)$$

$$=\text{O}(1)$$

$$\Sigma_2=\sum_{r=r+1}^{\infty}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{r}\frac{v}{n+v}sin(n+v)t\right|^{\text{K}}$$

$$=\ \sum_{r=r+1}^{\infty}\frac{1}{r(r+1)^k}\left|\sum_{v=1}^{r}\frac{v}{n+v}(n+v)t\right|$$

$$=\text{O}(\tau^k)\left(\frac{r}{n+r}\right)^k\left|\sum_{v=1}^{r} sin(n+v)t\right|^{\text{k}}$$

$\leq O(\tau^k)\sum_{v=\tau+1}^{\infty}\frac{r^{k-1}}{(v+1)^k}(n+v)^k$
$\leq O(\tau^k)\sum_{v=\tau+1}^{\infty}\frac{1}{(r+1)^{k+1}}$
$\leq O(\tau^k)\,(\tau^{k-})$
$=O(1)$

$\therefore \Sigma < \infty$ & uniformly in n

Hence $\sum A_n(x)$ is absolute Banach indexed summable.
This proves the theorem.

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